*Dedicated to Professor Philippe G. Ciarlet*
*on his 70th birthday*

# COMBINED EFFECTS IN NONLINEAR SINGULAR ELLIPTIC PROBLEMS WITH CONVECTION

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We establish some bifurcation results for the boundary value problem $-\Delta u = \lambda u^{-a}|\nabla u|^2 + \mu$ in $\Omega$, $u > 0$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, while $a, \lambda\,\mu > 0$. Our approach relies on finding explicit sub- and super-solutions combined with various techniques related to the maximum principle for elliptic equations with singular nonlinearities.

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*Key words:* singular elliptic equation, nonlinear perturbation, bifurcation problem, maximum principle.

## 1. INTRODUCTION

This paper is motivated by recent advances in the study of nonlinear elliptic problems involving both singular nonlinearities and convection (gradient) terms. We start with an elementary example. Consider the nonlinear boundary value problem

$$\begin{cases} \Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $p > 1$. Then the Gelfand transform $v = u^{-1}$ yields

$$\begin{cases} -\Delta v = v^{2-p} - \frac{2}{v}\,|\nabla v|^2 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The above equation contains both singular nonlinearities (like $v^{-1}$ or $v^{2-p}$, if $p > 2$) and a convection term (denoted by $|\nabla v|^2$). These nonlinearities make more difficult to handle problems like (1). We recall the pioneering paper [6] that contains one of the first existence results for singular elliptic problems. In fact, it is proved in [6] that the boundary value problem

$$\begin{cases} -\Delta u - u^{-\alpha} = -u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, for any $\alpha > 0$. Let us now consider the problem

$$\begin{cases} -\Delta u - u^{-\alpha} = \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\lambda \geq 0$ and $\alpha, p \in (0,1)$. In [5] it is proved that problem (2) has at least one solution for all $\lambda \geq 0$ and $0 < p < 1$. Moreover, if $p \geq 1$, then there exists $\lambda^*$ such that problem (2) has a solution for $\lambda \in [0, \lambda^*)$ and no solution for $\lambda > \lambda^*$. In [5] it is also proved a related non–existence result. More exactly, the problem

$$\begin{cases} -\Delta u + u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no solution, provided that $0 < \alpha < 1$ and $\lambda_1 \geq 1$, that is, if $\Omega$ is "small", where $\lambda_1$ denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

Problems related to multiplicity and uniqueness become difficult even in simple cases. In [24] is studied the existence of radial symmetric solutions to the problem

$$\begin{cases} \Delta u + \lambda(u^p - u^{-\alpha}) = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $\alpha > 0$, $0 < p < 1$, $\lambda > 0$, and $B_1$ is the unit ball in $\mathbb{R}^N$. Using a bifurcation theorem of Crandall and Rabinowitz, it has been shown in [24] that there exists $\lambda_1 > \lambda_0 > 0$ such that the above problem has no solutions for $\lambda < \lambda_0$, exactly one solution for $\lambda = \lambda_0$ or $\lambda > \lambda_1$, and two solutions for $\lambda_0 < \lambda \leq \lambda_1$.

Singular elliptic problems have been intensively studied in the last decades. Stationary problems involving singular nonlinearities, as well as the associated evolution equations, describe naturally several physical phenomena. At our best knowledge, the first study in this direction is due to Fulks and Maybee [11], who proved existence and uniqueness results by using a fixed point argument; moreover, they showed that solutions of the associated parabolic problem tend to the unique solution of the corresponding elliptic equation. A different approach (see Coclite and Palmieri [5], Crandall, Rabinowitz, and Tartar [6], Stuart [25]) consists in approximating the singular equation by a regular problem, where the standard techniques (e.g., monotonicity methods) can be applied. Then, passing to the limit, the solution of the original equation can be obtained. Nonlinear singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids (for more details we refer to Caffarelli, Hardt, and L. Simon [2], Callegari and Nachman [3], Díaz [8], Díaz, Morel, and Oswald [9] and the more recent papers by Ghergu and Rădulescu [12, 13, 14], Haitao [17], Hernández, Mancebo, and Vega [18], Meadows [20], Shi and Yao [24]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases. For instance, problems of this type characterize some reaction-diffusion processes where $u \geq 0$ is viewed as the density of a reactant and the region where $u = 0$ is called the *dead core*, where no reaction takes place (see Aris [1] for the study of a single, irreversible steady-state reaction). Nonlinear singular elliptic equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents (see Callegari and Nachman [3] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence).

