

## Research Article

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# Nonlinear Dirichlet problems with unilateral growth on the reaction

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**Abstract:** We consider a nonlinear Dirichlet problem driven by the  $p$ -Laplace differential operator with a reaction which has a subcritical growth restriction only from above. We prove two multiplicity theorems producing three nontrivial solutions, two of constant sign and the third nodal. The two multiplicity theorems differ on the geometry near the origin. In the semilinear case (that is,  $p = 2$ ), using Morse theory (critical groups), we produce a second nodal solution for a total of four nontrivial solutions. As an illustration, we show that our results incorporate and significantly extend the multiplicity results existing for a class of parametric, coercive Dirichlet problems.

**Keywords:** Unilateral growth, constant sign and nodal solutions, multiplicity theorems, critical groups

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## 1 Introduction

Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded domain with a  $C^2$ -boundary  $\partial\Omega$ . In this paper, we study the following nonlinear Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) = f(z, u(z)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here,  $\Delta_p$  denotes the  $p$ -Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \quad \text{for all } u \in W_0^{1,p}(\Omega), \quad 1 < p < \infty.$$

Usually such problems are examined under the assumption that the reaction  $f(z, \cdot)$  exhibits subcritical growth from above and below. In contrast, we assume here that  $f(z, \cdot)$  is subcritical only from above, while from below no growth restriction is imposed on  $f(z, \cdot)$ . In this setting, we prove a multiplicity theorem producing at least three nontrivial solutions, two of constant sign (one positive and one negative) and the third nodal (that is, sign-changing). Our multiplicity result compares with those proved by Liu and Liu [14], Liu [15] and Papageorgiou and Papageorgiou [17] who proved three solutions theorems for certain classes of coercive  $p$ -Laplacian equations. We also refer to Papageorgiou, Rădulescu and Repovš [19, 20] for multiplicity properties in the context of Robin problems with superlinear reaction and super-diffusive mixed problems.

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In all the aforementioned works, the reaction has bilateral subcritical growth and no nodal solutions are produced. In addition, in the present work, for the semilinear problem ( $p = 2$ ), using Morse theory (critical groups), we produce a second nodal solution, for a total of four nontrivial solutions. Finally, we mention the works of Villegas [23] and Filippakis, Gasinski and Papageorgiou [8] who proved existence theorems for unilaterally restricted scalar problems (that is,  $N = 1$ ). Villegas [23] studied semilinear (that is,  $p = 2$ ) Neumann problems and Filippakis, Gasinski and Papageorgiou [8] considered nonlinear (that is,  $1 < p < \infty$ ) periodic problems with a nonsmooth potential.

## 2 Mathematical background

Let  $X$  be a Banach space, let  $X^*$  be its topological dual and let  $\langle \cdot, \cdot \rangle$  denote the duality brackets for the pair  $(X^*, X)$ . We say that a function  $\varphi \in C^1(X)$  satisfies the *Palais–Smale condition* (*PS-condition*, for short), if the following property holds.

**Property 1.** Every sequence  $\{u_n\}_{n \geq 1} \subseteq X$  such that  $\{\varphi(u_n)\}_{n \geq 1} \subseteq X$  is bounded and

$$\varphi'(u_n) \rightarrow 0 \quad \text{in } X^* \text{ as } n \rightarrow \infty,$$

admits a strongly convergent subsequence.

This is a compactness-type condition on the functional  $\varphi$ , which leads to a deformation theorem from which one can derive the minimax theory of the critical values of  $\varphi$ . A basic result in this theory is the so-called “mountain pass theorem”, due to Ambrosetti and Rabinowitz [4].

**Theorem 2.1.** Assume that  $X$  is a Banach space,  $\varphi \in C^1(X)$  and it satisfies the PS-condition,  $u_0, u_1 \in X$ ,  $\|u_1 - u_0\| > \rho > 0$ ,

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho$$

and  $c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t))$ , where  $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$ . Then  $c \geq m_\rho$  and  $c$  is a critical value of  $\varphi$ .

In the study of problem (1.1), we will use the Sobolev space  $W_0^{1,p}(\Omega)$  (when  $p = 2$ , we will write  $H_0^1(\Omega)$ ) and the ordered Banach space  $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ , with the order cone

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega \right\}.$$

Here we denote the outward unit normal on  $\partial\Omega$  by  $n(\cdot)$ .

Let  $f_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that

$$|f_0(z, x)| \leq a(z)(1 + |x|^{r-1}) \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R},$$

with  $a \in L^\infty(\Omega)$  and

$$1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N \end{cases} \quad (\text{the critical Sobolev exponent}).$$

We set  $F_0(z, x) = \int_0^x f_0(z, s) ds$  and consider the  $C^1$ -functional  $\varphi_0 : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\varphi_0(u) = \frac{1}{p} \|Du\|_p^p - \int_\Omega F_0(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From Garcia Azorero, Manfredi and Peral Alonso [10], we recall the following result.

**Proposition 2.2.** Assume that  $u_0 \in W_0^{1,p}(\Omega)$  is a local  $C_0^1(\overline{\Omega})$ -minimizer of  $\varphi_0$ , that is, there exists  $\rho_0 > 0$  such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0.$$

Then  $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$  for some  $\alpha \in (0, 1)$  and  $u_0$  is also a local  $W_0^{1,p}(\Omega)$ -minimizer of  $\varphi_0$ , that is, there exists  $\rho_1 > 0$  such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \|h\| \leq \rho_1.$$

Hereafter, we denote the norm of the Sobolev space  $W_0^{1,p}(\Omega)$  by  $\|\cdot\|$ . By the Poincaré inequality we have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We will also use some basic facts about the spectrum of  $(-\Delta_p, W_0^{1,p}(\Omega))$ . So, we consider the following nonlinear eigenvalue problem:

$$-\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.1}$$

We say that  $\hat{\lambda}$  is an *eigenvalue* of  $(-\Delta_p, W_0^{1,p}(\Omega))$  if problem (2.1) admits a nontrivial solution  $\hat{u} \in W_0^{1,p}(\Omega)$ , which is an eigenfunction corresponding to the eigenvalue  $\hat{\lambda}$ . We know that there is a smallest eigenvalue  $\hat{\lambda}_1 > 0$  which is simple, isolated and admits the following variational characterization:

$$\hat{\lambda}_1 = \inf \left\{ \frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0 \right\}. \tag{2.2}$$

The infimum in (2.2) is realized on the corresponding one-dimensional eigenspace (recall that  $\hat{\lambda}_1 > 0$  is simple). It is clear from (2.2) that the elements of this eigenspace do not change sign. Let  $\hat{u}_1$  be the  $L^p$ -normalized, positive eigenfunction corresponding to  $\hat{\lambda}_1 > 0$ . From the nonlinear regularity theory and the nonlinear maximum principle (see, for example, [11, pp. 737–738]), we have  $\hat{u}_1 \in \text{int } C_+$ . From the Ljusternik–Schnirelmann minimax scheme we can obtain a whole strictly increasing sequence  $\{\hat{\lambda}_k\}_{k \geq 1}$  of eigenvalues such that  $\hat{\lambda}_k \rightarrow +\infty$ . We do not know if this sequence exhausts the spectrum of  $(-\Delta_p, W_0^{1,p}(\Omega))$ . This is the case if  $p = 2$  (linear eigenvalue problem) or  $N = 1$  (scalar eigenvalue problem). Since  $\hat{\lambda}_1$  is isolated, the second eigenvalue  $\hat{\lambda}_2^* > \hat{\lambda}_1$  is well-defined by

$$\hat{\lambda}_2^* = \inf \{ \hat{\lambda} : \hat{\lambda} > \hat{\lambda}_1, \hat{\lambda} \text{ is an eigenvalue of } (-\Delta_p, W_0^{1,p}(\Omega)) \}.$$

We know that  $\hat{\lambda}_2^* = \hat{\lambda}_2$ , that is, the second eigenvalue and the second Ljusternik–Schnirelmann eigenvalue coincide. For  $\hat{\lambda}_2$  we have the following minimax characterization due to Cuesta, de Figueiredo and Gossez [6].

**Proposition 2.3.** We have

$$\hat{\lambda}_2 = \inf_{\hat{\gamma} \in \hat{\Gamma}} \max_{-1 \leq t \leq 1} \|D\hat{\gamma}(t)\|_p^p,$$

where

$$\hat{\Gamma} = \{ \hat{\gamma} \in C([-1, 1], M) : \hat{\gamma}(-1) = -\hat{u}_1, \hat{\gamma}(1) = \hat{u}_1 \}$$

with

$$M = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p} \quad \text{and} \quad \partial B_1^{L^p} = \{ u \in L^p(\Omega) : \|u\|_p = 1 \}.$$

As we already said, in the case  $p = 2$  (linear eigenvalue problem), the spectrum of  $(-\Delta, H_0^1(\Omega))$  consists of a sequence  $\{\hat{\lambda}_k\}_{k \geq 1}$  of eigenvalues such that  $\hat{\lambda}_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ . We denote the corresponding eigenspace by  $E(\hat{\lambda}_k)$ . We have

$$H_0^1(\Omega) = \bigoplus_{k \geq 1} E(\hat{\lambda}_k).$$

In this case, we have nice variational characterizations for all the eigenvalues. Namely, we have

$$\hat{\lambda}_1 = \inf \left\{ \frac{\|Du\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0 \right\} \quad (\text{see (2.2)}), \tag{2.3}$$

and for  $k \geq 2$ ,

$$\hat{\lambda}_k = \inf \left\{ \frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \bar{H}_k = \overline{\bigoplus_{n \geq k} E(\hat{\lambda}_n)}, u \neq 0 \right\} = \sup \left\{ \frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \bar{H}_k = \bigoplus_{n=1}^k E(\hat{\lambda}_n), u \neq 0 \right\}. \tag{2.4}$$

Both the infimum and the supremum in (2.4) are realized on the corresponding eigenspace  $E(\hat{\lambda}_k)$ . Every such space has the so-called “unique continuation property” (UCP for short), which means that if  $u \in E(\hat{\lambda}_k)$  and  $u$  vanishes on a set of positive measure, then  $u \equiv 0$ . Note that by standard regularity theory,  $E(\hat{\lambda}_k) \subseteq C_0^1(\bar{\Omega})$  and  $E(\hat{\lambda}_k)$  is finite-dimensional. Invoking (2.3), (2.4) and the UCP, we have the following property.

