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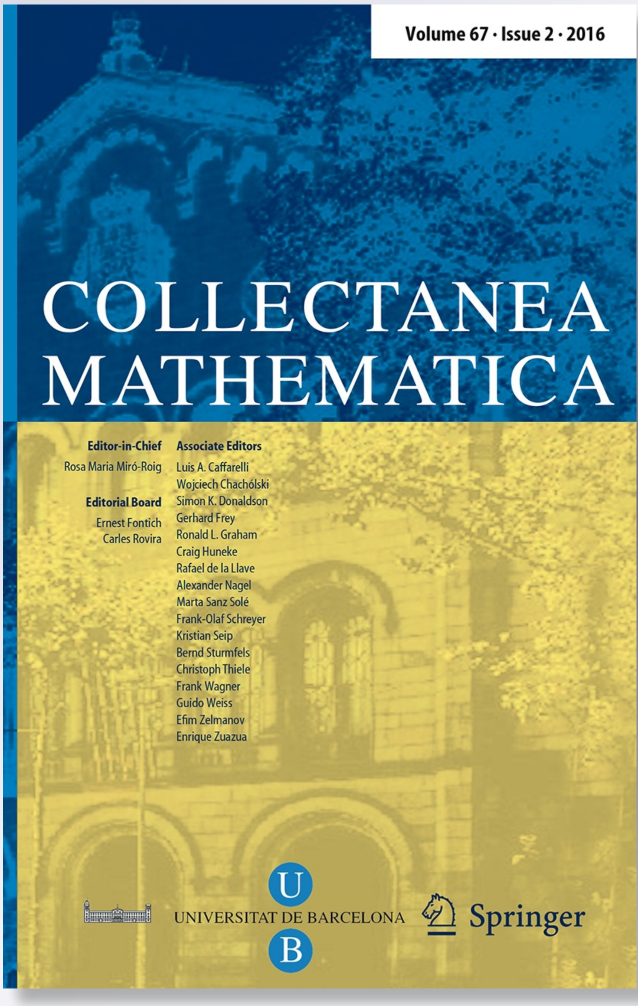

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Nonlocal fourth-order Kirchhoff systems with variable growth: low and high energy solutions

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Abstract This paper is concerned with the existence and multiplicity of solutions for a class of nonlocal fourth-order $(p(x), q(x))$-Kirchhoff systems. By means of a variational analysis, we obtain conditions for the existence of infinitely many solutions with high (resp., low) energies. The arguments combine related critical point theory arguments with a careful analysis of the energy levels.

Keywords Variational method · Nonlinear elliptic systems · $(p(x), q(x))$-Kirchhoff system · Nonlocal condition

Mathematics Subject Classification 35J60 · 35B30 · 35B40 · 58E05

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1 Introduction

In this paper, we study the following nonlocal elliptic system

$$\begin{cases} -M_1\left(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) = F_u(x,u,v) & \text{in } \Omega, \\ -M_2\left(\displaystyle\int_\Omega \frac{1}{q(x)}|\Delta v|^{q(x)}dx\right)\Delta\left(|\Delta v|^{q(x)-2}\Delta v\right) = F_v(x,u,v) & \text{in } \Omega, \\ u = v = \Delta u = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p,\ q \in C(\overline{\Omega})$, $p,\ q > 1$ in $\overline{\Omega}$, and M_1, M_2 are continuous functions. The function F satisfies Carathéodory conditions and is of class C^1 in $u, v \in \mathbb{R}$. The functions F_u, F_v represent *source forces*, while M_1, M_2 are *Kirchhoff dissipative terms*.

For simplicity reasons, in the present paper we reduce to the case where $M_1 = M_2 =: M$. Notice that the results we establish in what follows remain valid for $M_1 \neq M_2$ by adding some slight changes.

Boundary value problems like (1.1) model several physical and biological systems where u and v describe a process depending on the average of itself, as for example, population densities. We refer the reader, for instance, to Alves and Figueiredo [1], Autuori and Pucci [5–7], Autuori et al. [8], Molica Bisci and Rădulescu [22], Molica Bisci et al. [23], Rădulescu [25], Rădulescu and Repovš [26], and Vasconcellos [27].

Problem (1.1) is called a nonlocal problem because of the presence of the term M, which implies that the equation in (1.1) is no longer a pointwise identity. This provokes some mathematical difficulties which make the study of such a problem particularly interesting. Nonlocal differential equations are also called Kirchhoff-type equations because Kirchhoff [17] has investigated an equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where E is the Young modulus of the material, ρ is the mass density, L is the length of the string, h is the area of the cross-section, and ρ_0 is the initial tension. Equation (1.2) extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguished feature of Eq. (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, and hence the equation is no longer a pointwise identity.

We point out that the Kirchhoff model takes into account the length changes of the string produced by transverse vibrations. We refer to Bernstein [10] and Pohozaev [24] as pioneering papers dedicated to Kirchhoff equations. However, Eq. (1.2) received much attention only after the paper by Lions [21], where an abstract framework to the problem was proposed. D'Ancona and Spagnolo [13] considered Kirchhoff's equation as a quasi-linear hyperbolic Cauchy problem that describes the transverse oscillations of a stretched string. For completeness we refer the reader to some recent interesting results obtained by Autuori and Pucci in [5–7] studying Kirchhoff equations by using different approaches. We also recall that nonhomogeneous $p(x)$-Kirchhoff operators have been used in the last decades to model various phenomena, see [16,32] and the references therein. Indeed, recently, there has been an increasing interest in studying systems involving somehow nonhomogeneous $p(x)$-Laplace operators, motivated by the image restoration problem, by the modeling of *electro-rheological* *fluids* (sometimes referred to as *smart fluids*), as well as the *thermo-convective flows* of non-Newtonian fluids: details and further references can be found in [4].

The investigation of existence and multiplicity of solutions for problems with $p(x)$-biharmonic operators has drawn the attentions of many authors, see [3,14,18,20] for some recent work on this subject. Motivated by the above references, we establish the existence of infinitely many low or high energy solutions for system (1.1).

Our paper is organized as follows. We first present some necessary preliminary results on variable exponent Sobolev spaces. Next, we give the main results about the existence of weak solutions. The final part of this paper is concerned with the existence of infinitely many low or high energy solutions.

2 Functional setting

In this section, we recall some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. Denote

$$C_+(\overline{\Omega}) = \{h(x);\ h(x) \in C(\overline{\Omega}),\ h(x) > 1, \quad \text{for all} \quad x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max\{h(x);\ x \in \overline{\Omega}\}, h^- = \min\{h(x);\ x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R};\ u \text{ is a measurable real-valued function and } \int_\Omega |u(x)|^{p(x)} dx < \infty\right\},$$

endowed with the *Luxemburg norm*

$$|u|_{p(x)} = |u|_{L^{p(x)}(\Omega)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space, cf. [19].