## 2. THE MAIN RESULT

In the present paper we continue the bifurcation analysis developed in our previous papers [13, 15, 22, 23] for a large class of semilinear elliptic equations with singular nonlinearity and Dirichlet boundary condition. In these papers we have been concerned with various boundary value problems involving singular nonlinearities and with the existence or nonexistence of solutions under various assumptions.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary. Consider the nonlinear singular problem

$$\begin{cases} -\Delta u = u^{-a} + \lambda|\nabla u|^2 + \mu & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $a > 0$ and $\lambda$, $\mu \geq 0$. If $\lambda_1$ denotes the first eigenvalue of the Laplace operator in $H_0^1(\Omega)$, in [13, 22, 23] we have established the following results:

(i) Problem (3) has a solution if and only if $\lambda\mu < \lambda_1$.

(ii) Assume $\mu > 0$ is fixed and let $\lambda^* = \lambda_1/\mu$. Then problem (3) has a unique solution $u_\lambda$ for every $\lambda < \lambda^*$ and the sequence $(u_\lambda)_{\lambda<\lambda^*}$ is increasing with respect to $\lambda$. Moreover, if $a < 1$ then the sequence of solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ has the following properties:

($\mathrm{ii}_1$) for all $0 < \lambda < \lambda^*$ there exist two positive constants $c_1, c_2$ depending on $\lambda$ such that $c_1 \operatorname{dist}(x, \partial\Omega) \le u_\lambda \le c_2 \operatorname{dist}(x, \partial\Omega)$ in $\Omega$;

($\mathrm{ii}_2$) $u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;

($\mathrm{ii}_3$) $u_\lambda \to +\infty$ as $\lambda \nearrow \lambda^*$, uniformly on compact subsets of $\Omega$.

Our purpose in the present paper is to study a related nonlinear elliptic equation involving both a singular nonlinearity and a quadratic convection (gradient) term.

We are concerned with the boundary value problem

$$
\begin{cases} -\Delta u = \lambda u^{-a}\, |\nabla u|^2 + \mu & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}
$$

where $a$, $\lambda$, and $\mu$ are positive real numbers.

Our main result is as follows.

THEOREM 2.1. *There exists $\lambda_0 > 0$ such that problem* (4) *has a solution $u \in C^2(\Omega) \cap C(\partial\Omega)$ for any $0 < \lambda \le \lambda_0$.*

As remarked in [4, 19], the requirement that the nonlinearity grows at most quadratically in $|\nabla u|$ is natural in order to apply the maximum principle.

The proof of Theorem 2.1 relies on comparison arguments which are based on the method of lower and upper solutions for elliptic equations involving both singular nonlinearities and gradient terms. The roots of the method of lower and upper solutions go back to Picard who, in the early 1880s, applied the method of successive approximations to argue the existence of solutions for nonlinear elliptic equations that are suitable perturbations of uniquely solvable linear problems. Picard's techniques were applied later by Poincaré [21] in connection with problems arising in astrophysics. In particular, Poincaré established that the nonlinear equation $\Delta u = e^u$ has at least one solution. Because of the major contributions of Lyapunov (1906), Schmidt (1908), and Leray and Schauder (1934), the method of lower and upper solutions became a powerful tool in the modern nonlinear analysis. Due to the singular character of problem (4), in this paper we apply a recent version of the lower and upper solutions method which is due to Cui [7] and reads as follows. Suppose that $\underline{u}$, $\overline{u} \in C^2(\Omega) \cap C(\partial\Omega)$ satisfy $0 < \underline{u} \le \overline{u}$ in $\Omega$, $\underline{u} = \overline{u} = 0$ on $\partial\Omega$ and

$$
\begin{aligned} -\Delta \underline{u} &\le \lambda \underline{u}^{-a}\, |\nabla \underline{u}|^2 + \mu && \text{in } \Omega, \\ -\Delta \overline{u} &\ge \lambda \overline{u}^{-a}\, |\nabla \overline{u}|^2 + \mu && \text{in } \Omega. \end{aligned}
$$

Then problem (4) has at least one solution $u \in C^2(\Omega) \cap C(\partial\Omega)$ and, moreover, $\underline{u} \le u \le \overline{u}$ in $\Omega$.