**Proposition 2.4.** (a) *If  $\xi \in L^\infty(\Omega)$  with  $\xi(z) \geq \hat{\lambda}_k$  for almost all  $z \in \Omega$  with strict inequality on a set of positive measure, then*

$$\|Du\|_2^2 - \int_{\Omega} \xi(z)u^2 dz \leq -\hat{c}\|u\|^2 \quad \text{for all } u \in \bar{H}_k = \bigoplus_{n=1}^k E(\hat{\lambda}_k).$$

(b) *If  $\xi \in L^\infty(\Omega)$  with  $\xi(z) \leq \hat{\lambda}_k$  for almost all  $z \in \Omega$  with strict inequality on a set of positive measure, then there exists  $\tilde{c} > 0$  such that*

$$\|Du\|_2^2 - \int_{\Omega} \xi(z)u^2 dz \geq \tilde{c}\|u\|^2 \quad \text{for all } u \in \overline{\bigoplus_{n \geq k} E(\hat{\lambda}_k)}.$$

In what follows, we denote by

$$A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \quad \left( \frac{1}{p} + \frac{1}{p'} = 1 \right)$$

the nonlinear map corresponding to the  $p$ -Laplace differential operator and defined by

$$\langle A(u), v \rangle = \int_{\Omega} |Du|^{p-2} (Du, Dv)_{\mathbb{R}^N} dz \quad \text{for all } u, v \in W_0^{1,p}(\Omega). \tag{2.5}$$

From Papageorgiou and Kyritsi [18, p. 314], we have the following proposition.

**Proposition 2.5.** *The operator  $A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$  defined by (2.5) is continuous, strictly monotone (hence maximal monotone, too) and of type  $(S)_+$ , that is,*

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

implying

$$u_n \rightarrow u \quad \text{in } W_0^{1,p}(\Omega)$$

As before, let  $X$  be a Banach space, let  $\varphi \in C^1(X)$  and let  $c \in \mathbb{R}$ . We introduce the following sets:

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}, \quad K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}, \quad \varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

Let  $(Y_1, Y_2)$  be a topological pair such that  $Y_2 \subseteq Y_1 \subseteq X$ , and let  $k$  be a nonnegative integer. We denote by  $H_k(Y_1, Y_2)$  the  $k$ th-relative singular homology group of the topological pair  $(Y_1, Y_2)$  with integer coefficients. The critical groups of  $\varphi$  at an isolated  $u \in K_\varphi^c$  are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k,$$

with  $U$  being a neighborhood of  $u$  such that  $K_\varphi \cap \varphi^c \cap U = \{u\}$ . The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighborhood  $U$  of  $u$ .

Suppose that  $\varphi$  satisfies the PS-condition and  $-\infty < \inf \varphi(K_\varphi)$ . Let  $c < \inf \varphi(K_\varphi)$ . The critical groups of  $\varphi$  at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k.$$

The second deformation theorem (see, for example, [11, p. 628]) implies that the above definition of critical groups at infinity is independent of the choice of the level  $c < \inf \varphi(K_\varphi)$ .

Suppose that  $\varphi \in C^1(X)$  satisfies the *PS*-condition and  $K_\varphi$  is finite. We define

$$M(t, u) = \sum_{k \geq 0} \text{rank } C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, u \in K_\varphi,$$

$$P(t, \infty) = \sum_{k \geq 0} \text{rank } C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1+t)Q(t), \quad (2.6)$$

where  $Q(t) = \sum_{k \geq 0} \beta_k t^k$  is a formal series in  $t \in \mathbb{R}$  with nonnegative integer coefficients.

Finally, if  $x \in \mathbb{R}$ , we set  $x^\pm = \max\{\pm x, 0\}$ . Then for  $u \in W_0^{1,p}(\Omega)$  we set  $u^\pm(\cdot) = u(\cdot)^\pm$ . We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Also, if  $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function (for example, a Carathéodory function), then we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

(the Nemytskii map corresponding to  $h$ ). Note that  $z \mapsto N_h(u)(z)$  is measurable. We denote by  $|\cdot|_N$  the Lebesgue measure on  $\mathbb{R}^N$ .

### 3 The nonlinear equation ( $1 < p < \infty$ )

In this section, we deal with the general equation (1.1) and prove two multiplicity theorems producing three nontrivial solutions, all with sign information. The two multiplicity theorems differ in the geometry near the origin. In the first one, the reaction is  $(p-1)$ -sublinear near zero, while in the second it is  $(p-1)$ -superlinear (we have the presence of a concave term).

For the first multiplicity theorem, we start with the following hypotheses on the reaction  $f(z, x)$ . Using them, we will generate two nontrivial constant sign solutions.

(H<sub>1</sub>):  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function such that for almost all  $z \in \Omega$  one has  $f(z, 0) = 0$ ,  $f(z, \cdot)$  is locally  $\alpha$ -Hölder continuous with  $\alpha \in (0, 1]$  and local Hölder constant  $k \in L^\infty(\Omega)_+$  and the following assertions hold:

(i) For every  $\rho > 0$ , there exists  $a_\rho \in L^\infty(\Omega)$  such that

$$|f(z, x)| \leq a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \rho.$$

(ii) One has

$$\limsup_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \xi < \hat{\lambda}_1 \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) There exists a function  $\eta \in L^\infty(\Omega)$  such that  $\eta(z) \geq \hat{\lambda}_1$  for almost all  $z \in \Omega$ , the inequality is strict on a set of positive measure and

$$\liminf_{x \rightarrow 0} \frac{f(z, x)}{|x|^{p-2}x} \geq \eta(z) \quad \text{uniformly for almost all } z \in \Omega.$$

(iv) There exists  $M_0 > 0$  such that for almost all  $z \in \Omega$ ,

$$x \mapsto \frac{f(z, x)}{x^{p-1}} \quad \text{is nondecreasing on } [M_0, +\infty),$$

$$x \mapsto \frac{f(z, x)}{|x|^{p-2}x} \quad \text{is nonincreasing on } (-\infty, -M_0].$$

**Remark 3.1.** We stress that the above conditions do not impose any global growth condition from below on the reaction  $f(z, \cdot)$ .

Hypothesis (H<sub>1</sub>) (ii) implies that we can find  $\xi_1 \in (\xi, \hat{\lambda}_1)$  and  $M \geq M_0$  such that

$$f(z, x)x \leq \xi_1|x|^p \quad \text{for almost all } z \in \Omega \text{ and all } |x| \geq M. \tag{3.1}$$

Also, let  $\beta \in L^\infty(\Omega)_+$  such that

$$\beta(z) \geq a_M(z) + 1 \quad \text{for almost all } z \in \Omega \text{ (see hypothesis (H}_1\text{) (i)).} \tag{3.2}$$

Let  $\{t_n\}_{n \geq 1} \subseteq [1, +\infty)$  and assume that  $t_n \rightarrow +\infty$ . We define

$$h_n(z) = \begin{cases} \hat{\lambda}_1(t_n \hat{u}_1(z))^{p-1} & \text{if } z \in \{t_n \hat{u}_1 > M\}, \\ t_n^{p-1} \beta(z) & \text{if } z \in \{t_n \hat{u}_1 \leq M\}. \end{cases}$$

Evidently,  $h_n \in L^\infty(\Omega)$  for all  $n \geq 1$ . Recall that  $\hat{u}_1 \in \text{int } C_+$ . Hence  $\{t_n \hat{u}_1 \leq M\} \downarrow \emptyset$  as  $n \rightarrow \infty$ . So, for every  $r \in [1, \infty)$  we have

$$\left\| \frac{h_n}{t_n^{p-1}} - \hat{\lambda}_1 \hat{u}_1^{p-1} \right\|_r \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.3}$$

On the other hand, by [7, p. 477] we know that

$$\left\| \frac{h_n}{t_n^{p-1}} - \hat{\lambda}_1 \hat{u}_1^{p-1} \right\|_r \rightarrow \left\| \frac{h_n}{t_n^{p-1}} - \hat{\lambda}_1 \hat{u}_1^{p-1} \right\|_\infty \quad \text{as } r \rightarrow \infty, \text{ for every } n \geq 1. \tag{3.4}$$

Then from (3.4) we see that, given  $\epsilon > 0$ , we can find  $r_0 = r_0(\epsilon) \in \mathbb{N}$  such that

$$\left\| \frac{h_n}{t_n^{p-1}} - \hat{\lambda}_1 \hat{u}_1^{p-1} \right\|_\infty \leq \left\| \frac{h_n}{t_n^{p-1}} - \hat{\lambda}_1 \hat{u}_1^{p-1} \right\|_r + \frac{\epsilon}{2} \quad \text{for all } r \geq r_0. \tag{3.5}$$

Fix  $r \geq r_0$ . From (3.3) we see that we can find  $n_0 = n_0(\epsilon) \in \mathbb{N}$  such that

$$\left\| \frac{h_n}{t_n^{p-1}} - \hat{\lambda}_1 \hat{u}_1^{p-1} \right\|_r \leq \frac{\epsilon}{2} \quad \text{for all } n \geq n_0. \tag{3.6}$$

For the fixed  $r \geq r_0$ , using (3.6) in (3.5), we obtain

$$\left\| \frac{h_n}{t_n^{p-1}} - \hat{\lambda}_1 \hat{u}_1^{p-1} \right\|_\infty \leq \epsilon \quad \text{for all } n \geq n_0,$$

which implies

$$\frac{h_n}{t_n^{p-1}} \rightarrow \hat{\lambda}_1 \hat{u}_1^{p-1} \quad \text{in } L^\infty(\Omega) \text{ as } n \rightarrow \infty. \tag{3.7}$$

Then for every  $n \geq 1$  we consider the following auxiliary Dirichlet problem

$$-\Delta_p u_n(z) = h_n(z) \quad \text{in } \Omega, \quad u_n|_{\partial\Omega} = 0.$$

This problem has a unique solution  $u_n \in W_0^{1,p}(\Omega)$ ,  $u_n \geq 0$ . The nonlinear regularity theory and the nonlinear maximum principle (see [11, pp. 737–738]) imply that  $u_n \in \text{int } C_+$  for all  $n \geq 1$ . Let  $v_n = \frac{u_n}{t_n}$  for all  $n \geq 1$ . We have

$$-\Delta_p v_n(z) = \frac{h_n(z)}{t_n^{p-1}} \quad \text{in } \Omega, \quad v_n|_{\partial\Omega} = 0.$$

From [11, p. 738] we know that we can find  $\theta \in (0, 1)$  and  $M_1 > 0$  such that

$$v_n \in C_0^{1,\theta}(\bar{\Omega}) \quad \text{and} \quad \|v_n\|_{C_0^{1,\theta}(\bar{\Omega})} \leq M_1 \quad \text{for all } n \geq 1. \tag{3.8}$$

Exploiting the compact embedding of  $C_0^{1,\theta}(\bar{\Omega})$  into  $C_0^1(\bar{\Omega})$  and using (3.7), we can infer from (3.8) that

$$v_n \rightarrow \hat{u}_1 \quad \text{in } C_0^1(\bar{\Omega}) \text{ as } n \rightarrow \infty. \tag{3.9}$$

Hence, by (3.9) we can find  $n_1 \in \mathbb{N}$  such that

$$\xi_1 v_n(z)^{p-1} \leq \hat{\lambda}_1 \hat{u}_1(z)^{p-1} \quad \text{for all } z \in \bar{\Omega}, n \geq n_1, \tag{3.10}$$

and if  $t_n \hat{u}_1(z) > M$ , then  $t_n v_n(z) > M$  for all  $n \geq n_1$ .