Proposition 2.1 (Fan and Zhao [15]) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, that is,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \quad \text{for all} \quad x \in \Omega.$$

For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\ D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \le k\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)},$$

becomes a separable and reflexive Banach space. For more details, we refer the reader to [15,29]. Denote for $x \in \overline{\Omega}$ and $k \ge 1$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } \ p(x) < N, \\ +\infty & \text{if } \ p(x) \ge N, \end{cases}$$

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } \ kp(x) < N, \\ +\infty & \text{if } \ kp(x) \ge N. \end{cases}$$

Proposition 2.2 (Fan and Zhao [15]) *Let $p, r \in C_+(\overline{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$. Then there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\le$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Then the function space $\left(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \|u\|_{p(x)}\right)$ is a separable and reflexive Banach space, where

$$\|u\|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Remark 2.3 According to [30], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta\cdot|_{p(x)}$ in the space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|_{p(x)}$ and $|\Delta\cdot|_{p(x)}$ are equivalent.

Consider the functional

$$\phi(u) = \int_\Omega |\Delta u|^{p(x)}\, dx.$$

Then we have the following properties (see for example [3, Proposition 3.2]): if $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, then

$$\|u\|_{p(x)} < 1\ (=1;\ >1) \Leftrightarrow \phi(u) < 1\ (=1;\ >1); \tag{2.1}$$

$$\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \le \phi(u) \le \|u\|_{p(x)}^{p^+}; \tag{2.2}$$

$$\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \le \phi(u) \le \|u\|_{p(x)}^{p^-}; \tag{2.3}$$

$$\|u\|_{p(x)} \to 0\ (\to \infty) \Leftrightarrow \phi(u) \to 0\ (\to \infty). \tag{2.4}$$

Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space

$$X = \left(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)\right) \times \left(W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega)\right)$$

equipped with the norm

$$\|(u,v)\| = \max\{\|u\|_{p(x)}, \|v\|_{q(x)}\}.$$

We denote by $(u_n, v_n) \rightharpoonup (u, v)$ and $(u_n, v_n) \to (u, v)$ the weak convergence and strong convergence of (u_n, v_n) to (u, v) in X, respectively, denote by c_i the positive constants. The dual space X is denoted by X^* and $\|\cdot\|_*$ stands for its norm. Therefore

$$\|J'(u,v)\|_* = \|D_1 J(u,v)\|_{*,p(x)} + \|D_2 J(u,v)\|_{*,q(x)},$$

where $\|\cdot\|_{*,p(x)}$ (respectively $\|\cdot\|_{*,q(x)}$) is the norm of $(W^{2,p(x)}(\Omega) \cap W^{1,p(x)}(\Omega))^*$ (respectively $(W^{2,q(x)}(\Omega) \cap W^{1,q(x)}(\Omega))^*$).

Set

$$\begin{aligned}
I_1(u) &= \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx,\\
I_2(v) &= \int_\Omega \frac{1}{q(x)} |\Delta v|^{q(x)}\, dx,\\
I(u,v) &= I_1(u) + I_2(v),\\
\mathcal{F}(u,v) &= \int_\Omega F(x,u,v)\, dx.
\end{aligned}$$

Then

$$\begin{aligned}
I'(u,v)(\varphi,\psi) &= D_1 I(u,v)(\varphi) + D_2 I(u,v)(\psi),\\
\mathcal{F}'(u,v)(\varphi,\psi) &= D_1 \mathcal{F}(u,v)(\varphi) + D_2 \mathcal{F}(u,v)(\psi),
\end{aligned}$$

where

$$\begin{aligned}
D_1 I(u,v)(\varphi) &= \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta\varphi\, dx = I_1'(u)(\varphi),\\
D_2 I(u,v)(\psi) &= \int_\Omega |\Delta v|^{q(x)-2} \Delta v \Delta\psi\, dx = I_2'(v)(\psi),\\
D_1 \mathcal{F}(u,v)(\varphi) &= \int_\Omega \frac{\partial F}{\partial u}(x,u,v)\varphi\, dx,\\
D_2 \mathcal{F}(u,v)(\psi) &= \int_\Omega \frac{\partial F}{\partial v}(x,u,v)\psi\, dx.
\end{aligned}$$

The Euler–Lagrange functional associated to problem (1.1) is given by

$$J(u,v) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx\right) + \widehat{M}\left(\int_\Omega \frac{1}{q(x)} |\Delta v|^{q(x)}\, dx\right) - \int_\Omega F(x,u,v)\, dx,$$

where $\widehat{M}(t) = \int_0^t M(\tau) d\tau$. Then $J \in C^1(X, \mathbb{R})$ and

$$J'(u,v)(\varphi,\psi) = D_1 J(u,v)(\varphi) + D_2 J(u,v)(\psi), \tag{2.5}$$

where

$$\begin{aligned}
D_1 J(u,v)(\varphi) &= M\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx\right) \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta\varphi\, dx - D_1\mathcal{F}(u,v)(\varphi),\\
D_2 J(u,v)(\psi) &= M\left(\int_\Omega \frac{1}{q(x)} |\Delta v|^{q(x)}\, dx\right) \int_\Omega |\Delta v|^{q(x)-2} \Delta v \Delta\psi\, dx - D_2\mathcal{F}(u,v)(\psi).
\end{aligned}$$

Hereafter, $F(x, s, t)$ and $M(t)$ are always supposed to verify the following assumption:

(M_1) There exist $m_2 \geq m_1 > 0$ and $\beta \geq \alpha > 1$ such that for all $t \in \mathbb{R}^+$, $m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\beta-1}$.
(M_2) For all $t \in \mathbb{R}^+$, $\widehat{M}(t) \geq M(t)t$.
(F_1) For all $(x, s, t) \in \Omega \times \mathbb{R}^2$, we assume

$$|F(x,s,t)| \leq c_1 \left(1 + |s|^{\sigma_1(x)} + |t|^{\sigma_2(x)} + |s|^{\sigma_3(x)}|t|^{\sigma_4(x)}\right),$$

where c_1 is a positive constant, $(\sigma_1(x), \sigma_2(x), \sigma_3(x), \sigma_4(x)) \in C_+(\overline{\Omega})^4$ such that

$$\sigma_1(x) < p_2^*(x),\ \sigma_2(x) < q_2^*(x),\ \frac{2\sigma_3(x)}{p_2^*(x)} + \frac{2\sigma_4(x)}{q_2^*(x)} < 1 \quad \text{in} \quad \overline{\Omega}.$$

(F_2) There are $M > 0$, $\theta_1 > \beta p^+$, $\theta_2 > \beta q^+$ such that for all $x \in \Omega$ and all $(s, t) \in \mathbb{R}^2$ with $|s|^{\theta_1} + |t|^{\theta_2} \geq 2M$, we have

$$0 < F(x,s,t) \leq \frac{s}{\theta_1}\frac{\partial F}{\partial s}(x,s,t) + \frac{t}{\theta_2}\frac{\partial F}{\partial t}(x,s,t),$$

where β comes from (M_1) above.
(F_3) $F(x, s, t) = o(|s|^{\alpha p^+} + |t|^{\alpha q^+})$ as $(s, t) \to (0, 0)$ uniformly with respect to $x \in \Omega$, where α comes from (M_1).
(F_4) $F(x, -s, -t) = -F(x, s, t)$ for all $x \in \Omega$ and $(s, t) \in \mathbb{R}^2$.
(F_5) We have

$$F(x,s,t) \geq c_2(|s|^{\gamma_1(x)} + |t|^{\gamma_2(x)}) \quad \text{as } (s,t) \to (0,0),$$

where $(\gamma_1(x), \gamma_2(x)) \in (C_+(\overline{\Omega}))^2$ such that $\gamma_1(x) < p_2^*(x)$, $\gamma_2(x) < q_2^*(x)$, $p^+ < \gamma_1^- \leq \gamma_1^+ < \beta p^-$, $q^+ < \gamma_2^- \leq \gamma_2^+ < \beta q^-$ for a.e. $x \in \Omega$, where β comes from (M_1).