## 3. A RELATED SINGULAR DIFFERENTIAL INEQUALITY

As in [10], we are first concerned with the inequality problem

$$\begin{cases} -\Delta u \ge \lambda u^{-a} |\nabla u|^2 + \mu & \text{in } B_R, \\ u > 0 & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{5}$$

where $R > 0$ is fixed.

The main result of this section is as follows.

Theorem 3.1. *Fix $\mu > 0$ and $R > 0$. Then there exists $\lambda^* > 0$ depending on $\mu$ and $R$ such that, for any $0 < \lambda \le \lambda^*$, problem* (5) *has at least one radially symmetric solution in $C^2(B_R) \cap C(\partial B_R)$.*

We start with the study of the related nonlinear differential equation with initial boundary conditions

$$\begin{cases} -(r^{N-1}z'(r))' = r^{N-1}[z^{-a}(r)z'(r)^2 + \alpha] & \text{if } 0 < r < T, \\ z(0) = \beta \\ z'(0) = 0 \\ z > 0 & \text{in } (0,T), \end{cases} \tag{6}$$

where $T, \alpha$ and $\beta$ are positive numbers.

Lemma 3.1. *Fix the positive numbers $\alpha$ and $\beta$. Then there exists $T > 0$ such that problem* (6) *has a solution $z \in C^2[0,T) \cap C[0,T]$. Moreover, $z$ is decreasing on $[0,T]$ and $z(T) = 0$.*

*Proof.* Problem (6) may be equivalently written in terms of the nonlinear integral equation

$$z(r) = \beta - \frac{\alpha}{2N}r^2 - \int_0^r t^{1-N} \int_0^t s^{N-1} z^{-a}(s) z'(s)^2 \mathrm{d}s\mathrm{d}t, \tag{7}$$

for $r \in [0,T)$.

Fix $\varepsilon > 0$. Consider the function space $C^1[0,\varepsilon]$ endowed with the standard norm

$$\|z\| = \max_{t\in[0,\varepsilon]} |z(t)| + \max_{t\in[0,\varepsilon]} |z'(t)|.$$

Let $F_\varepsilon \subset C^1[0,\varepsilon]$ be the closed ball defined by

$$F_\varepsilon = \{z \in C^1[0,\varepsilon];\ \|z - \beta\| \le \beta/2\}$$

and consider the nonlinear integral operator $T$ defined by

$$Tz(r) = \beta - \frac{\alpha}{2N} r^2 - \int_0^r t^{1-N} \int_0^t s^{N-1} z^{-a}(s) z'(s)^2 \mathrm{d}s \mathrm{d}t,$$

for any $r \in [0, \varepsilon]$.

A standard straightforward computation shows that $T$ maps $F_\varepsilon$ into $F_\varepsilon$ and $T$ is a contraction, that is,

$$\|Tz_1 - Tz_2\| \leq \frac{1}{2} \|z_1 - z_2\|, \quad z_1, z_2 \in F_\varepsilon,$$

provided $\varepsilon > 0$ is small enough.

Thus, by the Banach contraction principle, $T$ has a fixed point $z \in F_\varepsilon$, which is a solution of problem (7), hence of (6).

Set

$$T = \sup\{\varepsilon > 0;\ \text{problem (6) has a solution in } (0, \varepsilon)\}.$$

Using now the definition of $T$ we deduce that $z'(r) < 0$ for any $r \in (0, T)$.

We argue in what follows that $z(T) = 0$. For this purpose we first observe that since $z' < 0$ on $(0, T)$ there exists $\ell = \lim_{r \to T^-} z(r) \geq 0$. It suffices to show that $\ell = 0$. We argue by contradiction and assume that $\ell > 0$.