Also, by (3.9) and our hypothesis on  $f(z, \cdot)$  we can find  $n_2 \in \mathbb{N}$  such that

$$|f(z, \hat{u}_1(z)) - f(z, v_n(z))| \leq \|k\|_\infty \|\hat{u}_1 - v_n\|_\infty^\alpha \leq 1 \quad \text{for all } z \in \Omega, n \geq n_2. \tag{3.11}$$

Let  $n_0 = \max\{n_1, n_2\}$ . Then for  $n \geq n_0$  we have the following assertions:

- If  $z \in \{t_n \hat{u}_1 > M\}$ , then

$$-\Delta_p(t_n v_n)(z) = \hat{\lambda}_1 (t_n u_n(z))^{p-1} \geq \xi_1 (t_n v_n(z))^{p-1} \geq f(z, t_n v_n(z)) \quad (\text{see (3.1) and (3.10)}),$$

which implies  $-\Delta_p v_n(z) \geq f(z, v_n(z))$  (see hypotheses  $(H_1)$  (iv) and recall that  $t_n \geq 1$ ).

- If  $z \in \{t_n \hat{u}_1 \leq M\}$ , then

$$-\Delta_p v_n(z) = \frac{h(z)}{t_n^{p-1}} = \beta(z) \geq f(z, \hat{u}_1(z)) + 1 \geq f(z, v_n(z)) \quad (\text{see (3.2) and (3.11)}).$$

So, fixing  $n \geq n_0$  and setting  $\bar{u} = v_n \in \text{int } C_+$ , we have

$$-\Delta_p \bar{u}(z) \geq f(z, \bar{u}(z)) \quad \text{for almost all } z \in \Omega. \tag{3.12}$$

In a similar fashion, we produce  $\bar{v} \in -\text{int } C_+$  such that

$$-\Delta_p \bar{v}(z) \leq f(z, \bar{v}(z)) \quad \text{for almost all } z \in \Omega. \tag{3.13}$$

Now, we are ready to produce nontrivial constant sign solutions for problem (1.1).

**Proposition 3.2.** *Assume that hypotheses  $(H_1)$  hold. Then problem (1.1) admits at least two constant sign solutions*

$$u_0 \in [0, \bar{u}] \cap \text{int } C_+ \quad \text{and} \quad v_0 \in [\bar{v}, 0] \cap (-\text{int } C_+),$$

where

$$[0, \bar{u}] = \{u \in W_0^{1,p}(\Omega) : 0 \leq u(z) \leq \bar{u}(z) \text{ for almost all } z \in \Omega\},$$

$$[\bar{v}, 0] = \{u \in W_0^{1,p}(\Omega) : \bar{v}(z) \leq u(z) \leq 0 \text{ for almost all } z \in \Omega\}.$$

*Proof.* First, we produce the positive solution. To this end, we consider the following truncation of  $f(z, \cdot)$ :

$$\hat{f}_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ f(z, x) & \text{if } 0 \leq x \leq \bar{u}(z), \\ f(z, \bar{u}(z)) & \text{if } \bar{u}(z) < x. \end{cases} \tag{3.14}$$

This is a Carathéodory function. We set

$$\hat{F}_+(z, x) = \int_0^x \hat{f}_+(z, s) ds$$

and consider the  $C^1$ -functional  $\hat{\phi}_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\hat{\phi}_+(u) = \frac{1}{p} \|Du\|_p^p - \int_\Omega \hat{F}_+(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (3.14) it is clear that  $\hat{\phi}_+$  is coercive. Also, using the Sobolev embedding theorem, we can easily check that  $\hat{\phi}_+$  is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find  $u_0 \in W_0^{1,p}(\Omega)$  such that

$$\hat{\phi}_+(u_0) = \inf\{\hat{\phi}_+(u) : u \in W_0^{1,p}(\Omega)\} = \hat{m}_+. \tag{3.15}$$

By virtue of hypothesis (H<sub>1</sub>) (iii), given  $\epsilon > 0$ , we can find  $\delta = \delta(\epsilon) > 0$  such that

$$F(z, x) \geq \frac{1}{p}(\eta(z) - \epsilon)x^p \quad \text{for almost all } z \in \Omega \text{ and all } x \in [0, \delta]. \tag{3.16}$$

Here,

$$F(z, x) = \int_0^x f(z, s) \, ds.$$

Since  $\hat{u}_1 \in \text{int } C_+$ , we can find  $t \in (0, 1)$  small enough so that  $t\hat{u}_1(z) \in [0, \delta]$  for all  $z \in \bar{\Omega}$ . We have

$$\begin{aligned} \hat{\varphi}_+(t\hat{u}_1) &\leq \frac{t^p}{p} \hat{\lambda}_1 \|\hat{u}_1\|_p^p - \frac{t^p}{p} \int_{\Omega} \eta(z) \hat{u}_1^p \, dz + \frac{t^p}{p} \epsilon \quad (\text{recall that } \|\hat{u}_1\|_p = 1 \text{ and see (3.16)}) \\ &= \frac{t^p}{p} \left[ \int_{\Omega} (\hat{\lambda}_1 - \eta(z)) \hat{u}_1^p \, dz + \epsilon \right]. \end{aligned} \tag{3.17}$$

Note that

$$\xi_0 = \int_{\Omega} (\eta(z) - \hat{\lambda}_1) \hat{u}_1^p \, dz > 0.$$

So, if we choose  $\epsilon \in (0, \xi_0)$ , then from (3.17) we see that  $\hat{\varphi}_+(t\hat{u}_1) < 0$ , which implies (see (3.15))

$$\hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0),$$

and hence  $u_0 \neq 0$ .

From (3.15) we have  $\hat{\varphi}'_+(u_0) = 0$ , implying

$$A(u_0) = N_{\hat{f}_+}(u_0). \tag{3.18}$$

On (3.18) we first act with  $-u_0^- \in W_0^{1,p}(\Omega)$ . We obtain  $\|Du_0^-\|_p^p = 0$  (see (3.14)), implying  $u_0 \geq 0$  and  $u_0 \neq 0$ . Then we act on (3.18) with  $(u_0 - \bar{u})^+ \in W_0^{1,p}(\Omega)$ . We have

$$\begin{aligned} \langle A(u_0), (u_0 - \bar{u})^+ \rangle &= \int_{\Omega} \hat{f}_+(z, u_0)(u_0 - \bar{u})^+ \, dz \\ &= \int_{\Omega} f(z, \bar{u})(u_0 - \bar{u})^+ \, dz \quad (\text{see (3.14)}) \\ &\leq \langle A(\bar{u}), (u_0 - \bar{u})^+ \rangle \quad (\text{see (3.12)}), \end{aligned}$$

implying

$$\int_{\{u_0 > \bar{u}\}} (|Du_0|^{p-2} Du_0 - |D\bar{u}|^{p-2} D\bar{u}, Du_0 - D\bar{u})_{\mathbb{R}^N} \, dz \leq 0.$$

This implies  $|\{u_0 > \bar{u}\}|_N = 0$ , and hence  $u_0 \leq \bar{u}$ .

So, we have proved that

$$u_0 \in [0, \bar{u}] \quad \text{and} \quad u_0 \neq 0. \tag{3.19}$$

Then (3.18) becomes  $A(u_0) = N_f(u_0)$  (see (3.14) and (3.19)), implying

$$-\Delta_p u_0(z) = f(z, u_0(z)) \quad \text{for almost all } z \in \Omega, \quad u_0|_{\partial\Omega} = 0. \tag{3.20}$$

The nonlinear regularity theory (see [11, pp. 737–738]) now implies that  $u_0 \in C_+ \setminus \{0\}$ . Let  $\rho = \|u_0\|_{\infty}$ . Hypotheses (H<sub>1</sub>) (i) and (iii) imply that we can find  $\tilde{\xi}_\rho > 0$  such that

$$f(z, x) + \tilde{\xi}_\rho x^{p-1} \geq 0 \quad \text{for almost all } z \in \Omega \text{ and all } x \in [0, \rho]. \tag{3.21}$$

Then from (3.20) and (3.21) we have  $\Delta_p u_0(z) \leq \tilde{\xi}_\rho u_0(z)^{p-1}$  for almost all  $z \in \Omega$ , implying  $u_0 \in \text{int } C_+$  by the nonlinear maximum principle (see [10, p. 738]).

Similarly, for the negative solution we introduce the truncation

$$\hat{f}_-(z, x) = \begin{cases} f(z, \bar{v}(z)) & \text{if } x < \bar{v}(z), \\ f(z, x) & \text{if } \bar{v}(z) \leq x \leq 0, \\ 0 & \text{if } 0 < x. \end{cases}$$

This is a Carathéodory function. We set

$$\hat{F}_-(z, x) = \int_0^x \hat{f}_-(z, s) ds$$

and consider the  $C^1$ -functional  $\hat{\varphi}_- : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\hat{\varphi}_-(u) = \frac{1}{p} \|Du\|_p^p - \int_{\Omega} \hat{F}_-(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working with  $\hat{\varphi}_-$  as above, via the direct method and using (3.13), we produce a negative solution

$$v_0 \in [\bar{v}, 0] \cap (-\text{int } C_+).$$

The proof is now complete. □

In fact, we can produce extremal constant sign solutions, that is, a smallest positive and a biggest negative solution. These extremal solutions will be helpful in obtaining nodal ones.

