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Anisotropic Choquard problems with Stein–Weiss potential: nonlinear patterns and stationary waves

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Abstract. Weighted inequality theory for fractional integrals is a relatively less known branch of calculus that offers remarkable opportunities to simulate interdisciplinary processes. Basic weighted inequalities are often associated to Hardy, Littlewood and Sobolev [6, 11], Caffarelli, Kohn and Nirenberg [4], respectively to Stein and Weiss [12]. A key attempt in the present paper is to prove a Stein–Weiss inequality with lack of symmetry and variable exponents. We quantify the defect of symmetry of the potential by considering the gap between the minimum and the maximum of the variable exponent. We conclude our work with a section dealing with the existence of stationary waves for a class of nonlocal problems with Choquard nonlinearity and anisotropic Stein–Weiss potential.

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1. A weighted anisotropic Stein–Weiss inequality

This paper is concerned with the extension and generalization of a classic inequality first considered by Stein and Weiss [12], which is a two-weight counterpart of the Hardy–Littlewood–Sobolev inequality (see [6,11]). In the present work, this new weighted inequality is established in the nonsymmetric and anisotropic setting described by potentials with variable exponent. We offer a new look to anisotropic differential inequalities by controlling the gap between the minimum and the maximum of the variable exponent. This enables us to quantify the defect of symmetry of the potential. Finally, we apply the new anisotropic Stein–Weiss inequality to the study of a nonlocal Choquard problem with variable growth and lack of compactness.

The features of this paper are the following:

- (i) we establish a *non-symmetric* Stein–Weiss inequality with *variable potential*;
- (ii) in order to quantify the *defect of symmetry* of the potential, we prove more general estimates by considering the gap between the minimum and the maximum of the variable exponent;
- (iii) the analysis developed in this paper is concerned with the *combined effects* of a nonhomogeneous differential operator with unbalanced growth and a Choquard nonlinearity with variable exponent;
- (iv) our analysis combines the *nonlocal* nature of the Choquard nonlinearity with the *local* perturbation in the absorption term.

Let us first recall the following classical Stein–Weiss inequality [12].

Theorem 1. *Let $1 < p, q < +\infty$, $0 < \lambda < N$, $\alpha+\beta \geqslant 0$ and $\alpha+\beta+\lambda \leqslant N$. Then the following properties hold.*

- (i) *If $1/p + 1/q + (\alpha+\beta+\lambda)/N = 2$ and $1 - 1/p - \lambda/N < \alpha/N < 1 - 1/p$, then there exists a constant $C_0 = C_0(p,q,\alpha,\beta,\lambda,N) < \infty$ such that*
$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}\,\mathrm{d}x\mathrm{d}y\right| \leqslant C_0\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}, \tag{1}$$
for all $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, where C_0 is independent of f, g.
- (ii) *For all $f \in L^p(\mathbb{R}^N)$ there exists a constant $C_1 = C_1(p,q,\alpha,\beta,\lambda,N) < \infty$ independent of f, such that*
$$\left\|\int_{\mathbb{R}^N} \frac{f(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}\,\mathrm{d}y\right\|_{L^q(\mathbb{R}^N)} \leqslant C_1\|f\|_{L^p(\mathbb{R}^N)}, \tag{2}$$
where $1 + 1/q = 1/p + (\alpha+\beta+\lambda)/N$ and $\alpha/N < 1/q < (\alpha+\lambda)/N$.

All hypotheses in the previous theorem are sharp. In fact, these conditions are necessary either to ensure integrability or they follow from the scaling of the inequality, which is a special feature of the power-weights case. In the case of radially symmetric functions, the condition $\alpha+\beta \geqslant 0$ can be relaxed and $\alpha+\beta$ is allowed to assume negative values, for instance $\alpha+\beta \geqslant -(N-1)\,|p^{-1} - q^{-1}|$; see Rubin [10].

In what follows, we set $C^+(\mathbb{R}^N) := \left\{r \in C(\mathbb{R}^N) : 1 < r^- := \inf_{\mathbb{R}^N} r \leqslant r^+ := \sup_{\mathbb{R}^N} r < +\infty\right\}$.

The main result in this section establishes the following Stein–Weiss inequality with variable exponents.

Theorem 2. *Let $p, q \in C^+(\mathbb{R}^N)$, $f \in L^{p^+}(\mathbb{R}^N) \cap L^{p^-}(\mathbb{R}^N)$, $g \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$, $\alpha+\beta \geqslant 0$ and $\lambda : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ be a continuous function such that*
$$0 < \lambda^- := \inf_{\mathbb{R}^N\times\mathbb{R}^N} \lambda \leqslant \lambda^+ := \sup_{\mathbb{R}^N\times\mathbb{R}^N} \lambda < N$$
and $0 < \alpha+\beta+\lambda^- \leqslant \alpha+\beta+\lambda^+ \leqslant N$. Then the following properties hold.