Fix $\varepsilon > 0$ and consider the closed set

$$F_\varepsilon(\ell) = \{z \in C^1[T, T + \varepsilon];\ \|z - z_T\| \leq l/2\},$$

where $z_T$ is defined by

$$z_T(r) = \ell + \frac{\alpha}{2N}(T^2 - r^2), \quad r \in [T, T + \varepsilon],$$

and $\|\cdot\|$ denotes the norm on the space $C^1[T, T + \varepsilon]$.

For any $z \in F_\varepsilon(\ell)$, define the nonlinear operator

$$S_z(r) = z_T(r) - \int_T^r t^{1-N} \int_T^t s^{N-1} z^{-a}(s) z'(s)^2 \mathrm{d}s \mathrm{d}t,$$

for all $r \in [T, T + \varepsilon]$. Then $S$ maps $F_\varepsilon(\ell)$ into itself and

$$\|Sz_1 - Sz_2\| \leq \frac{1}{2} \|z_1 - z_2\|, \quad z_1, z_2 \in F_\varepsilon(\ell).$$

Again, by Banach's contraction principle, $S$ has a fixed point $z \in F_\varepsilon(\ell)$, that is,

$$z(r) = z_T(r) - \int_T^r t^{1-N} \int_T^t s^{N-1} z^{-a}(s) z'(s)^2 \mathrm{d}s \mathrm{d}t,$$

for $r \in [T, T + \varepsilon]$. This shows that problem (6) has a solution on $[0, T + \varepsilon]$, which contradicts the definition of $T$. Thus, $\ell = 0$, hence $z(T) = 0$.

It remains to argue the regularity of $z$. Integrating (6) on $[0, r]$ we find

$$-z'(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} z(t)^{-a} z'(t)^2 \mathrm{d}t + \frac{\alpha}{N} r.$$

Therefore,

$$z''(r) = -\frac{\alpha}{N} + (N-1)r^{-N} \int_0^r t^{N-1} z(t)^{-a} z'(t)^2 \mathrm{d}t - z(r)^{-a} z'(r)^2,$$

hence $z \in C^2(0,T) \cap C[0,T]$. We also deduce that

$$z''(0) = \lim_{r\to 0} z''(r) = -\frac{\alpha}{N},$$

which concludes that $z \in C^2[0,T) \cap C[0,T]$. $\square$

*Proof of Theorem* 3.1 *concluded.* Fix $0 < \beta < \alpha R^2/(2N)$ and consider the corresponding solution $z$ of problem (6).

The local existence theorem applied to (6) implies $\varepsilon \le T \le (2N\beta/\alpha)^{1/2}$. The above choice of $\alpha$ and $\beta$ implies that $T < R$. For any $r \in [0,R)$ define the mapping

$$w(r) = \frac{R^2}{T^2}\, z\left(\frac{rT}{R}\right).$$

Then $w(0) = R^2\beta/T^2$, $w(R) = 0$, and $w > 0$ on $[0,R)$. A straightforward computation shows that $w$ satisfies the nonlinear differential equation

$$-(r^{N-1}w'(r))' = r^{N-1}\left[z\left(\frac{rT}{R}\right)^{-a} z'\left(\frac{rT}{R}\right)^2 + \alpha\right].$$

Since $z$ decreases, we deduce that

$$-(r^{N-1}w'(r))' \ge r^{N-1}\left[\left(\frac{T}{R}\right)^2 w(r)^{-a} w'(r)^2 + \alpha\right].$$

Set $\lambda^* = T^2/R^2$. Thus, for any $\lambda \in [0,\lambda^*]$,

$$-(r^{N-1}w'(r))' \ge r^{N-1}\left[\lambda w(r)^{-a} w'(r)^2 + \alpha\right],$$

for all $r \in (0,r)$. Equivalently, $w$ is a solution of the inequality problem (5), provided $\alpha$ is chosen so that $\alpha \ge \mu$. $\square$

## 4. AN AUXILIARY APPROXIMATE PROBLEM

Inspired by the techniques developed by Crandall, Rabinowitz and Tartar, for any $\varepsilon > 0$ we consider the nonlinear problem

$$\begin{cases} -\Delta u = \lambda u^{-a}|\nabla u|^2 + \mu + \varepsilon & \text{in } \Omega, \\ u > \varepsilon & \text{in } \Omega, \\ u \ge \varepsilon & \text{on } \partial\Omega. \end{cases} \tag{8}$$

The main result of this section establishes that problem (8) has at least one solution, provided $\varepsilon$ and $\lambda$ are small enough. More precisely, the following property holds true.