Lemma 2.4 (El Amrouss et al. [3]) *We have the following assertions:*

(1) *I_1' is a bounded homeomorphism and strictly monotone operator.*
(2) *I_1' is a mapping of type (S_+), namely*

$$u_n \rightharpoonup u \quad \textit{and} \quad \limsup_{n\to+\infty} I_1'(u_n)(u_n - u) \leq 0, \quad \textit{implies} \quad u_n \to u.$$

Since X is a reflexive and separable Banach space, then X^* is too. There exist (see [31]) $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, \ldots\}}, \quad X^* = \overline{\text{span}\{e_j^* : j = 1, 2, \ldots\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denote the duality product between X and X^*. We define

$$X_j = \text{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

A central role in our arguments will be played by the fountain theorem, which is due to Bartsch [9]. This result is nicely presented in Willem [28] by using the quantitative deformation lemma. We also point out that the dual version of the fountain theorem is due to

Bartsch and Willem, see [28]. Both the fountain theorem and its dual form are effective tools for studying the existence of infinitely many large or small energy solutions. It should be noted that the Palais–Smale condition plays an important role for these theorems and their applications.

Lemma 2.5 (Fountain Theorem, see [28]). *Assume*

(A1) *X is a Banach space, $J \in C^1(X, \mathbb{R})$ is an even functional.*
Suppose that for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that
(A2) $\inf\{J(u) : u \in Z_k,\ \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$.
(A3) $\max\{J(u) : u \in Y_k,\ \|u\| = \rho_k\} \leq 0$.
(A4) *J satisfies the Palais–Smale condition for every $c > 0$.*

Then J has an unbounded sequence of critical points.

Lemma 2.6 (*Dual Fountain Theorem, see* [28]). *Assume (A1) is satisfied and there is $k_0 > 0$ so that, for each $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that*

(B1) $a_k = \inf\{J(u) : u \in Z_k,\ \|u\| = \rho_k\} \geq 0$.
(B2) $b_k = \max\{J(u) : u \in Y_k,\ \|u\| = r_k\} < 0$.
(B3) $d_k = \inf\{J(u) : u \in Z_k,\ \|u\| \leq \rho_k\} \to 0$ *as* $k \to +\infty$.
(B4) *J satisfies the $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0)$.*

Then J has a sequence of negative critical values converging to 0.

For every $a > 1$, $u, v \in L^a(\Omega)$, we define

$$|(u, v)|_a := \max\{|u|_a, |v|_a\}.$$

Set

$$a := \max_{x \in \overline{\Omega}}\{\sigma_1(x), \sigma_2(x), 2\sigma_3(x), 2\sigma_4(x)\}, \tag{2.6}$$

$$b := \min_{x \in \overline{\Omega}}\{\sigma_1(x), \sigma_2(x), 2\sigma_3(x), 2\sigma_4(x)\}. \tag{2.7}$$

Then we have the following result.

Lemma 2.7 [14] *Denote*

$$\beta_k = \sup\{|(u, v)|_a;\ \|(u, v)\| = 1,\ (u, v) \in Z_k\}, \tag{2.8}$$

$$\theta_k = \sup\{|(u, v)|_b;\ \|(u, v)\| = 1,\ (u, v) \in Z_k\}. \tag{2.9}$$

Then $\lim_{k\to\infty} \beta_k = \lim_{k\to\infty} \theta_k = 0$.

We conclude this preliminary section by recalling the definition of the *localized Palais–Smale condition*, which was introduced by Brezis and Nirenberg [12].

Definition 2.8 We say that J satisfies the $(PS)_c^*$ condition (with respect to (Y_n)), if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$, contain a subsequence converging to a critical point of J.

3 Existence results in a non-symmetric setting

In this section we establish two existence results under general hypotheses on the potential F.

Theorem 3.1 *Assume that M satisfies* (M_1) *and*

$$|F(x,s,t)| \le c_3(1+|s|^{\alpha_1}+|t|^{\beta_1}),$$

where α_1, β_1 are two constants with $1 \le \alpha_1 < \min\{\alpha p^-, \alpha q^-\}$, $1 \le \beta_1 < \min\{\alpha p^-, \alpha q^-\}$ [α comes from (M_1)*]. Then problem* (1.1) *has a nontrivial weak solution.*

Proof In view of (M_1), the functional J is weakly lower semi-continuous. In the following, we will prove that J is coercive, that is, $J(u,v) \to +\infty$ as $\|(u,v)\| \to +\infty$. From (M_1) we have $\widehat{M}(t) \ge \frac{m_1}{\alpha} t^\alpha$. For $(u,v) \in X$ such that $\|(u,v)\| \to +\infty$, we obtain

$$\begin{aligned}
J(u,v) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right) + \widehat{M}\left(\int_\Omega \frac{1}{q(x)}|\Delta v|^{q(x)}\,dx\right) - \int_\Omega F(x,u,v)dx\\
&\ge \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)^\alpha + \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{q(x)}|\Delta v|^{q(x)}\,dx\right)^\alpha - c_3\int_\Omega |u|^{\alpha_1}\,dx\\
&\quad -c_3\int_\Omega |v|^{\beta_1}dx - c_3|\Omega|\\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{p(x)}^{\alpha p^-} + \frac{m_1}{\alpha(q^+)^\alpha}\|v\|_{q(x)}^{\alpha q^-} - c_4\|u\|_{p(x)}^{\alpha_1} - c_5\|v\|_{q(x)}^{\beta_1} - c_3|\Omega|,
\end{aligned}$$

where $|\Omega|$ denote the measure of Ω. Without loss of generality, we may assume $\|u\|_{p(x)} \ge \|v\|_{q(x)}$.

If $\|v\|_{q(x)} > 1$ we have

$$J(u,v) \ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{p(x)}^{\alpha p^-} + \frac{m_1}{\alpha(q^+)^\alpha}\|v\|_{q(x)}^{\alpha q^-} - c_4\|u\|_{p(x)}^{\alpha_1} - c_5\|v\|_{q(x)}^{\beta_1} - c_3|\Omega|.$$

If $\|v\|_{q(x)} < 1$ we have

$$J(u,v) \ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{p(x)}^{\alpha p^-} - c_4\|u\|_{p(x)}^{\alpha_1} - c_6.$$

By the assumptions on α_1 and β_1, we deduce the coercivity of J and hence J attains its minimum on X, which yields a solution of problem (1.1). □

Lemma 3.2 *Let (u_n, v_n) be a Palais–Smale sequence for the Euler–Lagrange functional J. If conditions* (M_1), (M_2), (F_2) *are satisfied, then (u_n, v_n) is bounded.*