**Proposition 3.3.** *Assume that hypotheses  $(H_1)$  hold. Then problem (1.1) admits a smallest positive solution  $u_* \in \text{int } C_+$  and a biggest negative solution  $v_* \in -\text{int } C_+$ .*

*Proof.* First, we produce the smallest positive solution. Let  $S_+$  be the set of positive solutions of problem (1.1). From Proposition 3.2 and its proof we know that  $S_+ \cap [0, \bar{u}] \neq \emptyset$  and  $S_+ \subseteq \text{int } C_+$ . By Hu and Papageorgiou [12, p. 178], we know that we can find  $\{u_n\}_{n \geq 1} \subseteq S_+ \cap [0, \bar{u}]$  such that

$$\inf S_+ = \inf_{n \geq 1} u_n.$$

We have

$$A(u_n) = N_f(u_n), \quad u_n \leq \bar{u} \quad \text{for all } n \geq 1, \tag{3.22}$$

implying that  $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$  is bounded (see hypothesis  $(H_1)$  (i)).

So, we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \rightarrow u_* \quad \text{in } L^p(\Omega).$$

On (3.22) we act with  $u_n - u_* \in W_0^{1,p}(\Omega)$ , pass to the limit as  $n \rightarrow \infty$  and use (3.18). Then

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u_* \rangle = 0,$$

implying

$$u_n \rightarrow u_* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{(see Proposition 2.5)}. \tag{3.23}$$

So, if in (3.22) we pass to the limit as  $n \rightarrow \infty$  and use (3.23), then (nonlinear regularity theory, see [10, p. 738])  $A(u_*) = N_f(u_*)$ , implying that  $u_*$  is a nonnegative solution of (1.1) and  $u_* \in C_+$ .

We need to show that  $u_* \neq 0$ . Owing to hypotheses  $(H_1)$  (i) and (iii), given  $\epsilon > 0$ , we can find  $c_1 = c_1(\epsilon) > 0$  such that

$$f(z, x)x \geq (\eta(z) - \epsilon)|x|^p - c_1|x|^r \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \rho, \tag{3.24}$$

with  $r > p$  and  $\rho = \max\{\|\bar{u}\|_{\infty}, \|\bar{v}\|_{\infty}\}$ . We introduce the following Carathéodory functions:

$$g_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ (\eta(z) - \epsilon)x^{p-1} - c_1x^{r-1} & \text{if } 0 \leq x \leq \bar{u}(z), \\ (\eta(z) - \epsilon)\bar{u}(z)^{p-1} - c_1\bar{u}(z)^{r-1} & \text{if } \bar{u}(z) < x, \end{cases} \tag{3.25}$$

and

$$g_-(z, x) = \begin{cases} (\eta(z) - \epsilon)|\bar{v}(z)|^{p-2}\bar{v}(z) - c_1|\bar{v}(z)|^{r-2}\bar{v}(z) & \text{if } x < \bar{v}(z), \\ (\eta(z) - \epsilon)|x|^{p-2}x - c_1|x|^{r-2}x & \text{if } \bar{v}(z) \leq x \leq 0, \\ 0 & \text{if } 0 < x. \end{cases}$$

We consider the following auxiliary Dirichlet problems:

$$-\Delta_p u(z) = g_+(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{3.26}$$

$$-\Delta_p v(z) = g_-(z, v(z)) \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0. \tag{3.27}$$

**Claim 1.** Problem (3.26) (resp. problem (3.27)) for  $\epsilon > 0$  small admits a unique positive solution  $\bar{u} \in \text{int } C_+$  (resp. a unique negative solution  $\bar{v} \in -\text{int } C_+$ ).

First, we deal with problem (3.26). So, let  $\psi_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  be the  $C^1$ -functional defined by

$$\psi_+(u) = \frac{1}{p} \|Du\|_p^p - \int_{\Omega} G_+(z, u(z)) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where

$$G_+(z, x) = \int_0^x g_+(z, s) \, ds.$$

From (3.25) it is clear that  $\psi_+$  is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find  $\bar{u} \in W_0^{1,p}(\Omega)$  such that

$$\psi_+(\bar{u}) = \inf\{\psi_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{3.28}$$

Let  $t \in (0, 1)$  be small such that  $t\hat{u}_1 \leq \bar{u}$  (recall that  $\bar{u} \in \text{int } C_+$  and use [9, Lemma 3.3]). We have

$$\begin{aligned} \psi_+(t\hat{u}_1) &\leq \frac{t^p}{p} \hat{\lambda}_1 + \frac{c_1}{r} t^r \|\hat{u}_1\|_r^r - \frac{t^p}{p} \int_{\Omega} (\eta(z) - \epsilon) \hat{u}_1^p \, dz \quad (\text{see (3.25)}) \\ &= \frac{t^p}{p} \left[ \int_{\Omega} (\hat{\lambda}_1 - (\eta(z) - \epsilon)) \hat{u}_1^p \, dz \right] + \frac{c_1}{r} t^r \|\hat{u}_1\|_r^r. \end{aligned}$$

Note that  $\beta = \int_{\Omega} (\eta(z) - \hat{\lambda}_1) \hat{u}_1^p \, dz > 0$ . So, choosing  $\epsilon \in (0, \beta)$ , we obtain

$$\psi_+(t\hat{u}_1) \leq -\frac{t^p}{p} c_2 + \frac{t^r}{r} c_1 \|\hat{u}_1\|_r^r \quad \text{for some } c_2 > 0.$$

Since  $r > p$ , by choosing  $t \in (0, 1)$  even smaller if necessary, we obtain  $\psi_+(t\hat{u}_1) < 0$ , which implies  $\psi_+(\bar{u}) < 0 = \psi_+(0)$  (see (3.28)), and hence  $\bar{u} \neq 0$ .

From (3.28) we have  $\psi'_+(\bar{u}) = 0$ , implying

$$A(\bar{u}) = N_{g_+}(\bar{u}). \tag{3.29}$$

On (3.29) we first act with  $-\bar{u}^- \in W_0^{1,p}(\Omega)$ . Then  $\|D\bar{u}^-\|_p^p = 0$  (see (3.25)), implying  $\bar{u} \geq 0$  and  $\bar{u} \neq 0$ . Also, we act on (3.29) with  $(\bar{u} - \bar{u})^+ \in W_0^{1,p}(\Omega)$ . Then

$$\begin{aligned} \langle A(\bar{u}), (\bar{u} - \bar{u})^+ \rangle &= \int_{\Omega} [(\eta(z) - \epsilon)\bar{u}^{p-1} - c_1\bar{u}^{r-1}](\bar{u} - \bar{u})^+ \, dz \quad (\text{see (3.25)}) \\ &\leq \int_{\Omega} f(z, \bar{u})(\bar{u} - \bar{u})^+ \, dz \quad (\text{see (3.24)}) \\ &\leq \langle A(\bar{u}), (\bar{u} - \bar{u})^+ \rangle \quad (\text{see (3.12)}), \end{aligned}$$

which implies

$$\int_{\{\bar{u} > \bar{u}\}} (|D\bar{u}|^{p-2}D\bar{u} - |D\bar{u}|^{p-2}D\bar{u}, D\bar{u} - D\bar{u})_{\mathbb{R}^N} \, dz \leq 0.$$

Therefore,  $|\{\bar{u} > \bar{u}\}|_N = 0$ , and hence  $\bar{u} \leq \bar{u}$ .

So, we have proved that

$$\tilde{u} \in [0, \bar{u}] \quad \text{and} \quad \tilde{u} \neq 0. \tag{3.30}$$

By (3.25) and (3.30), equation (3.29) becomes

$$A(\tilde{u}) = (\eta(\cdot) - \epsilon)\tilde{u}^{p-1} - c_1\tilde{u}^{r-1},$$

implying that  $\tilde{u}$  is a positive solution of (3.26).

The nonlinear regularity theory and the nonlinear maximum principle (see [11, pp. 737–738]) imply

$$\tilde{u} \in \text{int } C_+.$$

Now we show that  $\tilde{u}$  is the unique positive solution of (3.26). To this end, let  $\tilde{y}$  be another positive solution of (3.26). As we did for  $\tilde{u}$ , we can show that  $\tilde{y} \in [0, \bar{u}] \cap \text{int } C_+$ . Note that we can find  $c_3 > 0$  such that for almost all  $z \in \Omega$  the function  $x \mapsto (\eta(z) + c_3 - \epsilon)x^{p-1} - c_1x^{r-1}$  is nondecreasing on  $[0, \rho]$  (recall that  $\rho = \max\{\|\bar{u}\|_\infty, \|\bar{v}\|_\infty\}$ ). Let  $t > 0$  be the biggest positive real such that

$$t\tilde{y} \leq \tilde{u} \quad (\text{see [9, Lemma 3.3]}).$$

Suppose  $t \in (0, 1)$ . We have

$$\begin{aligned} -\Delta_p(t\tilde{y}) + c_3(t\tilde{y})^{p-1} &= t^{p-1}[-\Delta_p\tilde{y} + c_3\tilde{y}^{p-1}] \\ &= t^{p-1}[(\eta(z) + c_3 - \epsilon)\tilde{y}^{p-1} - c_1\tilde{y}^{r-1}] \\ &< (\eta(z) - \epsilon)(t\tilde{y})^{p-1} - c_1(t\tilde{y})^{r-1} + c_3(t\tilde{y})^{p-1} \quad (\text{since } r > p, t \in (0, 1)) \\ &\leq (\eta(z) - \epsilon)\tilde{u}^{p-1} - c_1\tilde{u}^{r-1} + c_3\tilde{u}^{p-1} \quad (\text{since } t\tilde{y} \leq \tilde{u} \text{ and by the choice of } c_3) \\ &= -\Delta_p\tilde{u} + c_3\tilde{u}^{p-1} \quad (\text{since } \tilde{u} \in \text{int } C_+ \text{ is a solution of (3.26)}), \end{aligned}$$

thus implying

$$\tilde{u} - t\tilde{y} \in \text{int } C_+ \quad (\text{see [5, Proposition 2.6]}).$$

This contradicts the maximality of  $t > 0$ . Therefore  $t \geq 1$ , and so

$$\tilde{y} \leq \tilde{u}.$$

If in the above argument we interchange the roles of  $\tilde{y}$  and  $\tilde{u}$ , we also have  $\tilde{u} \leq \tilde{y}$ , implying  $\tilde{u} = \tilde{y}$ .