LEMMA 4.1. *There exists* $\varepsilon_0 > 0$ *and* $\Lambda > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $\lambda \in (0, \Lambda)$, *problem* (8) *has a solution in* $C^2(\overline{\Omega})$.

*Proof.* Our arguments are based on the method of lower and upper solutions. So, we construct functions $\overline{u}$ and $\underline{u}$ such that $\overline{u} \geq \underline{u} > \varepsilon$ in $\Omega$ and

$$-\Delta \underline{u} - \lambda \underline{u}^{-a}|\nabla \underline{u}|^2 - \mu - \varepsilon \leq 0 \leq -\Delta \overline{u} - \lambda \overline{u}^{-a}|\nabla \overline{u}|^2 - \mu - \varepsilon \quad \text{in } \Omega.$$

Set $d = \sup_{x\in\Omega}|x|$. For $R = 3d/2$, denote $\overline{u} = z + \varepsilon$, where $z$ is the solution of problem (5) given by Theorem 3.1. So, $\overline{u} \in C^2(\overline{\Omega})$ is an upper solution of problem (8).

In order to find a lower solution, fix $p > N$ and consider the unique solution $u_0 \in C^\infty(\overline{\Omega})$ of the linear problem

$$\begin{cases} -\Delta u_0 = \mu & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the function $\underline{u} = u_0 + \varepsilon$ is a linear solution of (8).

It remains to argue that $\underline{u} \leq \overline{u}$ in $\Omega$. For this purpose we observe that $\underline{u} \leq \overline{u}$ on $\partial\Omega$ and $-\Delta(\overline{u} - \underline{u}) \geq 0$ in $\Omega$. Thus, by the maximum principle, $\underline{u} \leq \overline{u}$ in $\Omega$. It follows that problem (8) has a smooth solution $u_\varepsilon$ such that $\underline{u} \leq u_\varepsilon \leq \overline{u}$ in $\Omega$. $\square$

## 5. PROOF OF THEOREM 2.1

We first prove that if $0 < \varepsilon_1 < \varepsilon_2$ then $u_{\varepsilon_1} \geq u_{\varepsilon_2}$, where $u_\varepsilon$ denotes the solution of problem (8). Arguing by contradiction, let $x_0 \in \Omega$ be such that $u_{\varepsilon_2}(x_0) < u_{\varepsilon_1}(x_0)$. Moreover, since $u_{\varepsilon_1} - u_{\varepsilon_2} = \varepsilon_1 - \varepsilon_2 < 0$ on $\partial\Omega$, we deduce that without loss of generality we can assume that $x_0 \in \Omega$ is a global maximum point of the function $u_{\varepsilon_1} - u_{\varepsilon_2}$ in $\overline{\Omega}$. Hence

$$\nabla u_{\varepsilon_1}(x_0) = \nabla u_{\varepsilon_2}(x_0) \quad \text{and} \quad \Delta(u_{\varepsilon_1} - u_{\varepsilon_2})(x_0) \leq 0.$$

Therefore,

$$\Delta u_{\varepsilon_1}(x_0) - \Delta u_{\varepsilon_2}(x_0) = \lambda \left(u_{\varepsilon_2}(x_0)^{-a} - u_{\varepsilon_1}(x_0)^{-a}\right)|\nabla u_{\varepsilon_1}(x_0)|^2 + \varepsilon_2 - \varepsilon_1 > 0.$$

This contradiction shows that $u_{\varepsilon_1} \geq u_{\varepsilon_2}$ in $\Omega$ provided $\varepsilon_1 < \varepsilon_2$.