Proof Let (u_n, v_n) be a Palais–Smale sequence for the functional J. This means that $J(u_n, v_n)$ is bounded and $\|J'(u_n, v_n)\|_* \to 0$ as $n \to +\infty$. Thus, there is a positive constant c_7 such that

$$\begin{aligned}
c_7 &\ge J(u_n, v_n)\\
&= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right) + \widehat{M}\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{q(x)}\,dx\right) - \int_\Omega F(x,u_n,v_n)\,dx\\
&\ge M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx - \int_\Omega \frac{u_n}{\theta_1}\frac{\partial F}{\partial u}(x,u_n,v_n)\,dx\\
&\quad +M\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{q(x)}\,dx\right)\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{p(x)}\,dx - \int_\Omega \frac{v_n}{\theta_2}\frac{\partial F}{\partial v}(x,u_n,v_n)\,dx - c_8,
\end{aligned}$$

where c_8 is some positive constant. Therefore

$$\begin{aligned}
c_7 &\geq J(u_n, v_n)\\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta_1}\right) M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)\int_\Omega |\Delta u_n|^{p(x)}\,dx - \frac{1}{\theta_1} D_1 J(u_n, v_n)(u_n)\\
&\quad + \left(\frac{1}{q^+} - \frac{1}{\theta_2}\right) M\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{q(x)}dx\right)\int_\Omega |\Delta v_n|^{q(x)}\,dx - \frac{1}{\theta_2} D_2 J(u_n, v_n)(v_n) - c_8\\
&\geq \frac{m_1}{(p^+)^{\alpha-1}}\left(\frac{1}{p^+} - \frac{1}{\theta_1}\right)\left(\int_\Omega |\Delta u_n|^{p(x)}\,dx\right)^\alpha + \frac{m_1}{(q^+)^{\alpha-1}}\left(\frac{1}{q^+} - \frac{1}{\theta_2}\right)\left(\int_\Omega |\Delta v_n|^{q(x)}\,dx\right)^\alpha\\
&\quad - \frac{1}{\theta_1}\|D_1 J(u_n, v_n)\|_{*,p(x)}\|u_n\| - \frac{1}{\theta_2}\|D_2 J(u_n, v_n)\|_{*,q(x)}\|v_n\| - c_8.
\end{aligned}$$

Now, we suppose that the sequence (u_n, v_n) is not bounded. Without loss of generality, we may assume $\|u_n\|_{p(x)} \geq \|v_n\|_{q(x)}$.

Therefore, for n large enough that $\|u_n\|_{p(x)} > 1$, we obtain

$$\begin{aligned}
c_7 \geq\ & \frac{m_1}{(p^+)^{\alpha-1}}\left(\frac{1}{p^+} - \frac{1}{\theta_1}\right)\|u_n\|_{p(x)}^{\alpha p^-}\\
& - \left(\frac{1}{\theta_1}\|D_1 J(u_n, v_n)\|_{*,p} + \frac{1}{\theta_2}\|D_2 J(u_n, v_n)\|_{*,q}\right)\|u_n\|_{p(x)}.
\end{aligned}$$

But this cannot hold true since $\alpha p^- > p^- > 1$. Hence, (u_n, v_n) is bounded. □

Lemma 3.3 *Let (u_n, v_n) be a bounded Palais–Smale sequence for the Euler–Lagrange functional J. If conditions* (M_1), (M_2), (F_1), (F_2) *are satisfied, then (u_n, v_n) contains a convergent subsequence.*

Proof Let (u_n, v_n) be a bounded Palais–Smale sequence for J. Then there is a subsequence still denoted by (u_n, v_n) which converges weakly in X. Without loss of generality, we assume that $(u_n, v_n) \rightharpoonup (u, v)$, then $J'(u_n, v_n)(u_n - u, v_n - v) \to 0$. We obtain

$$\begin{aligned}
& J'(u_n, v_n)(u_n - u, v_n - v)\\
&= M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta u)\,dx\\
&\quad + M\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{q(x)}dx\right)\int_\Omega |\Delta v_n|^{q(x)-2}\Delta v_n(\Delta v_n - \Delta v)\,dx\\
&\quad - \int_\Omega \frac{\partial F}{\partial u}(x, u_n, v_n)(u_n - u)\,dx - \int_\Omega \frac{\partial F}{\partial v}(x, u_n, v_n)(v_n - v)\,dx \to 0.
\end{aligned}$$

On the other hand, let $\tilde{\sigma_1}(x)$ and $\tilde{\sigma_2}(x)$ be two continuous and positive functions on $\overline{\Omega}$ such that for all $x \in \overline{\Omega}$

$$\frac{2\sigma_3(x) + \tilde{\sigma_1}(x)}{p_2^*(x)} + \frac{2\sigma_4(x) + \tilde{\sigma_2}(x)}{q_2^*(x)} = 1.$$

Using the Young inequality, we obtain

$$|s|^{\sigma_3(x)}|t|^{\sigma_4(x)} \leq |s|^{\frac{\sigma_3(x)p_2^*(x)}{2\sigma_3(x)+\tilde{\sigma_1}(x)}} + |t|^{\frac{\sigma_4(x)q_2^*(x)}{2\sigma_4(x)+\tilde{\sigma_2}(x)}} = |s|^{\sigma_5(x)} + |t|^{\sigma_6(x)},$$

where $\sigma_5(x) := \frac{\sigma_3(x)p_2^*(x)}{2\sigma_3(x)+\tilde{\sigma_1}(x)} < p_2^*(x)$ and $\sigma_6(x) := \frac{\sigma_4(x)q_2^*(x)}{2\sigma_4(x)+\tilde{\sigma_2}(x)} < q_2^*(x)$. From ($\mathrm{F}_1$), we obtain $\sigma_6(x), \sigma_7(x) \in C_+(\overline{\Omega})$ with $\sigma_6(x) < p_2^*(x)$, $\sigma_7(x) < q_2^*(x)$ in $\overline{\Omega}$ such that

$$|F(x,s,t)| \leq c_9(1+|s|^{\sigma_6(x)}+|t|^{\sigma_7(x)}).$$

From this inequality, Propositions (2.1) and (2.2), we deduce that

$$\int_\Omega \frac{\partial F}{\partial u}(x,u_n,v_n)(u_n-u)\,dx \to 0$$

and

$$\int_\Omega \frac{\partial F}{\partial v}(x,u_n,v_n)(v_n-v)\,dx \to 0.$$

Therefore

$$M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\right)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)\,dx \to 0,$$
$$M\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{q(x)}dx\right)\int_\Omega |\Delta v_n|^{q(x)-2}\Delta v_n(\Delta v_n-\Delta v)\,dx \to 0.$$

Since (u_n, v_n) is bounded in X, passing to a subsequence, if necessary, we may assume that when $n \to +\infty$

$$\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx \to t_0 \geq 0$$

and

$$\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{p(x)}\,dx \to t_1 \geq 0.$$

If $t_0 = 0 = t_1$ then (u_n, v_n) converges strongly to $(0,0)$ and the proof is finished. Otherwise, since the function M is continuous, when $n \to +\infty$ we find

$$M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right) \to M(t_0)$$

and

$$M\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{p(x)}\,dx\right) \to M(t_1).$$

Thus, by (M_1), for sufficiently large n, we have

$$0 < c_9 \leq M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right) \leq c_{10}, \tag{3.1}$$
$$0 < c_{11} \leq M\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{q(x)}\,dx\right) \leq c_{12}. \tag{3.2}$$

From (3.1) and (3.2), we deduce that

$$\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)\,dx \to 0,$$
$$\int_\Omega |\Delta v_n|^{q(x)-2}\Delta v_n(\Delta v_n-\Delta v)\,dx \to 0.$$

Using Lemma 2.4, we have $u_n \to u$ in $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$ and $v_n \to v$ in $W^{2,q(x)}(\Omega)\cap W_0^{1,q(x)}(\Omega)$, which implies that $(u_n, v_n) \to (u, v)$ in X. $\square$

Theorem 3.4 *Assume that M satisfies* (M_1), (M_2) *and F verifies* $(\mathrm{F}_1)-(\mathrm{F}_3)$ *and*

(F_6) $\sigma_1^-, 2\sigma_3^- > \alpha p^+$ *and* $\sigma_2^-, 2\sigma_4^- > \alpha q^+$.