(i) *There exists a sharp constant* $C_3 = C_3(p^{\pm}, q^{\pm}, \alpha, \beta, \lambda^{\pm}, N) < \infty$, *independent of* f, g, *such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda(x,y)}|y|^{\beta}}\,\mathrm{d}x\mathrm{d}y\right| \leqslant C_3\|f\|_{L^{p^+}(\mathbb{R}^N)}\|g\|_{L^{q^+}(\mathbb{R}^N)} + C_3\|f\|_{L^{p^-}(\mathbb{R}^N)}\|g\|_{L^{q^-}(\mathbb{R}^N)}, \tag{3}$$

where

$$\max\left\{1-\frac{1}{p^+}-\frac{\lambda^+}{N}, 1-\frac{1}{p^-}-\frac{\lambda^-}{N}\right\} < \frac{\alpha}{N} < 1-\frac{1}{p^-}$$

and

$$\frac{1}{p(x)}+\frac{1}{q(y)}+\frac{\alpha+\beta+\lambda(x,y)}{N} = 2, \quad \forall\ x, y \in \mathbb{R}^N.$$

(ii) *Moreover, there exist constants*

$$C_4 = C_4(p^+, q^+, \alpha, \beta, \lambda^+, N) < \infty \quad \textit{and} \quad C_5 = C_5(p^-, q^-, \alpha, \beta, \lambda^-, N) < \infty,$$

independent of f, *such that*

$$\left\|\int_{\mathbb{R}^N} \frac{f(y)}{|x|^{\alpha}|x-y|^{\lambda^+}|y|^{\beta}}\,\mathrm{d}y\right\|_{L^{q^+}(\mathbb{R}^N)} \leqslant C_4\|f\|_{L^{p^+}(\mathbb{R}^N)} \tag{4}$$

and

$$\left\|\int_{\mathbb{R}^N} \frac{f(y)}{|x|^{\alpha}|x-y|^{\lambda^-}|y|^{\beta}}\,\mathrm{d}y\right\|_{L^{q^-}(\mathbb{R}^N)} \leqslant C_5\|f\|_{L^{p^-}(\mathbb{R}^N)}, \tag{5}$$

where

$$1+\frac{1}{q^+} = \frac{1}{p^+}+\frac{\alpha+\beta+\lambda^+}{N}, \quad 1+\frac{1}{q^-} = \frac{1}{p^-}+\frac{\alpha+\beta+\lambda^-}{N}$$

and

$$\frac{\alpha}{N} < \frac{1}{q^+} \leqslant \frac{1}{q^-} < \frac{\alpha+\lambda^-}{N}.$$

Proof. (i). We first observe that

$$\alpha+\beta+\lambda(x,y) \leqslant N\left(2-\frac{1}{p^+}-\frac{1}{q^+}\right), \quad \forall\ x, y \in \mathbb{R}^N,$$
$$\Longrightarrow \alpha+\beta+\lambda^+ \leqslant N\left(2-\frac{1}{p^+}-\frac{1}{q^+}\right).$$

On the other hand, we have

$$p^+ = \sup_{\mathbb{R}^N} p \text{ and } q^+ = \sup_{\mathbb{R}^N} q,$$
$$\Longrightarrow \text{there exist } \{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^N \text{ such that } p(x_n) \to p^+,\ q(y_n) \to q^+ \text{ as } n \to \infty.$$

It follows that

$$\alpha+\beta+\lambda(x_n,y_n) \to N\left(2-\frac{1}{p^+}-\frac{1}{q^+}\right) \text{ as } n \to \infty.$$

We conclude that

$$\frac{1}{p^+}+\frac{1}{q^+}+\frac{\alpha+\beta+\lambda^+}{N} = 2. \tag{6}$$

In a similar way, we obtain

$$\frac{1}{p^-}+\frac{1}{q^-}+\frac{\alpha+\beta+\lambda^-}{N} = 2. \tag{7}$$

Taking into account the elementary inequality

$$\frac{1}{|x|^{\alpha}|x-y|^{\lambda(x,y)}|y|^{\beta}} \leqslant \frac{1}{|x|^{\alpha}|x-y|^{\lambda^+}|y|^{\beta}} + \frac{1}{|x|^{\alpha}|x-y|^{\lambda^-}|y|^{\beta}}, \quad \forall\ x, y \in \mathbb{R}^N,$$

we deduce that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(x)||g(y)|}{|x|^{\alpha}|x-y|^{\lambda(x,y)}|y|^{\beta}}\mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(x)||g(y)|}{|x|^{\alpha}|x-y|^{\lambda^+}|y|^{\beta}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(x)||g(y)|}{|x|^{\alpha}|x-y|^{\lambda^-}|y|^{\beta}}\mathrm{d}x\mathrm{d}y. \tag{8}$$