This proves the uniqueness of the solution  $\tilde{u} \in \text{int } C_+$  of problem (3.26).

Similarly, using the  $C^1$ -functional  $\psi_- : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\psi_-(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega G_-(z, u(z)) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where  $G_-(z, x) = \int_0^x g_-(z, s) \, ds$  and reasoning as above, we show that problem (3.27) has a unique solution  $\tilde{v} \in -\text{int } C_+$ . This proves Claim 1.

**Claim 2.**  $\tilde{u} \leq u$  for all  $u \in S_+ \cap [0, \bar{u}]$ .

Let  $u \in S_+ \cap [0, \bar{u}] \subseteq [0, \bar{u}] \cap \text{int } C_+$  and consider the Carathéodory function

$$k_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ (\eta(z) - \epsilon)x^{p-1} - c_1x^{r-1} & \text{if } 0 \leq x \leq u(z), \\ (\eta(z) - \epsilon)u(z)^{p-1} - c_1u(z)^{r-1} & \text{if } u(z) < x. \end{cases} \tag{3.31}$$

Let  $K_+(z, x) = \int_0^x k_+(z, s) \, ds$  and consider the  $C^1$ -functional  $\sigma_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\sigma_+(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega K_+(z, u(z)) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (3.31) we see that  $\sigma_+$  is coercive. Also, it is sequentially weakly continuous. So, we can find  $\tilde{u}_* \in W_0^{1,p}(\Omega)$  such that

$$\sigma_+(\tilde{u}_*) = \inf\{\sigma_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{3.32}$$

As in the proof of Claim 1, we can show that for  $t \in (0, 1)$  small (at least such that  $t\tilde{u}_1 \leq u \in \text{int } C_+$ ) we have

$$\sigma_+(t\tilde{u}_1) < 0 = \sigma_+(0).$$

This implies  $\sigma_+(\tilde{u}_*) < 0 = \sigma_+(0)$  (see (3.32)), and hence  $\tilde{u}_* \neq 0$ .

As before, we can check that

$$K_{\sigma_+} \subseteq [0, u] \subseteq [0, \bar{u}],$$

which implies  $\tilde{u}_* \in [0, u] \setminus \{0\}$  (see (3.32)). Therefore,  $\tilde{u}_* = \bar{u} \in \text{int } C_+$  (see Claim 1 and (3.31)), and hence

$$\tilde{u} \leq u \quad \text{for all } u \in C_+ \cap [0, \bar{u}].$$

This proves Claim 2.

Because of Claim 2, we have

$$\tilde{u} \leq u_n \quad \text{for all } n \geq 1,$$

implying  $\tilde{u} \leq u_*$  (see (3.23)), and hence  $u_* \neq 0$ .

Hence we have

$$u_* \in S_+ \quad \text{and} \quad u_* = \inf S_+.$$

Similarly, if  $S_-$  is the set of negative solutions of (1.1), we produce  $v_* \in -\text{int } C_+$ , the biggest element of  $S_-$ . In this case, by Claim 2 we have  $v \leq \tilde{v}$  for all  $v \in S_- \cap [\tilde{v}, 0]$  with  $S_- \subseteq -\text{int } C_+$ . □

As we have already mentioned, we will use these extremal solutions to produce a nodal solution. To do this, we need to strengthen the condition on  $f(z, \cdot)$  near zero. Note that hypothesis (H<sub>1</sub>) (iii) permits that  $f(z, \cdot)$  near zero is either  $(p - 1)$ -linear or  $(p - 1)$ -superlinear. We consider both cases and for both we produce nodal solutions.

First, we deal with the  $(p - 1)$ -linear case. We impose the following conditions on the reaction  $f(z, x)$ .

(H<sub>2</sub>):  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function such that for almost all  $z \in \Omega$  one has  $f(z, 0) = 0$ ,  $f(z, \cdot)$  is locally  $\alpha$ -Hölder continuous with  $\alpha \in (0, 1]$  and local Hölder constant  $k \in L^\infty(\Omega)_+$  and the following assertions hold:

(i) For every  $\rho > 0$ , there exists  $a_\rho \in L^\infty(\Omega)_+$  such that

$$|f(z, x)| \leq a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \rho.$$

(ii) One has

$$\limsup_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \xi < \hat{\lambda}_1 \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) There exist  $\xi_* \geq \xi_0 > \hat{\lambda}_2$  such that

$$\xi_0 \leq \liminf_{x \rightarrow 0} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{x \rightarrow 0} \frac{f(z, x)}{|x|^{p-2}x} \leq \xi_* \quad \text{uniformly for almost all } z \in \Omega.$$

(iv) There exists  $M_0 > 0$  such that for almost all  $z \in \Omega$ ,

$$\begin{aligned} x \mapsto \frac{f(z, x)}{x^{p-1}} & \text{ is nondecreasing on } [M_0, +\infty), \\ x \mapsto \frac{f(z, x)}{|x|^{p-2}x} & \text{ is nonincreasing on } (-\infty, -M_0]. \end{aligned}$$

**Proposition 3.4.** *If hypotheses (H<sub>2</sub>) hold, then problem (1.1) admits a nodal solution  $y_0 \in [v_*, u_*] \cap C_0^1(\bar{\Omega})$  (here  $[v_*, u_*] = \{u \in W_0^{1,p}(\Omega) : v_*(z) \leq u(z) \leq u_*(z) \text{ for almost all } z \in \Omega\}$  with  $u_* \in \text{int } C_+$  and  $v_* \in -\text{int } C_+$  being the extremal constant sign solutions produced in Proposition 3.3).*

*Proof.* We consider the following Carathéodory function

$$h(z, x) = \begin{cases} f(z, v_*(z)) & \text{if } x < v_*(z), \\ f(z, x) & \text{if } v_*(z) \leq x \leq u_*(z), \\ f(z, u_*(z)) & \text{if } u_*(z) < x. \end{cases} \tag{3.33}$$

We set  $H(z, x) = \int_0^x h(z, s) ds$  and consider the  $C^1$ -functional  $\beta : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\beta(u) = \frac{1}{p} \|Du\|_p^p - \int_{\Omega} H(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We also consider the positive and negative truncations of  $h(z, \cdot)$ , namely the Carathéodory functions

$$h_{\pm}(z, x) = h(z, \pm x^{\pm}).$$

We set  $H_{\pm}(z, x) = \int_0^x h_{\pm}(z, s) ds$  and consider the  $C^1$ -functionals  $\beta_{\pm} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\beta_{\pm}(u) = \frac{1}{p} \|Du\|_p^p - \int_{\Omega} H_{\pm}(z, u(z)) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

**Claim 3.**  $K_{\beta} \subseteq [v_*, u_*]$ ,  $K_{\beta_+} = \{0, u_*\}$  and  $K_{\beta_-} = \{0, v_*\}$ .

Let  $u \in K_{\beta}$ . Then

$$A(u) = N_f(u) \tag{3.34}$$

On (3.34), we first act with  $(u - u_*)^+ \in W_0^{1,p}(\Omega)$ . Then

$$\begin{aligned} \langle A(u), (u - u_*)^+ \rangle &= \int_{\Omega} f(z, u_*)(u - u_*)^+ dz \quad (\text{see (3.33)}) \\ &= \langle A(u_*), (u - u_*)^+ \rangle \quad (\text{since } u_* \in S_+), \end{aligned}$$

which implies

$$\int_{\{u > u_*\}} (|Du|^{p-2} Du - |Du_*|^{p-2} Du_*, Du - Du_*)_{\mathbb{R}^N} dz = 0.$$

Therefore,  $|\{u > u_*\}|_N = 0$ , and hence  $u \leq u_*$ .

Similarly, acting on (3.34) with  $(v_* - u)^+ \in W_0^{1,p}(\Omega)$ , we obtain  $v_* \leq u$ . So, we have  $u \in [v_*, u_*]$ , implying  $K_{\beta} \subseteq [v_*, u_*]$ .

In a similar fashion, we show that

$$K_{\beta_+} \subseteq [0, u_*] \quad \text{and} \quad K_{\beta_-} \subseteq [v_*, 0].$$

The extremality of the solutions  $u_* \in \text{int } C_+$  and  $v_* \in -\text{int } C_+$  (see Proposition 3.3) implies that

$$K_{\beta_+} = \{0, u_*\} \quad \text{and} \quad K_{\beta_-} = \{0, v_*\}.$$

This proves Claim 3.

**Claim 4.**  $u_* \in \text{int } C_+$  and  $v_* \in -\text{int } C_+$  are local minimizers of  $\beta$ .

From (3.33) it is clear that  $\beta_+$  is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find  $\hat{u}_* \in W_0^{1,p}(\Omega)$  such that

$$\beta_+(\hat{u}_*) = \inf\{\beta_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{3.35}$$

As before, by virtue of hypothesis (H<sub>2</sub>) (iii), we have  $\beta_+(\hat{u}_*) < 0 = \beta_+(0)$ , implying

$$\hat{u}_* \neq 0. \tag{3.36}$$

From (3.35) and Claim 1 we have  $\hat{u}_* \in K_{\beta_+} = \{0, u_*\}$ , implying

$$\hat{u}_* = u_* \in \text{int } C_+ \quad (\text{see (3.36)}). \tag{3.37}$$

Note that  $\beta|_{C_+} = \beta_+|_{C_+}$ . Then from (3.37) we see that  $u_*$  is a local  $C_0^1(\bar{\Omega})$ -minimizer of  $\beta$ , implying that  $u_*$  is a local  $W_0^{1,p}(\Omega)$ -minimizer of  $\beta$  (see Proposition 2.2).

We handle  $v_* \in -\text{int } C_+$  similarly, using this time the functional  $\beta_-$ .

This proves Claim 4.