Then problem (1.1) *has a nontrivial weak solution.*

Proof Let us show that J satisfies the conditions of mountain pass theorem (see [2,28]). As pointed out by Brezis and Browder [11], the mountain pass theorem “extends ideas already present in Poincaré and Birkhoff”.

By Lemmas (3.2) and (3.3), J satisfies Palais–Smale condition in X.

For $\|(u, v)\| < 1$, using the Young inequality, the fact $\frac{2\sigma_3(x)}{p_2^*(x)} + \frac{2\sigma_4(x)}{q_2^*(x)} < 1$ in $\overline{\Omega}$, Proposition 2.2 and (2.3), we obtain

$$\int_\Omega |u|^{\sigma_3(x)}|v|^{\sigma_4(x)}\,dx \le \frac{1}{2}\int_\Omega |u|^{2\sigma_3(x)}\,dx + \frac{1}{2}\int_\Omega |v|^{2\sigma_4(x)}\,dx \le c_{13}\left(\|u\|_{p(x)}^{2\sigma_3^-} + \|v\|_{q(x)}^{2\sigma_4^-}\right).$$

Since $\alpha p^+ < p_2^*(x)$ and $\alpha q^+ < q_2^*(x)$, we deduce $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha p^+}(\Omega)$ and $W^{2,q(x)}(\Omega)\cap W_0^{1,q(x)}(\Omega) \hookrightarrow L^{\alpha q^+}(\Omega)$. Then there exist $c_{14}, c_{15} > 0$ such that

$$\begin{aligned} |u|_{\alpha p^+} &\le c_{14}\|u\|_{p(x)} \quad \text{for} \quad u \in W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega),\\ |v|_{\alpha q^+} &\le c_{15}\|v\|_{q(x)} \quad \text{for} \quad v \in W^{2,q(x)}(\Omega)\cap W_0^{1,q(x)}(\Omega), \end{aligned}$$

where $|\cdot|_r$ denote the norm on $L^r(\Omega)$. Let $\epsilon > 0$ be small enough such that $\epsilon c_{14}^{\alpha p^+} \le \frac{m_1}{2\alpha(p^+)^\alpha}$ and $\epsilon c_{15}^{\alpha q^+} \le \frac{m_1}{2\alpha(q^+)^\alpha}$. By the assumptions (F_1) and (F_3), we have

$$|F(x,s,t)| \le \epsilon(|s|^{\alpha p^+} + |t|^{\alpha q^+}) + c(\epsilon)(|s|^{\sigma_1(x)} + |t|^{\sigma_2(x)} + |s|^{\sigma_3(x)}|t|^{\sigma_4(x)})$$

for all $(x, s, t) \in \Omega \times \mathbb{R}^2$. In view of (M_1) and the above inequality, for $\|(u, v)\|$ sufficiently small, noting Proposition 2.2, we have

$$\begin{aligned} J(u,v) \ge{}& \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)^\alpha + \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{q(x)}|\Delta v|^{q(x)}\,dx\right)^\alpha\\ &- \epsilon\int_\Omega |u|^{\alpha p^+}\,dx - \epsilon\int_\Omega |v|^{\alpha q^+}\,dx\\ &- c(\epsilon)\int_\Omega \left(|u|^{\sigma_1(x)} + |v|^{\sigma_2(x)} + |u|^{\sigma_3(x)}|v|^{\sigma_4(x)}\right)\,dx\\ \ge{}& \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{p(x)}^{\alpha p^+} - \epsilon c_{14}^{\alpha p^+}\|u\|_{p(x)}^{\alpha p^+} + \frac{m_1}{\alpha(q^+)^\alpha}\|v\|_{q(x)}^{\alpha q^+} - \epsilon c_{15}^{\alpha q^+}\|v\|_{q(x)}^{\alpha q^+}\\ &- c(\epsilon)\left(\|u\|_{p(x)}^{\sigma_1^-} + \|v\|_{q(x)}^{\sigma_2^-} + c_{13}\|u\|_{p(x)}^{2\sigma_3^-} + c_{13}\|v\|_{q(x)}^{2\sigma_4^-}\right)\\ \ge{}& \frac{m_1}{2\alpha(p^+)^\alpha}\|u\|_{p(x)}^{\alpha p^+} + \frac{m_1}{2\alpha(q^+)^\alpha}\|v\|_{q(x)}^{\alpha q^+} - c(\epsilon)\left(\|u\|_{p(x)}^{\sigma_1^-} + \|v\|_{q(x)}^{\sigma_2^-} + c_{13}\|u\|_{p(x)}^{2\sigma_3^-}\right.\\ &\left. + c_{13}\|v\|_{q(x)}^{2\sigma_4^-}\right). \end{aligned}$$

Since $\sigma_1^-, 2\sigma_3^- > \alpha p^+$ and $\sigma_2^-, 2\sigma_4^- > \alpha q^+$, there exist $r > 0$ and $\delta > 0$ such that $J(u, v) \geq \delta > 0$ for every $(u, v) \in X$ satisfying $\|(u, v)\| = r$.

On the other hand, our assumption (F_2) implies the following assertion: for every $x \in \overline{\Omega}$, $s, t \in \mathbb{R}$, the inequality

$$F(x, s, t) \geq c_{16}(|s|^{\theta_1} + |t|^{\theta_2} - 1) \tag{3.3}$$

holds, see [14]. For $(\widetilde{u}, \widetilde{v}) \in X\backslash\{(0, 0)\}$ and $t > 1$, by (M_1) we have

$$\begin{aligned}
J(t\widetilde{u}, t\widetilde{v}) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}\left(|t\Delta\widetilde{u}|^{p(x)}\,dx\right) + \widehat{M}\left(\int_\Omega \frac{1}{q(x)}|t\Delta\widetilde{v}|^{q(x)}\,dx\right)\right.\\
&\quad - \int_\Omega F(x, t\widetilde{u}, t\widetilde{v})\,dx\\
&\leq \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{p(x)}|t\Delta\widetilde{u}|^{p(x)}\,dx\right)^\beta - c_{16}\int_\Omega |t\widetilde{u}|^{\theta_1}dx\\
&\quad + \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{q(x)}|t\Delta\widetilde{v}|^{q(x)}\,dx\right)^\beta - c_{16}\int_\Omega |t\widetilde{v}|^{\theta_2}\,dx - c_{17}\\
&\leq \frac{m_2}{\beta(p^-)^\beta}t^{\beta p^+}\left(\int_\Omega |\Delta\widetilde{u}|^{p(x)}\,dx\right)^\beta - c_{16}t^{\theta_1}\int_\Omega |\widetilde{u}|^{\theta_1}\,dx\\
&\quad + \frac{m_2}{\beta(q^-)^\beta}t^{\beta q^+}\left(\int_\Omega |\Delta\widetilde{v}|^{q(x)}\,dx\right)^\beta - c_{16}t^{\theta_2}\int_\Omega |\widetilde{v}|^{\theta_2}\,dx - c_{17}\\
&\to -\infty, \quad \text{as} \quad t \to +\infty,
\end{aligned}$$

due to $\theta_1 > \beta p^+$ and $\theta_2 > \beta q^+$. Since $J(0, 0) = 0$, considering Lemmas 3.2 and 3.3, we conclude that J satisfies the conditions of mountain pass lemma. So J admits at least one nontrivial critical point. □

4 Infinitely many low or high energy solutions

In this section we establish two multiplicity results, provided that the potential F has a suitable symmetry. The first property shows the existence of a sequence of *high energy solutions* while the second result deals with the existence of a sequence of solutions with negative energies that converge to zero (that is, *small energy solutions*).