For any $x \in \Omega$, denote

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x).$$

To conclude the proof, we show that $u$ is a solution of problem (4), that is, $u \in C^2(\Omega) \cap C(\partial\Omega)$ and

$$\begin{aligned} &-\Delta u = \lambda u^{-a}\,|\nabla u|^2 + \mu && \text{in } \overline{\Omega},\\ &u = 0 && \text{on } \partial\Omega. \end{aligned}$$

To obtain the regularity of $u$ we apply bootstrap arguments. We first observe that the definition of $u$ implies $u \in L^\infty(\Omega)$. Let $u_n$ be a solution of the problem

$$-\Delta u_n = \lambda u_n^{-a}\,|\nabla u_n|^2 + \mu + \frac{1}{n} \quad \text{in } \Omega.$$

Then $\underline{u} \le u_n \le C$ in $\overline{\Omega}$ for all $n \ge 1$, where $C > 0$ is a constant. Moreover, $u_n \in W^{2,p}(\Omega)$ for any $p > N$ and

$$-\int_\Omega u_n \Delta\varphi \mathrm{d}x = \int_\Omega \lambda u_n^{-a}\,|\nabla u_n|^2 \varphi \mathrm{d}x + \left(\mu + \frac{1}{n}\right)\int_\Omega \varphi \mathrm{d}x.$$

for all $\varphi \in C_0^\infty(\Omega)$.

Consider a sequence $(\Omega_m)_{m\ge1}$ of open sets with smooth boundary such that $\Omega = \bigcup_{m=1}^\infty \Omega_m$. Fix a positive integer $k \ge 1$ and a function $\varphi \in C_0^\infty(\Omega)$ satisfying $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $\overline{\Omega}_k$. We deduce that

$$\lambda \int_{\Omega_k} u_n^{-a}\,|\nabla u_n|^2 \varphi \mathrm{d}x \le -\int_\Omega u_n \Delta\varphi \mathrm{d}x - \left(\mu + \frac{1}{n}\right)\int_\Omega \varphi \mathrm{d}x,$$

hence

$$\lambda C^{-a} \int_{\Omega_k} |\nabla u_n|^2 \mathrm{d}x \le C \int_\Omega |\Delta\varphi| \mathrm{d}x + (\mu+1)|\Omega|.$$

It follows that the sequence $(u_n)_{n\ge1}$ is bounded in $H^1(\Omega_k)$. Thus, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup v \quad \text{in } H^1(\Omega_k),\\ &u_n \to v \quad \text{in } L^q(\Omega_k) \text{ for all } 1 \le q < \tfrac{2N}{N-2},\\ &u_n \to v \quad \text{a.e. in } \Omega_k. \end{aligned}$$

Hence, by the definition of $u$, we have $u = v$ a.e. in $\Omega_k$, for any $k \ge 1$. This shows that, in fact,

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } H^1_{loc}(\Omega),\\ &u_n \to u \quad \text{in } L^q_{loc}(\Omega) \text{ for any } 1 \le q < \tfrac{2N}{N-2}. \end{aligned}$$

By Schauder estimates we deduce that $(u_n)$ is bounded in $W^{2,p}_{loc}(\Omega)$, so in $C^{1,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$. It follows that for any compact set $K \subset \Omega$ we have

$$u_n \to u \quad \text{in } C^2(K)$$

and

$$-\Delta u = \lambda u^{-a}\,|\nabla u|^2 + \mu \quad \text{in } \Omega.$$

To show that $u \in C(\partial\Omega)$, let $x_0 \in \partial\Omega$ and consider a sequence of points $(x_j)_{j\geq 1} \subset \Omega$ such that $x_j \to x_0$. Then $0 \leq u(x_j) \leq u_n(x_j)$, hence

$$0 \leq \overline{\lim}_{j\to\infty} u(x_j) \leq \lim_{j\to\infty} u_n(x_j) = u_n(x_0) = \frac{1}{n}.$$

We obtain that $u$ is continuous at $x_0 \in \partial\Omega$ and $u(x_0) = 0$.

The proof of theorem is now complete. $\square$

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REFERENCES

[1] R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts.* Clarendon Press, Oxford, 1975.