Next, combining the inequality

$$\max\left\{1-\frac{1}{p^+}-\frac{\lambda^+}{N}, 1-\frac{1}{p^-}-\frac{\lambda^-}{N}\right\} < \frac{\alpha}{N} < 1-\frac{1}{p^-},$$

with relations (6)–(8) and Theorem 1, we infer that

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda(x,y)}|y|^{\beta}}\,\mathrm{d}x\mathrm{d}y\right| \leqslant C_3\|f\|_{L^{p+}(\mathbb{R}^N)}\|g\|_{L^{q+}(\mathbb{R}^N)} + C_3\|f\|_{L^{p-}(\mathbb{R}^N)}\|g\|_{L^{q-}(\mathbb{R}^N)}$$

for some constant $C_3 = C_3(p^{\pm}, q^{\pm}, \alpha, \beta, \lambda^{\pm}, N) < \infty$, which is independent of f, g.

(ii). It follows from Theorem 1. □

For $p \in C^+(\mathbb{R}^N)$, we consider the following Lebesgue space with variable exponent

$$L^{p(x)}(\mathbb{R}^N) := \left\{u : \mathbb{R}^N \mapsto \mathbb{R} \text{ is a measurable function; } \int_{\mathbb{R}^N}|u(x)|^{p(x)}\mathrm{d}x < \infty\right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\mathbb{R}^N)} = \inf\left\{\eta > 0 : \int_{\mathbb{R}^N}\left|\frac{u(x)}{\eta}\right|^{p(x)}\mathrm{d}x \leqslant 1\right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is the subspace of functions $u \in L^{p(x)}(\mathbb{R}^N)$ whose distributional gradient exists almost everywhere and satisfies $|\nabla u| \in L^{p(x)}(\mathbb{R}^N)$. More precisely, we have

$$W^{1,p(x)}(\mathbb{R}^N) := \left\{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\right\},$$

which is a Banach space under the norm

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \|u\|_{L^{p(x)}(\mathbb{R}^N)} + \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}.$$

In the sequel, we set $h \ll s$ if and only if $h, s \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf\{s(x) - h(x) : x \in \mathbb{R}^N\} > 0$.

Theorem 3. *Assume that $p : \mathbb{R}^N \mapsto \mathbb{R}$ is a Lipschitz function with $1 < p^- \leqslant p^+ < N$ and $t \in C^+(\mathbb{R}^N)$. Then $W^{1,p(x)}(\mathbb{R}^N)$ is continuously embedded into $L^{t(x)}(\mathbb{R}^N)$ for any $p \leqslant t \leqslant p^*$ and $W^{1,p(x)}(\mathbb{R}^N)$ is compactly embedded into $L^{t(x)}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $p \leqslant t \ll p^*$, where $p^*(x) := Np(x)/(N-p(x))$ for all $x \in \mathbb{R}^N$.*

We refer to [5] for a proof of this result and more details.

In what follows, we assume that $p : \mathbb{R}^N \mapsto \mathbb{R}$ is a Lipschitz continuous function with $1 < p^- \leqslant p^+ < N$.

Corollary 4. *Let $q \in C^+(\mathbb{R}^N)$, $\alpha \geqslant 0$, and assume that $\lambda : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function such that $0 < \lambda^- := \inf_{\mathbb{R}^N\times\mathbb{R}^N}\lambda \leqslant \lambda^+ := \sup_{\mathbb{R}^N\times\mathbb{R}^N}\lambda < N$, $0 < 2\alpha + \lambda^- \leqslant 2\alpha + \lambda^+ \leqslant N$,*

$$\max\left\{1-\frac{1}{q^+}-\frac{\lambda^+}{N}, 1-\frac{1}{q^-}-\frac{\lambda^-}{N}\right\} < \frac{\alpha}{N} < 1-\frac{1}{q^-}$$

and

$$\frac{1}{q(x)} + \frac{1}{q(y)} + \frac{2\alpha + \lambda(x,y)}{N} = 2, \ \forall\, x, y \in \mathbb{R}^N.$$

Let $u \in W^{1,p(x)}(\mathbb{R}^N)$ and $r \in \mathscr{G}$, where

$$\mathscr{G} := \left\{r \in C^+(\mathbb{R}^N) : p(x) \leqslant r(x)q^- \leqslant r(x)q^+ \leqslant p^*(x), \ \forall\, x \in \mathbb{R}^N\right\}. \tag{9}$$

Then $|u(\cdot)|^{r(\cdot)} \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$.