Theorem 4.1 *Assume that M satisfies* (M_1), (M_2) *and F fulfills hypotheses* (F_1), (F_2), (F_4), *and* (F_6). *Then problem* (1.1) *has a sequence of weak solutions $(\pm u_k)$ such that $J(\pm u_k) \to +\infty$ as $k \to \infty$.*

Theorem 4.2 *Assume that M satisfies* (M_1), (M_2) *and F fulfills hypotheses* (F_1), (F_2)$-$(F_5), *and*

(F_7) $\alpha p^+ > \sigma_1^-, 2\sigma_3^-$ *and* $\alpha q^+ > \sigma_2^-, 2\sigma_4^-$.

Then problem (1.1) *has a sequence of weak solutions $(\pm v_k)$ such that $J(\pm v_k) < 0$, $J(\pm v_k) \to 0$ as $k \to \infty$.*

Proof of Theorem 4.1 According to (F_4) and Lemmas 3.2 and 3.3, J is an even functional and satisfies the Palais–Smale condition. We prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A_2) and (A_3) are fulfilled. Thus, the assertion of conclusion can be obtained from fountain theorem.

(A_2) : For any $(u_k, v_k) \in Z_k$, $\|u_k\|_{p(x)} \geq 1$, $\|v_k\|_{q(x)} \geq 1$ and $\|(u_k, v_k)\| = r_k$ (r_k will be specified below), we have

$$\begin{aligned}
J(u_k, v_k) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u_k|^{p(x)}\,dx\right) + \widehat{M}\left(\int_\Omega \frac{1}{q(x)}|\Delta v_k|^{q(x)}\,dx\right) \\
&\quad - \int_\Omega F(x, u_k, v_k)\,dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\left(\int_\Omega |\Delta u_k|^{p(x)}\,dx\right)^\alpha + \frac{m_1}{\alpha(q^+)^\alpha}\left(\int_\Omega |\Delta v_k|^{q(x)}\,dx\right)^\alpha \\
&\quad - c_1\int_\Omega \left(1 + |u_k|^{\sigma_1(x)} + |v_k|^{\sigma_2(x)} + |u_k|^{\sigma_3(x)}|v_k|^{\sigma_4(x)}\right)dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u_k\|_{p(x)}^{\alpha p^-} + \frac{m_1}{\alpha(q^+)^\alpha}\|v_k\|_{q(x)}^{\alpha q^-} - c_1|u_k|_{\sigma_1(x)}^{\sigma_1(\xi_1^k)} - c_1|v_k|_{\sigma_2(x)}^{\sigma_2(\xi_2^k)} \\
&\quad - c_{18}|u_k|_{2\sigma_3(x)}^{2\sigma_3(\eta_1^k)} - c_{18}|v_k|_{2\sigma_4(x)}^{2\sigma_4(\eta_2^k)} - c_1|\Omega|,
\end{aligned}$$

where $\xi_1^k, \xi_2^k, \eta_1^k, \eta_2^k \in \Omega$. Therefore

$$\begin{aligned}
&J(u_k, v_k) \\
&\geq \frac{m_1}{\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\|(u_k, v_k)\|^{\min\{\alpha p^-, \alpha q^-\}} - c_1|u_k|_a^{\sigma_1(\xi_1^k)} - c_1|v_k|_a^{\sigma_2(\xi_2^k)} \\
&\quad - c_{18}|u_k|_a^{2\sigma_3(\eta_1^k)} - c_{18}|v_k|_a^{2\sigma_4(\eta_2^k)} - c_1|\Omega| \\
&\geq \frac{m_1}{\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\|(u_k, v_k)\|^{\min\{\alpha p^-, \alpha q^-\}} - c_1\left(\beta_k\|(u_k, v_k)\|\right)^{\sigma_1(\xi_1^k)} \\
&\quad - c_1\left(\beta_k\|(u_k, v_k)\|\right)^{\sigma_2(\xi_2^k)} - c_{18}\left(\beta_k\|(u_k, v_k)\|\right)^{2\sigma_3(\eta_1^k)} - c_{18}\left(\beta_k\|(u_k, v_k)\|\right)^{2\sigma_4(\eta_2^k)} \\
&\quad - c_1|\Omega| \\
&\geq \frac{m_1}{\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\|(u_k, v_k)\|^{\min\{\alpha p^-, \alpha q^-\}} - c_{19}\beta_k^b\|(u_k, v_k)\|^a - c_1|\Omega|,
\end{aligned}$$

where a, b are defined in (2.6) and (2.7). At this stage, we fix r_k as follows:

$$r_k := \left(\frac{m_1}{2c_{19}\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}\beta_k^b}\right)^{\frac{1}{a-\min\{\alpha p^-, \alpha q^-\}}} \to +\infty \quad \text{as} \quad k \to +\infty.$$

Consequently, if $\|(u_k, v_k)\| = r_k$ then

$$\begin{aligned}
J(u_k, v_k) \geq \frac{m_1}{2\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\|(u_k, v_k)\|^{\min\{\alpha p^-, \alpha q^-\}} - c_1|\Omega| \to +\infty \\
\text{as} \quad k \to +\infty.
\end{aligned}$$

(A_3) : From (F_2), we have $F(x, s, t) \geq c_{16}(|s|^{\theta_1} + |t|^{\theta_2} - 1)$ for every $x \in \Omega$ and $s, t \in \mathbb{R}$. Therefore, for any $(u, v) \in Y_k$ with $\|(u, v)\| = 1$ and $1 < \rho_k = t_k$ with $t_k \to +\infty$, using (M_1) we have

$$\begin{aligned}
&J(t_k u, t_k v)\\
&\quad= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta(t_k u)|^{p(x)}\,dx\right) + \widehat{M}\left(\int_\Omega \frac{1}{q(x)}|\Delta(t_k v)|^{q(x)}\,dx\right)\\
&\qquad - \int_\Omega F(x, t_k u, t_k v)\,dx\\
&\quad\le \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{p(x)}|\Delta(t_k u)|^{p(x)}\,dx\right)^\beta + \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{q(x)}|\Delta(t_k v)|^{q(x)}\,dx\right)^\beta\\
&\qquad - c_{16}\int_\Omega |t_k u|^{\theta_1}\,dx - c_{16}\int_\Omega |t_k v|^{\theta_2}\,dx + c_{20}\\
&\quad\le \frac{m_2}{\beta(p^-)^\beta} t_k^{\beta p^+}\left(\int_\Omega |\Delta u|^{p(x)}\,dx\right)^\beta - c_{16} t_k^{\theta_1}\int_\Omega |u|^{\theta_1}\,dx\\
&\qquad + \frac{m_2}{\beta(q^-)^\beta} t_k^{\beta q^+}\left(\int_\Omega |\Delta v|^{q(x)}\,dx\right)^\beta - c_{16} t_k^{\theta_2}\int_\Omega |v|^{\theta_2}\,dx + c_{20}.
\end{aligned}$$