[2] L. Caffarelli, R. Hardt and L. Simon, *Minimal surfaces with isolated singularities.* Manuscripta Math. **48** (1984), 1–18.

[3] A. Callegari and A. Nachman, *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids.* SIAM J. Appl. Math. **38** (1980), 275–281.

[4] Y. Choquet-Bruhat and J. Leray, *Sur le problème de Dirichlet quasilinéaire d'ordre* 2. C.R. Acad. Sci. Paris Sér. A-B **274** (1972), A81–A85.

[5] M.M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem.* Comm. Partial Differential Equations **14** (1989), 1315–1327.

[6] M.G. Crandall, P.H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity.* Comm. Partial Differential Equations **2** (1977), 193–222.

[7] S. Cui, *Existence and nonexistence of positive solutions for singular semilinear elliptic boundary value problems.* Nonlinear Anal. **41** (2000), 149–176.

[8] J.I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Vol. I. Elliptic Equations.* Research Notes in Mathematics **106**. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[9] J.I. Díaz, J.M. Morel and L. Oswald, *An elliptic equation with singular nonlinearity.* Comm. Partial Differential Equations **12** (1987), 1333–1344.

[10] L. Dupaigne, M. Ghergu and V. Rădulescu, *Lane-Emden-Fowler equations with convection and singular potential.* J. Math. Pures Appl. **87** (2007), 563–581.

[11] W. Fulks and J.S. Maybee, *A singular nonlinear equation.* Osaka J. Math. **12** (1960), 1–19.

[12] M. Ghergu and V. Rădulescu, *Sublinear singular elliptic problems with two parameters.* J. Differential Equations **195** (2003), 520–536.

[13] M. Ghergu and V. Rădulescu, *Multiparameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term.* Proc. Roy. Soc. Edinburgh Sect. A **135** (2005), 61–84.

[14] M. Ghergu and V. Rădulescu, *Ground state solutions for the singular Lane-Emden-Fowler equation with sublinear convection term.* J. Math. Anal. Appl. **333** (2007), 265–273.

[15] M. Ghergu and V. Rădulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis.* Oxford Lecture Series in Mathematics and its Applications **37**. Oxford Univ. Press, 2008.

[16] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition. Springer-Verlag, Berlin–Heidelberg–New York, 1983.

[17] Y. Haitao, *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem.* J. Differential Equations **189** (2003), 487–512.

[18] J. Hernández, F.J. Mancebo and J.M. Vega, *On the linearization of some singular nonlinear elliptic problems and applications.* Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 777–813.

[19] J. Kazdan and F.W. Warner, *Remarks on some quasilinear elliptic equations.* Comm. Pure Appl. Math. **28** (1975), 567–597.

[20] A. Meadows, *Stable and singular solutions of the equation* $\Delta u = 1/u$. Indiana Univ. Math. J. **53** (2004), 1681–1703.

[21] H. Poincaré, (1898). *Les fonctions fuchsiennes et l'équation* $\Delta u = e^u$. J. Math. Pures Appl. **4** (1898), 137–230.

[22] V. Rădulescu, *Bifurcation and asymptotics for elliptic problems with singular nonlinearity.* In: C. Bandle, H. Berestycki, B. Brighi, A. Brillard, M. Chipot, J.-M. Coron, C. Sbordone, I. Shafrir, V. Valente, G. Vergara Caffarelli (Eds.), *Studies in Nonlinear Partial Differential Equations: In Honor of Haim Brezis. Fifth European Conference on Elliptic and Parabolic Problems: A special tribute to the work of Haim Brezis, Gaeta, Italy, May* 30–*June* 3, 2004, pp. 349–362. Birkhäuser-Verlag, Basel, 2005.

[23] V. Rădulescu, *Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities.* In: M. Chipot (Ed.), *Handbook of Differential Equations: Stationary Partial Differential Equations*, Vol. 4, pp. 483–591. Elsevier, North-Holland, Amsterdam, 2007.

[24] J. Shi and M. Yao, *On a singular nonlinear semilinear elliptic problem.* Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 1389–1401.

[25] C.A. Stuart, *Existence and approximation of solutions of nonlinear elliptic equations.* Math. Z. **147** (1976), 53–63.

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