Moreover, we have

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{r(x)}|u(y)|^{r(y)}}{|x|^\alpha|x-y|^{\lambda(x,y)}|y|^\alpha}\mathrm{d}x\mathrm{d}y\right| \leqslant C_6\left(\||u(\cdot)|^{r(\cdot)}\|^2_{L^{q+}(\mathbb{R}^N)} + \||u(\cdot)|^{r(\cdot)}\|^2_{L^{q-}(\mathbb{R}^N)}\right) \tag{10}$$

for some $C_6 = C_6(q^\pm,\alpha,\lambda^\pm,N) < \infty$, *where* C_6 *does not depend on* $u \in W^{1,p(x)}(\mathbb{R}^N)$.

Proof. Taking into account that $r \in \mathscr{G}$ and using Theorem 3, we obtain that $|u(\cdot)|^{r(\cdot)} \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$. Then we can use Theorem 2 to obtain the desired result. □

2. Choquard problems with weighted anisotropic Stein–Weiss potential

In this section, we focus on the existence of solutions for the following nonlocal problem with anisotropic Stein–Weiss convolution term:

$$\begin{cases} -\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u = \dfrac{1}{|x|^\alpha}\left(\displaystyle\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|x-y|^{\lambda(x,y)}|y|^\alpha}\mathrm{d}y\right)f(x,u(x)) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{P_λ}$$

where $\alpha \geqslant 0$, $0 < \lambda^- \leqslant \lambda^+ < N$, $0 < 2\alpha+\lambda^- \leqslant 2\alpha+\lambda^+ \leqslant N$, $V \in C(\mathbb{R}^N,\mathbb{R})$, $\lambda \in C(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R})$, $f \in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, $F(x,t) := \int_0^t f(x,\tau)\mathrm{d}\tau$, and $\Delta_{p(x)}u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$.

We assume that the positive potential V is bounded from below and is coercive, that is,

$$V \in C(\mathbb{R}^N,\mathbb{R}), \quad \inf_{\mathbb{R}^N} V := V_0 > 0 \quad \text{and} \quad V(x) \to +\infty \text{ as } |x| \to +\infty. \tag{V_0}$$

Next, we introduce the weighted Sobolev space $W^{1,p(x)}_V(\mathbb{R}^N)$ defined by

$$W^{1,p(x)}_V(\mathbb{R}^N) := \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

where

$$\|u\| = \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)} + \|V(\cdot)^{1/p(\cdot)}u\|_{L^{p(x)}(\mathbb{R}^N)} := \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)} + \|u\|_{L^{p(x)}(\mathbb{R}^N),V}.$$

From Lemma 4.2 of [1], we know that $W^{1,p(x)}_V$ is compactly embedded into $L^{t(x)}(\mathbb{R}^N)$ for all $t \in C^+(\mathbb{R}^N)$ and $p \ll t \ll p^*$.

In addition, the following assumptions are required on the reaction $f : \mathbb{R}^N\times\mathbb{R} \mapsto \mathbb{R}$:

- (f_1) $|f(x,t)| \leqslant C_7\left(|t|^{r(x)-1} + |t|^{s(x)-1}\right)$ for all $(x,t) \in \mathbb{R}^N\times\mathbb{R}$, where $C_7 > 0$ and r, $s \in \mathscr{G}$ ($\mathscr{G}$ is defined by (9)) with $p \ll rq^- \leqslant rq^+ \ll q^*$, $p \ll sq^- \leqslant sq^+ \ll q^*$ and r^-, $s^- > p^+/2$;
- (f_2) there exists $\vartheta > p^+$ such that $0 < \vartheta F(x,t) \leqslant 2f(x,t)t$ for all $t > 0$;
- (f_3) there exist constants $\ell > 0$ and $C_\ell > 0$ such that $F(x,\ell) \geqslant C_\ell$ for all $x \in \mathbb{R}^N$.

The main result of this section establishes the following existence property for problem (P_λ).

Theorem 5. *Assume that* (V_0) *and* (f_1)–(f_3) *are fulfilled, and that* $p : \mathbb{R}^N \mapsto \mathbb{R}$ *is a Lipschitz function with* $1 < p^- \leqslant p^+ < N$. *Then problem* (P_λ) *has at least one nontrivial solution.*

The proof of Theorem 5 relies on a variational method. For this purpose, we introduce the energy (Euler) functional $\mathscr{E} : W^{1,p(x)}_V(\mathbb{R}^N) \mapsto \mathbb{R}$ defined by

$$\mathscr{E}(u) := \int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(x,u(x))F(y,u(y))}{|x|^\alpha|x-y|^{\lambda(x,y)}|y|^\alpha}\mathrm{d}x\mathrm{d}y$$

for all $u \in W^{1,p(x)}_V(\mathbb{R}^N)$.