By $\theta_1 > \beta p^+$, $\theta_2 > \beta q^+$ and dim $Y_k < \infty$, we deduce that $J(u_k, v_k) \to -\infty$ as $\|(t_k u, t_k v)\| \to +\infty$ for $(u, v) \in Y_k$. The conclusion of Theorem 4.1 is reached by the fountain theorem. □

Proof of Theorem 4.2 From (F_4), we know that J satisfies (A_1), the assertion of conclusion can be obtained from the dual fountain theorem.

(B_1): For any $(u_k, v_k) \in Z_k$, $\|u_k\|_{p(x)} < 1$, $\|v_k\|_{q(x)} < 1$ and $\|(u_k, v_v)\| = \rho_k$ (ρ_k will be specified below), we have

$$\begin{aligned}
&J(u_k, v_k)\\
&\quad\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u_k\|_{p(x)}^{\alpha p^+} + \frac{m_1}{\alpha(q^+)^\alpha}\|v_k\|_{q(x)}^{\alpha q^+} - \epsilon c_{14}^{\alpha p^+}\|u_k\|_{p(x)}^{\alpha p^+} - \epsilon c_{15}^{\alpha q^+}\|v_k\|_{q(x)}^{\alpha q^+}\\
&\qquad - c(\epsilon)|u_k|_{\sigma_1(x)}^{\sigma_1(\xi_3^k)} - c(\epsilon)|v_k|_{\sigma_2(x)}^{\sigma_2(\xi_4^k)} - c(\epsilon)|u_k|_{2\sigma_3(x)}^{2\sigma_3(\eta_3^k)} - c(\epsilon)|v_k|_{2\sigma_4(x)}^{\sigma_4(\eta_4^k)},
\end{aligned}$$

where $\xi_3^k, \xi_4^k, \eta_3^k, \eta_4^k \in \Omega$. Therefore

$$\begin{aligned}
&J(u_k, v_k)\\
&\quad\ge \frac{m_1}{\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\|(u_k, v_k)\|^{\max\{\alpha p^+, \alpha q^+\}} - c(\epsilon)|u_k|_b^{\sigma_1(\xi_3^k)} - c(\epsilon)|v_k|_b^{\sigma_2(\xi_4^k)}\\
&\qquad - c(\epsilon)|u_k|_b^{2\sigma_3(\eta_3^k)} - c(\epsilon)|v_k|_b^{\sigma_4(\eta_4^k)}\\
&\quad\ge \frac{m_1}{\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\|(u_k, v_k)\|^{\max\{\alpha p^+, \alpha q^+\}} - c(\epsilon)(\theta_k\|(u_k, v_k)\|)^{\sigma_1(\xi_3^k)}\\
&\qquad - c(\epsilon)(\theta_k\|(u_k, v_k)\|)^{\sigma_2(\xi_4^k)} - c(\epsilon)(\theta_k\|(u_k, v_k)\|)^{2\sigma_3(\eta_3^k)} - c(\epsilon)(\theta_k\|(u_k, v_k)\|)^{2\sigma_4(\eta_4^k)}\\
&\quad\ge \frac{m_1}{\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\|(u_k, v_k)\|^{\max\{\alpha p^+, \alpha q^+\}} - c_{21}\theta_k^b\|(u_k, v_k)\|^b.
\end{aligned}\tag{4.1}$$

Choose $\rho_k = \left(2c_{21}\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}\theta_k^b m_1^{-1}\right)^{\frac{1}{\max\{\alpha p^+, \alpha q^+\} - b}}$. Then

$$\begin{aligned}
J(u_k, v_k) &\ge \frac{m_1}{2\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\rho_k^{\max\{\alpha p^+, \alpha q^+\}}\\
&\quad - \frac{m_1}{2\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}}\rho_k^{\max\{\alpha p^+, \alpha q^+\}} = 0.
\end{aligned}$$

Since $\alpha p^+ > \sigma_1(x), 2\sigma_3(x)$ and $\alpha q^+ > \sigma_2(x), 2\sigma_4(x)$, we have $\rho_k \to 0$ as $k \to +\infty$.
(B_2) : For $(u, v) \in Y_k$, $\|(u, v)\| = 1$ and $0 < t_k < \rho_k < 1$, using (F_1), (F_5) we obtain

$$\begin{aligned}
&J(t_k u, t_k v)\\
&\quad\le \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{p(x)}|t_k\Delta u|^{p(x)}\,dx\right)^\beta + \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{q(x)}|t_k\Delta v|^{q(x)}\,dx\right)^\beta\\
&\qquad - c_2\int_\Omega |t_k u|^{\gamma_1(x)}\,dx - c_2\int_\Omega |t_k v|^{\gamma_2(x)}\,dx\\
&\quad\le \frac{m_2}{\beta(p^-)^\beta} t_k^{\beta p^-}\left(\int_\Omega |\Delta u|^{p(x)}\,dx\right)^\beta - c_2 t_k^{\gamma_1^+}\int_\Omega |u|^{\gamma_1(x)}\,dx\\
&\qquad + \frac{m_2}{\beta(q^-)^\beta} t_k^{\beta q^-}\left(\int_\Omega |\Delta v|^{q(x)}\,dx\right)^\beta - c_2 t_k^{\gamma_2^+}\int_\Omega |u|^{\gamma_2(x)}\,dx.
\end{aligned}$$

Conditions $\gamma_1^+ < \beta p^-$ and $\gamma_2^+ < \beta q^-$ imply that there exists $r_k \in (0, \rho_k)$ such that $J(t_k u, t_k v) < 0$ when $t_k = r_k$. Hence, we deduce that

$$b_k := \max_{(u,v)\in Y_k,\ \|(u,v)\|=r_k} J(u, v) < 0.$$

(B_3) : Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k = \inf\{J(u) : u \in Z_k,\ \|u\| \le \rho_k\} \le b_k = \max\{J(u) : u \in Y_k,\ \|u\| = r_k\} < 0.$$

From (4.1), for $(u', v') \in Z_k$, $\|(u', v')\| = 1$, $0 \le t \le \rho_k$ and $(u, v) = (tu', tv')$, we have

$$J(u, v) = J(tu', tv') \ge \frac{m_1}{\max\{\alpha(p^+)^\alpha, \alpha(q^+)^\alpha\}} t^{\max\{\alpha p^+,\alpha q^+\}} - c_{21}t^b\theta_k^b \ge -c_{21}t^b\theta_k^b,$$

hence, $d_k \to 0$, that is, (B_3) is satisfied.