Because of Claim 1, we may assume that  $K_\beta$  is finite (otherwise we already have an infinity of nodal solutions, see (3.33) and recall the extremality of  $u_* \in \text{int } C_+$  and of  $v_* \in -\text{int } C_+$ ). Also, without any loss of generality, we may assume that

$$\beta(v_*) \leq \beta(u_*).$$

The reasoning is similar if the opposite inequality holds. Because of Claim 2, we can find  $\rho \in (0, 1)$  small such that (see [1, proof of Proposition 29])

$$\beta(v_*) \leq \beta(u_*) < \inf\{\beta(u) : \|u - u_*\| = \rho\} = m_\rho, \quad \|v_* - u_*\| > \rho. \tag{3.38}$$

Since  $\beta(\cdot)$  is coercive (see (3.33)), it satisfies the PS-condition. This fact and (3.38) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find  $y_0 \in W_0^{1,p}(\Omega)$  such that

$$m_\rho \leq \beta(y_0) \quad \text{and} \quad y_0 \in K_\beta \subseteq [v_*, u_*] \quad (\text{see Claim 1}). \tag{3.39}$$

From (3.38) and (3.39), it follows that

$$y_0 \notin \{u_*, v_*\}.$$

So, if we can show that  $y_0 \neq 0$ , then  $y_0$  will be nodal (see (3.39)). By the mountain pass theorem (see Theorem 2.1), we have

$$\beta(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \beta(\gamma(t)), \tag{3.40}$$

with  $\Gamma = \{\gamma \in C([0, 1], W_0^{1,p}(\Omega)) : \gamma(0) = v_*, \gamma(1) = u_*\}$ . According to (3.40), in order to show the nontriviality of  $y_0$  it suffices to construct a path  $\gamma_* \in \Gamma$  such that  $\beta|_{\gamma_*} < 0 = \beta(0)$ .

To this end, note that hypothesis (H<sub>2</sub>) (iii) implies that we can find  $\xi_1 \in (\hat{\lambda}_2, \xi_0)$  and  $\delta > 0$  such that

$$F(z, x) \geq \frac{1}{p} \xi_1 |x|^p \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \delta. \tag{3.41}$$

Let

$$\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_p = 1\}, \quad \widehat{M} = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p} \quad \widehat{M}_c = \widehat{M} \cap C_0^1(\bar{\Omega}).$$

We introduce the following sets of paths:

$$\begin{aligned} \hat{\Gamma} &= \{\hat{\gamma} \in C([-1, 1], \widehat{M}) : \hat{\gamma}(-1) = -\hat{u}_1, \hat{\gamma}(1) = \hat{u}_1\}, \\ \hat{\Gamma}_c &= \{\hat{\gamma} \in C([-1, 1], \widehat{M}_c) : \hat{\gamma}(-1) = -\hat{u}_1, \hat{\gamma}(1) = \hat{u}_1\}. \end{aligned}$$

**Claim 5.**  $\hat{\Gamma}_c$  is dense in  $\hat{\Gamma}$  for the  $C([-1, 1], W_0^{1,p}(\Omega))$ -topology.

Let  $\hat{\gamma} \in \hat{\Gamma}$ . For every  $n \geq 1$  we consider the multifunction  $T_n : [-1, 1] \rightarrow 2^{C_0^1(\bar{\Omega})}$  defined by

$$T_n(t) = \begin{cases} \{-\hat{u}_1\} & \text{if } t = -1, \\ \left\{ u \in C_0^1(\bar{\Omega}) : \|u - \hat{\gamma}(t)\| < \frac{1}{n} \right\} & \text{if } t \in (-1, 1), \\ \{\hat{u}_1\} & \text{if } t = 1. \end{cases}$$

Evidently,  $T_n(\cdot)$  has nonempty convex values, which are open sets if  $t \in (-1, 1)$ . Also, by Papageorgiou and Kyritsi [18, pp. 458–463], we have that  $T_n(\cdot)$  is a lower semicontinuous multifunction. So, we can

apply [16, Theorem 3.1'''] (see also [12, p. 97]) and find a continuous map  $\tau_n : [-1, 1] \rightarrow C_0^1(\bar{\Omega})$  such that  $\tau_n(t) \in T_n(t)$  for all  $t \in [-1, 1]$  and all  $n \geq 1$ . We have

$$\|\tau_n(t) - \hat{y}(t)\| < \frac{1}{n} \quad \text{for all } t \in [-1, 1] \text{ and all } n \geq 1, \tag{3.42}$$

which implies

$$\|\tau_n(t)\|_p \rightarrow \|\hat{y}(t)\|_p \quad \text{uniformly in } t \in [-1, 1] \text{ as } n \rightarrow \infty.$$

So, for  $n \geq 1$  big enough, we can define

$$\hat{y}_n(t) = \frac{\tau_n(t)}{\|\tau_n(t)\|_p} \quad \text{for all } t \in [-1, 1] \text{ (recall that } \hat{y}(t) \in \partial B_1^{L^p} \text{ for all } t \in [-1, 1]).$$

Then we have

$$\begin{aligned} \|\hat{y}_n(t) - \hat{y}(t)\| &\leq \|\hat{y}_n(t) - \tau_n(t)\| + \|\tau_n(t) - \hat{y}(t)\| \\ &\leq |1 - \|\tau_n(t)\|_p| \frac{\|\tau_n(t)\|}{\|\tau_n(t)\|_p} + \frac{1}{n} \quad \text{for all } t \in [-1, 1], n \geq 1 \text{ (see (3.42)).} \end{aligned} \tag{3.43}$$

Also, since  $\|\hat{y}(t)\|_p = 1$  for all  $t \in [-1, 1]$ , we can write

$$\begin{aligned} |1 - \|\tau_n(t)\|_p| &= \|\hat{y}(t)\|_p - \|\tau_n(t)\|_p \\ &\leq \|\hat{y}(t) - \tau_n(t)\|_p \\ &\leq c_4 \|\hat{y}(t) - \tau_n(t)\| \quad \text{for some } c_4 > 0 \text{ and all } t \in [-1, 1], n \geq 1, \end{aligned}$$

which implies

$$\max_{-1 \leq t \leq 1} |1 - \|\tau_n(t)\|_p| \leq c_4 \frac{1}{n} \quad \text{for all } n \geq 1 \text{ (see (3.42)).} \tag{3.44}$$

Returning to (3.43) and using (3.44), we obtain

$$\max_{-1 \leq t \leq 1} \|\hat{y}_n(t) - \hat{y}(t)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Evidently,  $\hat{y}_n \in \hat{\Gamma}_c$  for all  $n \geq 1$ . So, we have proved Claim 5.

Using Claim 5 and Proposition 2.3, given  $\eta \in (0, \xi_1 - \hat{\lambda}_2)$ , we can find  $\bar{y}_0 \in \hat{\Gamma}_c$  such that

$$\|D\bar{y}_0(t)\|_p^p \leq \hat{\lambda}_2 + \eta \quad \text{for all } t \in [-1, 1]. \tag{3.45}$$

The set  $\bar{y}_0([-1, 1])$  is compact in  $C_0^1(\bar{\Omega})$ . Also,  $u_* \in \text{int } C_+$  and  $v_* \in -\text{int } C_+$  (see Proposition 3.3). So, using also [9, Lemma 3.3], we can find  $\vartheta \in (0, 1)$  small such that (see (3.41))

$$\vartheta \bar{y}_0(t) \in [v_*, u_*] \quad \text{for all } t \in [-1, 1] \quad \text{and} \quad |\vartheta \bar{y}_0(t)(z)| \leq \delta \quad \text{for all } t \in [-1, 1], z \in \bar{\Omega} \tag{3.46}$$

Let  $\hat{y}_0 = \vartheta \bar{y}_0$ . Then  $\hat{y}_0$  is a path in  $W_0^{1,p}(\Omega)$  connecting  $-\vartheta \hat{u}_1$  and  $\vartheta \hat{u}_1$ , and we have

$$\begin{aligned} \beta(\hat{y}_0(t)) &= \frac{1}{p} \|D\hat{y}_0(t)\|_p^p - \int_{\Omega} F(z, \hat{y}_0(t)) dz \quad \text{(see (3.33) and (3.46))} \\ &\leq \frac{1}{p} [\hat{\lambda}_2 + \eta - \xi_1] \|\hat{y}_0(t)\|_p^p \quad \text{for all } t \in [-1, 1] \text{ (see (3.41), (3.45) and (3.46))} \\ &< 0 \quad \text{for all } t \in [-1, 1] \text{ (recall that } 0 < \eta < \xi_1 - \hat{\lambda}_2), \end{aligned}$$

which implies

$$\beta|_{\hat{y}_0} < 0. \tag{3.47}$$

Next, we produce a path in  $W_0^{1,p}(\Omega)$  connecting  $\vartheta \hat{u}_1$  and  $u_*$  and along which  $\beta$  is negative.

To this end, let  $a = \beta_+(u_*)$ . From the proof of Claim 4 we know that  $a < 0$ , and because of Claim 3, we see that

$$K_{\beta_+}^a = \{u_*\}. \tag{3.48}$$

Applying the second deformation theorem (see, for example, [11, p. 628]), we can find a deformation  $h : [0, 1] \times (\beta_+^0 \setminus \{0\}) \rightarrow \beta_+^0$  such that

$$h(0, u) = u \quad \text{for all } u \in \beta_+^0 \setminus \{0\}, \tag{3.49}$$

$$h(1, u) = u_* \quad \text{for all } u \in \beta_+^0 \setminus \{0\} \text{ (see (3.48))}, \tag{3.50}$$

$$\beta_+(h(t, u)) \leq \beta_+(h(s, u)) \quad \text{for all } t, s \in [0, 1], s < t, \text{ and all } u \in \beta_+^0 \setminus \{0\}. \tag{3.51}$$

We define

$$\hat{y}_+(t) = h(t, \vartheta \hat{u}_1)^+ \quad \text{for all } t \in [0, 1].$$

Evidently, this is a path in  $W_0^{1,p}(\Omega)$  and

$$\hat{y}_+(0) = \vartheta \hat{u}_1 \quad \text{(see (3.49) and recall that } \vartheta \hat{u}_1 \in \text{int } C_+),$$

$$\hat{y}_+(1) = u_* \quad \text{(see (3.50) and recall that } u_* \in \text{int } C_+).$$

Also, since  $\hat{y}_+(t)(z) \geq 0$  for all  $z \in \bar{\Omega}$  and all  $t \in [0, 1]$ , we have

$$\beta(\hat{y}_+(t)) = \beta_+(\hat{y}_+(t)) \leq \beta_+(\vartheta \hat{u}_1) = \beta(\vartheta \hat{u}_1) < 0 \quad \text{for all } t \in [0, 1] \text{ (see (3.47) and (3.51))},$$

which implies

$$\beta|_{\hat{y}_+} < 0. \tag{3.52}$$

In a similar way, we can produce a path  $\hat{y}_-$  in  $W_0^{1,p}(\Omega)$  which connects  $-\vartheta \hat{u}_1$  and  $v_*$  and such that

$$\beta|_{\hat{y}_-} < 0. \tag{3.53}$$

We concatenate  $\hat{y}_-, \hat{y}_0, \hat{y}_+$  and generate a path  $\gamma_* \in \Gamma$  such that  $\beta|_{\gamma_*} < 0$  (see (3.47), (3.52) and (3.53)). This implies  $\gamma_0 \neq 0$ , and hence  $\gamma_0 \in C_0^1(\bar{\Omega})$  (nonlinear regularity) is a nodal solution of (1.1).  $\square$

So, we can state our first multiplicity theorem.