Arguing as in the proof of Lemma 3.2 in [2], Theorem 2 combined with hypotheses (V_0), (f_1) and the definition of F implies that $\mathscr{E}$ is well-defined and $\mathscr{E} \in C^1(W_V^{1,p(x)}(\mathbb{R}^N), \mathbb{R})$, with

$$\begin{aligned}\langle \mathscr{E}'(u), \varphi\rangle = &\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + V(x)|u|^{p(x)-2} u\varphi\right) \mathrm{d}x \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u(y)) f(x,u(x))\varphi(x)}{|x|^\alpha |x-y|^{\lambda(x,y)}|y|^\alpha} \mathrm{d}x\mathrm{d}y \text{ for all } \varphi \in W_V^{1,p(x)}(\mathbb{R}^N).\end{aligned}$$

As in many situations, the energy $\mathscr{E}$ is unbounded from above and below, so that it has no maximum or minimum. This forces us to look for saddle points, which are obtained by minimax arguments that go back to the mountain pass theorem of Ambrosetti and Rabinowitz [3] (see also Pucci and Rădulescu [9] for a survey). The original proof of this classical result relies on some deep deformation techniques developed by Palais and Smale [8], who put the main ideas of the Morse theory into the framework of differential topology on infinite dimensional manifolds.

We first show that the energy (Euler) functional $\mathscr{E}$ has a mountain pass geometry. The next property establishes the existence of a "mountain" near the origin, while the second property implies the existence of a "valley" on the other side of this mountain.

Lemma 6. *The following properties are fulfilled for the functional* $\mathscr{E}$:

(a) *there exist* $\delta, \varrho > 0$ *such that* $\mathscr{E}(u) \geqslant \delta$ *for* $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ *with* $\|u\| = \varrho$;

(b) *there exists* $e \in W_V^{1,p(x)}(\mathbb{R}^N)$ *such that* $\|e\| > \varrho$ *and* $\mathscr{E}(e) < 0$.

Proof. (a). We first observe that, using condition (f_1) and the definition of F, we have, for all $u \in W_V^{1,p(x)}(\mathbb{R}^N)$:

$$\begin{aligned}\|F(\cdot,u)\|^2_{L^{q^+}(\mathbb{R}^N)} &\leqslant C_8 \left(\int_{\mathbb{R}^N}\left(|u|^{q^+ r(x)} + |u|^{q^+ s(x)}\right)\mathrm{d}x\right)^{2/q^+} \\ &\leqslant C_9\left(\int_{\mathbb{R}^N}|u|^{q^+ r(x)}\mathrm{d}x\right)^{2/q^+} + C_9\left(\int_{\mathbb{R}^N}|u|^{q^+ s(x)}\mathrm{d}x\right)^{2/q^+},\end{aligned}$$

for some constants $C_8, C_9 > 0$.

It follows that

$$\begin{aligned}\|F(\cdot,u)\|^2_{L^{q^+}(\mathbb{R}^N)} &\leqslant C_9 \max\left\{\|u\|^{2r^+}_{L^{q^+ r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^+ r(x)}(\mathbb{R}^N)}\right\} \\ &\quad + C_9 \max\left\{\|u\|^{2s^+}_{L^{q^+ s(x)}(\mathbb{R}^N)}, \|u\|^{2s^-}_{L^{q^+ s(x)}(\mathbb{R}^N)}\right\} \\ &\leqslant C_{10} \max\left\{\|u\|^{2r^+}_{W^{1,p(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{W^{1,p(x)}(\mathbb{R}^N)}\right\} \\ &\quad + C_{10}\max\left\{\|u\|^{2s^+}_{W^{1,p(x)}(\mathbb{R}^N)}, \|u\|^{2s^-}_{W^{1,p(x)}(\mathbb{R}^N)}\right\} \quad \text{(by Theorem 3)}, \qquad (11)\end{aligned}$$

where C_{10} is a finite constant.

In a similar way, we have

$$\begin{aligned}\|F(\cdot,u)\|^2_{L^{q^-}(\mathbb{R}^N)} &\leqslant C_{11} \max\left\{\|u\|^{2r^+}_{W^{1,p(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{W^{1,p(x)}(\mathbb{R}^N)}\right\} \\ &\quad + C_{11}\max\left\{\|u\|^{2s^+}_{W^{1,p(x)}(\mathbb{R}^N)}, \|u\|^{2s^-}_{W^{1,p(x)}(\mathbb{R}^N)}\right\}, \qquad (12)\end{aligned}$$

for some finite constant C_{11}.