Finally, we verify the $(PS)_c^*$ condition. Suppose $(u_{n_j}, v_{n_j}) \subset X$ such that $n_j \to +\infty$, $(u_{n_j}, v_{n_j}) \in Y_{n_j}$ and $(J|_{Y_{n_j}})'(u_{n_j}, v_{n_j}) \to 0$. Similar to the process in the proof of Lemma 3.2, we deduce the boundedness of $\|(u_{n_j}, v_{n_j})\|$. Going if necessary to a subsequence, we can assume $(u_{n_j}, v_{n_j}) \rightharpoonup (u, v)$ in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose (u'_{n_j}, v'_{n_j}) such that $(u'_{n_j}, v'_{n_j}) \to (u, v)$. Hence

$$\begin{aligned}
&\lim_{n_j\to+\infty} J'(u_{n_j}, v_{n_j})(u_{n_j} - u, v_{n_j} - v)\\
&\quad = \lim_{n_j\to+\infty} J'(u_{n_j}, v_{n_j})(u_{n_j} - u'_{n_j}, v_{n_j} - v'_{n_j}) + \lim_{n_j\to+\infty} J'(u_{n_j}, v_{n_j})(u'_{n_j} - u, v'_{n_j} - v)\\
&\quad = \lim_{n_j\to+\infty} (J|_{Y_{n_j}})'(u_{n_j}, v_{n_j})(u_{n_j} - u'_{n_j}, v_{n_j} - v'_{n_j})\\
&\quad = 0.
\end{aligned}$$

Similar to the process of verifying the Palais–Smale condition in the proof of Lemma 3.3, we conclude $u_{n_j} \to u$ in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, and $v_{n_j} \to v$ in $W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega)$, which implies that $(u_{n_j}, v_{n_j}) \to (u, v)$ in X. Furthermore, we have $J'(u_{n_j}, v_{n_j}) \to J'(u, v)$.

Let us prove that $J'(u, v) = 0$. Taking $(\omega_k, \omega'_k) \in Y_k$, notice that when $n_j \ge k$ we have

$$\begin{aligned}
J'(u, v)(\omega_k, \omega'_k) &= (J'(u, v) - J'(u_{n_j}, v_{n_j}))(\omega_k, \omega'_k) + J'(u_{n_j}, v_{n_j})(\omega_k, \omega'_k)\\
&= (J'(u, v) - J'(u_{n_j}, v_{n_j}))(\omega_k, \omega'_k) + (J|_{Y_{n_j}})'(u_{n_j}, v_{n_j})(\omega_k, \omega'_k).
\end{aligned}$$

Going to the limit we obtain

$$J'(u, v)(\omega_k, \omega'_k) = 0, \quad \text{for all} \quad (\omega_k, \omega'_k) \in Y_k,$$

so $J'(u, v) = 0$, this show that J satisfies the $(PS)^*_c$ condition for every $c \in \mathbb{R}$. The conclusion of Theorem 4.2 is reached by the dual fountain theorem. $\square$

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ERRATUM

Erratum to: Nonlocal fourth-order Kirchhoff systems with variable growth: low and high energy solutions

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The goal of this erratum is to correct a mistake that appears in the assumption (M_2) in the original article. In the correct version, the hypothesis (M_2) should be removed. In such a case, we restate the following assumption:

(M_1) There exist $m_2 \geq m_1 > 0$ and $\alpha > 1$ such that $m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}$, for all $t \in \mathbb{R}^+$.

We point out that the original assumption (M_1) implies $\alpha_1 = \alpha_2$, so we rename constant α. In conditions (F_2) and (F_5), we replace β by α.

The correct statement of Lemma 3.2 is the following.

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Lemma 3.2 *Let (u_n, v_n) be a Palais–Smale sequence for the Euler–Lagrange functional J. Assume that conditions (M_1), (F_2) are satisfied and*

$$m_1\theta_1(p^-)^{\alpha-1} > \alpha m_2, \quad m_1\theta_2(q^-)^{\alpha-1} > \alpha m_2. \tag{0.1}$$

Then the sequence (u_n, v_n) is bounded.

In the proof of Lemma 3.2, by hypotheses (0.1), (M_1) and (F_2), we can write for n large enough

$$\begin{aligned} c_7 \geq J(u_n, v_n) \geq \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)^\alpha - \int_\Omega \frac{u_n}{\theta_1}\frac{\partial F}{\partial u}(x, u_n, v_n)\,dx \\ + \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{q(x)}\,dx\right)^\alpha - \int_\Omega \frac{v_n}{\theta_2}\frac{\partial F}{\partial v}(x, u_n, v_n)\,dx - c_8, \end{aligned}$$

where c_8 is a positive constant. Therefore

$$\begin{aligned} c_7 &\geq J(u_n, v_n) \\ &\geq \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)^\alpha - \frac{m_2}{\theta_1}\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)^{\alpha-1}\int_\Omega |\Delta u_n|^{p(x)}\,dx \\ &\quad + \frac{1}{\theta_1}D_1 J(u_n, v_n)(u_n) \\ &\quad + \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{q(x)}dx\right)^\alpha - \frac{m_2}{\theta_2}\left(\int_\Omega \frac{1}{q(x)}|\Delta v_n|^{p(x)}\,dx\right)^{\alpha-1}\int_\Omega |\Delta v_n|^{p(x)}\,dx \\ &\quad + \frac{1}{\theta_2}D_2 J(u_n, v_n)(v_n) - c_8 \\ &\geq \left(\frac{m_1}{\alpha} - \frac{m_2}{\theta_1(p^-)^{\alpha-1}}\right)\left(\int_\Omega |\Delta u_n|^{p(x)}\,dx\right)^\alpha + \left(\frac{m_1}{\alpha} - \frac{m_2}{\theta_2(q^-)^{\alpha-1}}\right)\left(\int_\Omega |\Delta v_n|^{q(x)}dx\right)^\alpha \\ &\quad - \frac{1}{\theta_1}\|D_1 J(u_n, v_n)\|_{*,p(x)}\|u_n\| - \frac{1}{\theta_2}\|D_2 J(u_n, v_n)\|_{*,q(x)}\|v_n\| - c_8. \end{aligned}$$

Now, we suppose that the sequence (u_n, v_n) is not bounded. Without loss of generality, we may assume $\|u_n\|_{p(x)} \geq \|v_n\|_{q(x)}$. Therefore, for n large enough so that $\|u_n\|_{p(x)} > 1$, we obtain

$$\begin{aligned} c_7 \geq \left(\frac{m_1}{\alpha} - \frac{m_2}{\theta_1(p^-)^{\alpha-1}}\right)\|u_n\|_{p(x)}^{\alpha p^-} \\ - \left(\frac{1}{\theta_1}\|D_1 J(u_n, v_n)\|_{*,p} + \frac{1}{\theta_2}\|D_2 J(u_n, v_n)\|_{*,q}\right)\|u_n\|_{p(x)}. \end{aligned}$$

But this cannot hold since $\alpha p^- > p^- > 1$. Hence, (u_n, v_n) is bounded.

Theorem 3.1 and Lemma 3.3 remain unchanged. However, Theorems 3.4, 4.1, 4.2 and Lemmas 3.2, 3.3 need to be stated without assumption (M_2). Hypothesis (0.1) should be also added in the statement of Theorems 3.4 and 4.1. The proofs of Theorems 3.4 , 4.1 and 4.2 are similar to the original proofs, but replacing β by α.