**Theorem 3.5.** *If hypotheses (H<sub>2</sub>) hold, then problem (1.1) admits at least three nontrivial solutions*

$$u_0 \in \text{int } C_+, \quad v_0 \in -\text{int } C_+, \quad \gamma_0 \in [v_0, u_0] \cap C_0^1(\bar{\Omega}) \quad \text{nodal.}$$

Next, we change the geometry near the origin by introducing a concave term. So, now the hypotheses on the reaction  $f(z, x)$  are the following.

(H<sub>3</sub>):  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function such that for almost all  $z \in \Omega$  one has  $f(z, 0) = 0$ ,  $f(z, \cdot)$  is locally  $\alpha$ -Hölder continuous with  $\alpha \in (0, 1]$  and local Hölder constant  $k \in L^\infty(\Omega)_+$  and the following assertions hold:

(i) For every  $\rho > 0$ , there exists  $a_\rho \in L^\infty(\Omega)_+$  such that

$$|f(z, x)| \leq a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \rho.$$

(ii) One has

$$\limsup_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \xi < \hat{\lambda}_1 \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) There exist  $q \in (1, p)$  and  $\delta > 0$  such that

$$0 < f(z, x)x \leq qF(z, x) \quad \text{for almost all } z \in \Omega \text{ and all } 0 < |x| \leq \delta,$$

$$0 < \text{ess inf}_\Omega F(\cdot, \pm\delta),$$

where  $F(z, x) = \int_0^x f(z, s) ds$ .

(iv) There exists  $M_0 > 0$  such that for almost all  $z \in \Omega$ ,

$$x \mapsto \frac{f(z, x)}{x^{p-1}} \quad \text{is nondecreasing on } [M_0, +\infty),$$

$$x \mapsto \frac{f(z, x)}{|x|^{p-2}x} \quad \text{is nonincreasing on } (-\infty, -M_0].$$

**Remark 3.6.** For example, we can think of a reaction of the form

$$f(z, x) = k_0(z)|x|^{q-2}x + f_0(z, x),$$

with  $1 < q < 2$ ,  $k_0 \in L^\infty(\Omega)$  and  $f_0(z, x)$  being a measurable function such that for almost all  $z \in \Omega$  the function  $f_0(z, \cdot)$  is locally  $\alpha$ -Hölder continuous with  $\alpha \in (0, 1)$  and local Hölder constant  $k \in L^\infty(\Omega)_+$ , and

$$\limsup_{x \rightarrow \pm\infty} \frac{f_0(z, x)}{|x|^{p-2}x} \leq \xi_1 < \hat{\lambda}_1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{f_0(z, x)}{|x|^{p-2}x} = 0 \quad \text{uniformly for almost all } z \in \Omega.$$

We are ready to state and prove our second multiplicity theorem.

**Theorem 3.7.** *If hypotheses (H<sub>3</sub>) hold, then problem (1.1) admits at least three nontrivial solutions*

$$u_0 \in \text{int } C_+, \quad v_0 \in -\text{int } C_+, \quad y_0 \in [v_0, u_0] \cap C_0^1(\bar{\Omega}) \quad \text{nodal.}$$

*Proof.* The two constant sign solutions come from Proposition 3.2.

Let  $u_* \in \text{int } C_+$  and  $v_* \in -\text{int } C_+$  be the two extremal constant sign solutions produced in Proposition 3.3. Using them and reasoning as in the first part of the proof of Proposition 3.4, via the functional  $\beta$  and the mountain pass theorem (see Theorem 2.1), we obtain a third solution

$$y_0 \in [v_*, u_*] \cap C_0^1(\bar{\Omega}).$$

Since  $y_0$  is a critical point of mountain pass type for the functional  $\beta$ , we have

$$C_1(\beta, y_0) \neq 0. \quad (3.54)$$

On the other hand it is well-known that hypothesis (H<sub>3</sub>) (iii) implies that

$$C_k(\beta, 0) = 0 \quad \text{for all } k \geq 0. \quad (3.55)$$

Comparing (3.54) and (3.55), we infer that  $y_0 \neq 0$ . This means that  $y_0 \in [v_*, u_*] \cap C_0^1(\bar{\Omega})$  is a nodal solution of problem (1.1).  $\square$

## 4 The semilinear equation ( $p = 2$ )

In this section, we focus on the semilinear equation (that is,  $p = 2$ ). So, the problem under consideration is the following:

$$-\Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (4.1)$$

By improving the regularity on the reaction  $f(z, \cdot)$ , we can produce a second nodal solution for a total of four nontrivial solutions for problem (4.1).

The hypotheses on the reaction  $f(z, x)$  are the following.

(H<sub>4</sub>):  $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function such that for almost all  $z \in \Omega$  one has that  $f(z, 0) = 0$ ,  $f(z, \cdot) \in C^1(\mathbb{R})$  and the following assertions hold:

(i) For every  $\rho > 0$ , there exists  $a_\rho \in L^\infty(\Omega)$  such that

$$|f'_x(z, x)| \leq a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \rho.$$

(ii) One has

$$\limsup_{x \rightarrow \pm\infty} \frac{f(z, x)}{x} \leq \xi < \hat{\lambda}_1 \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) One has

$$f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x} \quad \text{uniformly for almost all } z \in \Omega,$$

and there exists an integer  $m \geq 2$  such that

$$\hat{\lambda}_m \leq f'_x(z, 0) \leq \hat{\lambda}_{m+1} \quad \text{for almost all } z \in \Omega,$$

with the first inequality being strict on a set of positive measure and for  $F(z, x) = \int_0^x f(z, s) ds$  we have

$$F(z, x) \leq \frac{\hat{\lambda}_{m+1}}{2} x^2 \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}.$$

(iv) There exists  $M_0 > 0$  such that for almost all  $z \in \Omega$ ,

$$\begin{aligned} x \mapsto \frac{f(z, x)}{x^{p-1}} & \text{ is nondecreasing on } [M_0, +\infty), \\ x \mapsto \frac{f(z, x)}{|x|^{p-2}x} & \text{ is nonincreasing on } (-\infty, -M_0]. \end{aligned}$$

**Remark 4.1.** The differentiability of  $f(z, \cdot)$  and hypothesis  $(H_4)$  (i) imply that  $f(z, \cdot)$  is locally Lipschitz with locally Lipschitz constant in  $L^\infty(\Omega)_+$ .

From Proposition 3.3 we know that we have extremal constant sign solutions

$$u_* \in \text{int } C_+ \quad \text{and} \quad v_* \in -\text{int } C_+.$$

Using these extremal constant sign solutions, we consider the functional  $\beta : H_0^1(\Omega) \rightarrow \mathbb{R}$  introduced in the proof of Proposition 3.4 (now  $p = 2$ ). We have  $\beta \in C^{2-0}(H_0^1(\Omega))$  (that is,  $\beta$  is in  $C^1(H_0^1(\Omega))$  with locally Lipschitz derivative).

**Proposition 4.2.** *If hypotheses  $(H_4)$  hold, then  $C_k(\beta, 0) = \delta_{k,d_m} \mathbb{Z}$  for all  $k \geq 0$  with  $d_m = \dim \bigoplus_{k=1}^m E(\hat{\lambda}_k) \geq 2$ .*

*Proof.* If in hypothesis  $(H_4)$  (iii) the inequality  $f'_x(z, 0) \leq \hat{\lambda}_{m+1}$  is also strict on a set (not necessarily the same) of positive measure, then  $u = 0$  is a nondegenerate critical point of  $\beta$ , and so from [13] we have

$$C_k(\beta, 0) = \delta_{k,d_m} \mathbb{Z} \quad \text{for all } k \geq 0.$$

So, suppose that  $f'_x(z, 0) = \hat{\lambda}_{m+1}$  for almost all  $z \in \Omega$ . Using hypothesis  $(H_4)$  (iii) and (2.4), we have

$$\beta(u) \geq \frac{1}{2} \|Du\|_2^2 - \frac{\hat{\lambda}_{m+1}}{2} \|u\|^2 \geq 0 \quad \text{for all } u \in \hat{H}_{m+1} = \overline{\bigoplus_{k \geq m+1} E(\hat{\lambda}_k)}. \tag{4.2}$$

On the other hand, given  $\epsilon > 0$ , we can find  $\delta = \delta(\epsilon) > 0$  such that

$$F(z, x) \geq \frac{1}{2} (f'_x(z, 0) - \epsilon)x^2 \quad \text{for almost all } z \in \Omega \text{ and all } x \in [-\delta, \delta]. \tag{4.3}$$

Since  $\bar{H}_m = \bigoplus_{k=1}^m E(\hat{\lambda}_k)$  is finite-dimensional, all norms are equivalent, and so we can find  $\rho > 0$  small enough such that if  $\bar{B}_\rho = \{u \in H_0^1(\Omega) : \|u\| \leq \rho\}$ , then  $u \in \bar{H}_m \cap B_\rho$ , implying

$$|u(z)| \leq \delta \quad \text{for all } z \in \bar{\Omega} \text{ and } u \in [v_*, u_*]. \tag{4.4}$$

Let  $u \in \bar{H}_m \cap B_\rho$ . Then we have

$$\begin{aligned} \beta(u) & \leq \frac{1}{2} \|Du\|_2^2 - \frac{1}{2} \int_\Omega f'_x(z, 0) u^2 dz + \frac{\epsilon}{2} \|u\|_2^2 \quad (\text{see (3.33), (4.3) and (4.4)}) \\ & \leq -\frac{c_5 - \epsilon}{2} \|u\|^2 \quad \text{for some } c_5 > 0 \text{ (see Proposition 2.4)}. \end{aligned}$$

Choosing  $\epsilon \in (0, c_5)$ , we infer that

$$\beta(u) \leq 0 \quad \text{for all } u \in \bar{H}_m \cap \bar{B}_\rho. \tag{4.5}$$

From (4.2) and (4.5) we see that  $\beta$  has local linking at the origin and of course it is locally Lipschitz there. Therefore,

$$C_{d_m}(\beta, 0) \neq 0.$$

Invoking the shifting theorem for  $C^{2-0}$  functionals due to Li, Li and Liu [13], we conclude that

$$C_k(\beta, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0.$$

The proof is now complete. □

Now we are ready for our third multiplicity theorem concerning problem (4.1).