Using Theorem 2 and relations (9), (11) and (12), we deduce that, for all $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \leqslant 1$, we have

$$\begin{aligned}\mathscr{E}(u) \geqslant\ & C_{12}\left(\|\nabla u\|^{p^+}_{L^{p(x)}(\mathbb{R}^N)} + \|u\|^{p^+}_{L^{p(x)}(\mathbb{R}^N)}\right) - C_{13}\left(\|\nabla u\|^{2r^+}_{L^{p(x)}(\mathbb{R}^N)} + \|u\|^{2r^+}_{L^{p(x)}(\mathbb{R}^N)}\right) \\ &- C_{13}\left(\|\nabla u\|^{2r^-}_{L^{p(x)}(\mathbb{R}^N)} + \|u\|^{2r^-}_{L^{p(x)}(\mathbb{R}^N)}\right) - C_{13}\left(\|\nabla u\|^{2s^+}_{L^{p(x)}(\mathbb{R}^N)} + \|u\|^{2s^+}_{L^{p(x)}(\mathbb{R}^N)}\right) \\ &- C_{13}\left(\|\nabla u\|^{2s^-}_{L^{p(x)}(\mathbb{R}^N)} + \|u\|^{2s^-}_{L^{p(x)}(\mathbb{R}^N)}\right),\end{aligned}$$

for some finite positive constants C_{12}, C_{13} that are independent of u.

Using $2r^-, 2s^- > p^+$ and taking $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \leqslant C_{14}\|u\| := C_{14}\varrho$ (where $C_{14} > 0$ is independent of u) with $\varrho \leqslant 1/C_{14}$ sufficiently small, we see that (a) holds true.

(b). Using hypotheses (f_2)–(f_3), there exists a constant $C_{15} = C_{15}(\ell, \vartheta) > 0$ such that $F(x,t) \geqslant C_{15} t^{\frac{\vartheta}{2}}$ for all $(x,t) \in \mathbb{R}^N \times [\ell, +\infty)$. Then we can choose a function $v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ and use the last inequality to infer that $\mathscr{E}(tv) < 0$ for $t > 0$ sufficiently large. The proof of the lemma is complete. □

Using Lemma 6, we can define the following minimax level:

$$0 < c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathscr{E}(\gamma(t)), \tag{13}$$

where

$$\Gamma := \left\{\gamma \in C([0,1], W_V^{1,p(x)}(\mathbb{R}^N)) : \gamma(0) = 0, \mathscr{E}(\gamma(1)) < 0\right\}.$$

Lemma 7. *The Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_V^{1,p}(\mathbb{R}^N)$ of the functional $\mathscr{E}$ at the level c is bounded.*

Proof. Since $\{u_n\}_{n\in\mathbb{N}} \subset W_V^{1,p}(\mathbb{R}^N)$ is a Palais–Smale sequence at the level c, we have

$$\mathscr{E}(u_n) = c + o_n(1) \text{ and } \mathscr{E}'(u_n) = o_n(1) \text{ as } n \to \infty.$$

Using (f_2), for large enough $n \in \mathbb{N}$ we see that

$$\begin{aligned}
c(1 + \|u_n\|) &\geqslant \mathscr{E}(u_n) - \frac{1}{\vartheta}\langle \mathscr{E}'(u_n), u_n\rangle \\
&\geqslant \int_{\mathbb{R}^N} \left(\frac{1}{p^+} - \frac{1}{\vartheta}\right)\left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u_n(y))(2f(x,u_n(x))u_n(x) - \vartheta F(x,u_n(x)))}{2\vartheta|x|^\alpha|x-y|^{\lambda(x,y)}|y|^\alpha}\mathrm{d}x\mathrm{d}y \\
&\geqslant \left(\frac{1}{p^+} - \frac{1}{\vartheta}\right)\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right)\mathrm{d}x,
\end{aligned}$$

which implies that the sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_V^{1,p}(\mathbb{R}^N)$ is bounded. □

2.1. *Proof of Theorem 5 completed*

Let $\{u_n\}_{n\in\mathbb{N}} \subset W_V^{1,p}(\mathbb{R}^N)$ be a Palais–Smale sequence at the level c, that is, $\mathscr{E}(u_n) = c + o_n(1)$ and $\mathscr{E}'(u_n) = o_n(1)$ as $n \to \infty$. From Lemma 7 it follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$. So, we may suppose that $u_n \xrightarrow{w} u$ in $W_V^{1,p}(\mathbb{R}^N)$, $u_n \to u$ in $L^{t(x)}(\mathbb{R}^N)$ for all $t \in C^+(\mathbb{R}^N)$ and $p \ll t \ll p^*$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Since $\mathscr{E}(u_n) \to c > 0$ and $\mathscr{E}'(u_n) \to 0$ as $n \to \infty$, then we can deduce that $u \neq 0$. Assume by contradiction that $u = 0$. Using the fact that $u_n \to 0$ in $L^{t(x)}(\mathbb{R}^N)$ for all $t \in C^+(\mathbb{R}^N)$ and $p \ll t \ll p^*$, together with (f_1) and the definition of F, we deduce that

$$\begin{aligned}
&F(\cdot, u_n) \to 0 \ \text{ in } \ L^{q+}(\mathbb{R}^N) \ \text{ as } \ n \to \infty, \\
&F(\cdot, u_n) \to 0 \ \text{ in } \ L^{q-}(\mathbb{R}^N) \ \text{ as } \ n \to \infty, \\
&f(\cdot, u_n)u_n \to 0 \ \text{ in } \ L^{q+}(\mathbb{R}^N) \ \text{ as } \ n \to \infty, \\
&f(\cdot, u_n)u_n \to 0 \ \text{ in } \ L^{q-}(\mathbb{R}^N) \ \text{ as } \ n \to \infty.
\end{aligned}$$

The above limits and Theorem 2 yield

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u_n(y))f(x,u_n(x))u_n(x)}{|x|^\alpha|x-y|^{\lambda(x,y)}|y|^\alpha}\mathrm{d}x\mathrm{d}y \to 0 \ \text{ as } \ n \to \infty.$$

Now, we can use the above limit and the fact that $\langle \mathscr{E}'(u_n), u_n\rangle = o_n(1)$ as $n\to\infty$ to conclude that

$$\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right)\mathrm{d}x \to 0 \text{ as } n\to\infty.$$

So, it follows that $u_n \to 0$ in $W_V^{1,p}(\mathbb{R}^N)$ as $n\to\infty$. Therefore, $\mathscr{E}(u_n)\to 0$ as $n\to\infty$. Since $\mathscr{E}(u_n)\to c>0$ as $n\to\infty$, we reach a contradiction.

Next, we prove that the weak limit u is a critical point of the functional $\mathscr{E}$.

We first show that the following limit holds true up to a subsequence:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left(F(y,u_n(y)) - F(y,u(y))\right)f(x,u(x))\varphi(x)}{|x|^{\alpha}|x-y|^{\lambda(x,y)}|y|^{\alpha}}\mathrm{d}x\mathrm{d}y \to 0 \tag{14}$$

as $n\to\infty$, for all $\varphi\in C_0^{\infty}(\mathbb{R}^N)$. In fact, since $L^{q+}(\mathbb{R}^N)$ and $L^{q-}(\mathbb{R}^N)$ are uniformly convex, then the Banach space $\left(L^{q^+}(\mathbb{R}^N)\cap L^{q^-}(\mathbb{R}^N), \max\left\{\|\cdot\|_{L^{q+}(\mathbb{R}^N)}, \|\cdot\|_{L^{q-}(\mathbb{R}^N)}\right\}\right)$ is also uniformly convex (and therefore, reflexive). In addition, from the definition of F and the boundedness of $\{u_n\}_{n\in\mathbb{N}}\subset W_V^{1,p(x)}(\mathbb{R}^N)$ it follows that the sequence $\{F(\cdot,u_n)\}_{n\in\mathbb{N}}\subset L^{q^+}(\mathbb{R}^N)\cap L^{q^-}(\mathbb{R}^N)$ is bounded. Then we may assume that there exists a function $w\in L^{q^+}(\mathbb{R}^N)\cap L^{q^-}(\mathbb{R}^N)$ such that $F(\cdot,u_n)\xrightarrow{w} w$ (up to a subsequence) in $L^{q^+}(\mathbb{R}^N)\cap L^{q^-}(\mathbb{R}^N)$ as $n\to\infty$. Using Proposition 5.4.7 of Willem [13], we see that $F(\cdot,u_n)\xrightarrow{w} F(\cdot,u)$ in $L^{q+}(\mathbb{R}^N)$ as $n\to\infty$. Then we can conclude that $w(\cdot) = F(\cdot,u(\cdot))$ a.e. in $\mathbb{R}^N$.

For some fixed $\varphi\in C_0^{\infty}(\mathbb{R}^N)$, we define the following continuous linear functional

$$I_{\varphi}(v) := \int_{\mathbb{R}^N} v\varphi\,\mathrm{d}x \text{ for all } v\in L^{q^+}(\mathbb{R}^N)\cap L^{q^-}(\mathbb{R}^N).$$

Consequently, we have

$$I_{\varphi}(F(\cdot,u_n)) \to \int_{\mathbb{R}^N} w\varphi\,\mathrm{d}x \text{ as } n\to\infty.$$

By Theorem 2, we can define the following continuous linear functional:

$$\mathscr{L}(v) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v(y)f(x,u(x))\varphi(x)}{|x|^{\alpha}|x-y|^{\lambda(x,y)}|y|^{\alpha}}\mathrm{d}x\mathrm{d}y \text{ for all } v\in L^{q^+}(\mathbb{R}^N)\cap L^{q^-}(\mathbb{R}^N).$$

Finally, since $\mathscr{L}$ is linear, continuous and $F(\cdot,u_n)\xrightarrow{w} F(\cdot,u)$ in $L^{q^+}(\mathbb{R}^N)\cap L^{q^-}(\mathbb{R}^N)$ as $n\to\infty$, we obtain (14).