**Theorem 4.3.** *If hypotheses  $(H_4)$  hold, then problem (4.1) admits at least four nontrivial solutions*

$$u_0 \in \text{int } C_+, \quad v_0 \in -\text{int } C_+, \quad y_0, \hat{y} \in \text{int}_{C_0^1(\bar{\Omega})}[v_0, u_0] \quad \text{nodal.}$$

*Proof.* From Proposition 3.2 we already have two nontrivial constant sign solutions

$$u_0 \in \text{int } C_+ \quad \text{and} \quad v_0 \in -\text{int } C_+.$$

Moreover, by virtue of Proposition 3.3 we may assume that  $u_0$  and  $v_0$  are extremal (that is,  $u_0 = u_* \in \text{int } C_+$  and  $v_0 = v_* \in -\text{int } C_+$ ). The differentiability of  $f(z, \cdot)$  and hypothesis  $(H_4)$  (i) imply that if  $\rho = \max\{\|\bar{u}\|_\infty, \|\bar{v}\|_\infty\}$ , then we can find  $\tilde{\xi}_\rho > 0$  such that  $x \rightarrow f(z, x) + \tilde{\xi}_\rho x$  is nondecreasing on  $[-\rho, \rho]$  for almost all  $z \in \Omega$ .

As in the proof of Proposition 3.4, using the functional  $\beta \in C^{2-0}(H_0^1(\Omega))$  and the mountain pass theorem (see Theorem 2.1), we can find  $y_0 \in [v_0, u_0] \cap C_0^1(\bar{\Omega})$ , which is a solution of problem (4.1). We have

$$\begin{aligned} -\Delta y_0(z) + \tilde{\xi}_\rho y_0(z) &= f(z, y_0(z)) + \tilde{\xi}_\rho y_0(z) \\ &\leq f(z, u_0(z)) + \tilde{\xi}_\rho u_0(z) \quad (\text{since } y_0 \leq u_0) \\ &= -\Delta u_0(z) + \tilde{\xi}_\rho u_0(z) \quad \text{for almost all } z \in \Omega, \end{aligned}$$

This implies

$$\Delta(u_0 - y_0)(z) \leq \tilde{\xi}_\rho(u_0 - y_0)(z) \quad \text{for almost all } z \in \Omega,$$

and hence  $u_0 - y_0 \in \text{int } C_+$  by the strong maximum principle.

Similarly, we show that  $y_0 - v_0 \in \text{int } C_+$ . Therefore,

$$y_0 \in \text{int}_{C_0^1(\bar{\Omega})}[v_0, u_0].$$

Since  $y_0$  is a critical point of mountain pass type for  $\beta$ , from [13, Theorem 2.7] we have

$$C_k(\beta, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.6}$$

From Proposition 4.2 we know that

$$C_k(\beta, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0 \text{ with } d_m \geq 2. \tag{4.7}$$

Comparing (4.6) and (4.7), we infer that  $y_0 \neq 0$ , and so

$$y_0 \in \text{int}_{C_0^1(\bar{\Omega})}[v_0, u_0] \quad \text{is a nodal solution of (4.1).}$$

Recall that  $u_0, v_0$  are local minimizers of  $\beta$  (see Claim 4 in the proof of Proposition 3.4). Hence we have

$$C_k(\beta, u_n) = C_k(\beta, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.8}$$

Moreover, the coercivity of  $\beta(\cdot)$  (see (3.33)) implies that

$$C_k(\beta, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.9}$$

Suppose that  $K_\beta = \{0, u_0, v_0, y_0\}$ . Then from (4.6)–(4.9) and the Morse relation (2.6) with  $t = -1$ , we have

$$(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0,$$

implying  $(-1)^{d_m} = 0$ , a contradiction.

So, there exists  $\hat{y} \in K_\beta$ ,  $\hat{y} \notin \{0, u_0, v_0, y_0\}$ . Then  $\hat{y} \in [v_0, u_0] \cap C_0^1(\bar{\Omega})$  is nodal (see Claim 3 in the proof of Proposition 3.4 and use standard regularity theorem). In fact, as we did in the beginning of the proof for  $y_0$ , we can show that

$$\hat{y} \in \text{int}_{C_0^1(\bar{\Omega})}[v_0, u_0].$$

The proof is now complete. □

## 5 A special case

In this section, we consider a special case of problem (1.1) under hypotheses  $(H_2)$ , which we encounter in the literature.

So, we deal with the following parametric nonlinear Dirichlet problem:

$$-\Delta_p u(z) = \lambda |u(z)|^{p-2} u(z) - g(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \lambda > 0. \tag{5.1}$$

We impose the following conditions on the perturbation  $g(z, x)$ .

$(H_5)$ :  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function such that for almost all  $z \in \Omega$  one has  $g(z, 0) = 0$ ,  $g(z, \cdot)$  is locally  $\alpha$ -Hölder continuous with  $\alpha \in (0, 1]$  and local Hölder constant  $k \in L^\infty(\Omega)_+$  and the following assertions hold:

(i) For every  $\rho > 0$ , there exists  $a_\rho \in L^\infty(\Omega)_+$  such that

$$|g(z, x)| \leq a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \rho.$$

(ii) One has

$$\liminf_{x \rightarrow \pm\infty} \frac{g(z, x)}{|x|^{p-2}x} \geq \xi^* > \lambda - \hat{\lambda}_1 \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) There exist  $\xi_0, \xi_* \in \mathbb{R}$ ,  $\xi_* < \lambda - \hat{\lambda}_2$ , such that

$$\xi_0 \leq \liminf_{x \rightarrow 0} \frac{g(z, x)}{|x|^{p-2}x} \leq \limsup_{x \rightarrow 0} \frac{g(z, x)}{|x|^{p-2}x} \leq \xi_* \quad \text{uniformly for almost all } z \in \Omega.$$

(iv) There exists  $M_0 > 0$  such that for almost all  $z \in \Omega$ ,

$$\begin{aligned} x \mapsto \frac{g(z, x)}{x^{p-1}} & \text{ is nondecreasing on } [M_0, +\infty), \\ x \mapsto \frac{g(z, x)}{|x|^{p-2}x} & \text{ is nonincreasing on } (-\infty, -M_0]. \end{aligned}$$

Setting  $f(z, x) = \lambda |x|^{p-2}x - g(z, x)$  and using Theorem 3.5, we can state the following multiplicity theorem for problem (5.1).

**Theorem 5.1.** *If hypotheses  $(H_5)$  hold and  $\lambda > \hat{\lambda}_2$ , then problem (5.1) admits at least three nontrivial solutions*

$$u_0 \in \text{int } C_+, \quad v_0 \in -\text{int } C_+, \quad y_0 \in [v_0, u_0] \cap C_0^1(\bar{\Omega}) \quad \text{nodal.}$$

**Remark 5.2.** This theorem complements the multiplicity result of Papageorgiou and Papageorgiou [17].

In the semilinear case ( $p = 2$ ), we can say more. So, now the problem under consideration is the following:

$$-\Delta u(z) = \lambda u(z) - g(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \lambda > 0. \tag{5.2}$$

The hypotheses on the perturbation  $g(z, x)$  are the following.

(H<sub>6</sub>):  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function such that for almost all  $z \in \Omega$  one has that  $g(z, 0) = 0$ ,  $g(z, \cdot) \in C^1(\mathbb{R})$  and the following assertions hold:

(i) For every  $\rho > 0$ , there exists  $a_\rho \in L^\infty(\Omega)_+$  such that

$$|g'_x(z, x)| \leq a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \rho.$$

(ii) One has

$$\liminf_{x \rightarrow \pm\infty} \frac{g(z, x)}{x} \geq \xi^* > \lambda - \hat{\lambda}_1 \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) One has

$$g'_x(z, 0) = \lim_{x \rightarrow 0} \frac{g(z, x)}{x} = 0 \quad \text{uniformly for almost all } z \in \Omega.$$

(iv) There exists  $M_0 > 0$  such that for almost all  $z \in \Omega$ ,

$$\begin{aligned} x \mapsto \frac{g(z, x)}{x^{p-1}} & \text{ is nondecreasing on } [M_0, +\infty), \\ x \mapsto \frac{g(z, x)}{|x|^{p-2}x} & \text{ is nonincreasing on } (-\infty, -M_0]. \end{aligned}$$

Again, setting  $f(z, x) = \lambda x - g(z, x)$  and using Theorem 4.3, we can state the following multiplicity theorem for problem (5.2).

**Theorem 5.3.** *If hypotheses (H<sub>6</sub>) hold and  $\lambda > \hat{\lambda}_2$ , then problem (5.2) has at least four nontrivial solutions*

$$u_0 \in \text{int } C_+, \quad v_0 \in -\text{int } C_+, \quad y_0, \hat{y} \in \text{int}_{C_0^1(\bar{\Omega})} [v_0, u_0] \quad \text{nodal}.$$

**Remark 5.4.** This theorem complements the multiplicity results of Ambrosetti and Lupo [2], Ambrosetti and Mancini [3] and Struwe [21, 22], which produce only three solutions and there are no nodal solutions among them.

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