Combining the fact that f has a subcritical growth with Theorem 2, the boundedness of $\{u_n\}_{n\in\mathbb{N}}\subset W_V^{1,p(x)}(\mathbb{R}^N)$, and the Lebesgue Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,u_n(y))(f(x,u_n(x)) - f(x,u(x)))\varphi(x)}{|x|^{\alpha}|x-y|^{\lambda(x,y)}|y|^{\alpha}}dxdy \to 0 \tag{15}$$

as $n\to\infty$, for all $\varphi\in C_0^{\infty}(\mathbb{R}^N)$.

Let $R>0$ and $\psi\in C_0^{\infty}(\mathbb{R}^N)$ such that $\psi\in[0,1]$ and $\psi(x)=1$ for $x\in B_R(0)$. Using $\langle\mathscr{E}'(u_n), u_n\psi\rangle\to 0$ and $\langle\mathscr{E}'(u_n), u\psi\rangle\to 0$ as $n\to\infty$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\psi\,\mathrm{d}x + o_n(1)\\
&\quad= \langle\mathscr{E}'(u_n), u_n\psi\rangle - \langle\mathscr{E}'(u_n), u\psi\rangle - \int_{\mathbb{R}^N}(u_n - u)|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi\,\mathrm{d}x\\
&\qquad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,u_n(y))f(x,u_n(x))(u_n(x)-u(x))\psi(x)}{|x|^{\alpha}|x-y|^{\lambda(x,y)}|y|^{\alpha}}\mathrm{d}x\mathrm{d}y\\
&\qquad - \int_{\mathbb{R}^N}V(x)|u_n|^{p(x)-2}u_n(u_n-u)\psi\,\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u(\nabla u_n - \nabla u)\psi\,\mathrm{d}x.
\end{aligned}$$

It is easy to check that

$$\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\psi\,\mathrm{d}x \to 0 \text{ as } n\to\infty. \tag{16}$$

From Proposition 3.3 of [7], we know that the following estimate

$$(|\xi|^{p(x)-2}\xi-|\zeta|^{p(x)-2}\zeta,\xi-\zeta)_{\mathbb{R}^N}\geqslant\begin{cases}(|\xi|+|\zeta|)^{p(x)-2}|\xi-\zeta|^2 & \text{if } 1<p\,(x)<2,\\ 4^{1-p^+}|\xi-\zeta|^{p(x)} & \text{if } p\,(x)\geqslant 2\end{cases}\tag{17}$$

holds true for all $\xi,\zeta\in\mathbb{R}^N$.

Combining relations (16) and (17), we deduce that for all $R>0$ we have $\nabla u_n\to\nabla u$ in $L^{p(x)}(B_R(0))$ as $n\to\infty$. Since $R>0$ is arbitrary, passing to a subsequence we can infer that $\nabla u_n(x)\to\nabla u(x)$ a.e. in $\mathbb{R}^N$ as $n\to\infty$.

In addition, since $\left\{|\nabla u_n|^{p(x)-2}\nabla u_n\right\}_{n\in\mathbb{N}}\subset L^{p(x)/(p(x)-1)}(\mathbb{R}^N)$ is bounded, arguing as the proof of Proposition 5.4.7 of Willem [13], we have

$$|\nabla u_n|^{p(x)-2}\nabla u_n\xrightarrow{w}|\nabla u|^{p(x)-2}\nabla u\ \text{ in }\ (L^{p(x)/(p(x)-1)}(\mathbb{R}^N))_{\mathbb{R}^N}\ \text{ as }\ n\to\infty.\tag{18}$$

Additionally, applying the Lebesgue Dominated Convergence Theorem, we also get

$$\int_{\mathbb{R}^N}V(x)|u_n|^{p(x)-2}u_n\varphi\,\mathrm{d}x\to\int_{\mathbb{R}^N}V(x)|u|^{p(x)-2}u\varphi\,\mathrm{d}x\ \text{ as }\ n\to\infty.\tag{19}$$

Combining relations (14), (15), (18), (19) and the fact that

$$\langle\mathscr{E}'(u_n),\varphi\rangle\to 0\ \text{ for all }\ \varphi\in C_0^\infty(\mathbb{R}^N)\ \text{ as }\ n\to\infty,$$

we deduce that

$$\langle\mathscr{E}'(u),\varphi\rangle=0\ \text{ for all }\ \varphi\in C_0^\infty(\mathbb{R}^N).$$

Since $\overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|}=W_V^{1,p(x)}(\mathbb{R}^N)$, we conclude that u is a critical point of $\mathscr{E}$, hence a nontrivial solution of problem (P_λ). □

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