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Partial Differential Equations

Nonhomogeneous boundary value problems in Orlicz–Sobolev spaces

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Abstract

We study the nonlinear Dirichlet problem $-\operatorname{div}(\log(1+|\nabla u|^q)|\nabla u|^{p-2}\nabla u) = -\lambda |u|^{p-2}u + |u|^{r-2}u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, while p, q and r are real numbers satisfying $p, q > 1$, $p + q < \min\{N, r\}$, $r < (Np - N + p)/(N - p)$. The main result of this Note establishes that for any $\lambda > 0$ this boundary value problem has infinitely many solutions in the Orlicz–Sobolev space $W_0^1 L_\Phi(\Omega)$, where $\Phi(t) = \int_0^t \log(1 + |s|^q) \cdot |s|^{p-2} s \,\mathrm{d}s$. ***To cite this article: M. Mihăilescu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Problèmes aux limites non homogènes dans les espaces d'Orlicz–Sobolev. On étudie le problème de Dirichlet non linéaire $-\operatorname{div}(\log(1+|\nabla u|^q)|\nabla u|^{p-2}\nabla u) = -\lambda |u|^{p-2}u + |u|^{r-2}u$ dans Ω, $u = 0$ sur $\partial\Omega$, où Ω est un domaine borné, régulier et p, q, r sont des nombres réels tels que $p, q > 1$, $p + q < \min\{N, r\}$, $r < (Np - N + p)/(N - p)$. Le résultat principal de cette Note montre que pour tout $\lambda > 0$ ce problème admet une infinité de solutions dans l'espace d'Orlicz–Sobolev $W_0^1 L_\Phi(\Omega)$, où $\Phi(t) = \int_0^t \log(1 + |s|^q) \cdot |s|^{p-2} s \,\mathrm{d}s$. ***Pour citer cet article : M. Mihăilescu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

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Soit $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) un domaine borné et régulier. On considère le problème non linéaire

$$\begin{cases} -\operatorname{div}(\log(1+|\nabla u|^q)|\nabla u|^{p-2}\nabla u) = -\lambda |u|^{p-2}u + |u|^{r-2}u, & \text{pour } x \in \Omega, \\ u = 0, & \text{pour } x \in \partial\Omega. \end{cases} \tag{1}$$

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Soit $\Phi(t) := \int_0^t \log(1+|s|^q)\cdot|s|^{p-2}s\,\mathrm{d}s$. On définit la classe d'Orlicz par $K_\Phi(\Omega) := \{u\colon\ \Omega\to\mathbb{R};\ u$ mésurable et $\int_\Omega \Phi(|u(x)|)\,\mathrm{d}x < \infty\}$ et soit $L_\Phi(\Omega)$ l'espace linéaire engendré par $K_\Phi(\Omega)$. L'espace d'Orlicz $L_\Phi(\Omega)$ muni avec l'une des normes équivalentes

$$\|u\|_\Phi := \inf\left\{k>0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\mathrm{d}x \leqslant 1\right\} \quad \text{(norme de Luxemburg)}$$

ou

$$\|u\|_{(\Phi)} := \sup\left\{\left|\int_\Omega uv\,\mathrm{d}x\right|;\ v\in K_{\bar\Phi}(\Omega),\ \int_\Omega \bar\Phi\big(|v|\big)\,\mathrm{d}x\leqslant 1\right\} \quad \text{(norme d'Orlicz)}$$

devient un espace de Banach, où $\bar\Phi$ est la conjugée de Young de Φ.

Soit $W^1L_\Phi(\Omega)$ l'espace d'Orlicz–Sobolev défini par

$$W^1L_\Phi(\Omega) := \left\{u\in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i}\in L_\Phi(\Omega),\ i=1,\dots,N\right\}.$$

Cet espace, muni avec la norme $\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi$ est un espace de Banach. On désigne par $W_0^1L_\Phi(\Omega)$ l'adhérence de $C_0^\infty(\Omega)$ dans $W^1L_\Phi(\Omega)$ (voir [1,6,12] pour plus de détails).

Le résultat principal de cette Note est le suivant :

Théorème 0.1. *On suppose que p, q et r sont des nombres réels tels que $p,\ q>1,\ p+q<\min\{N,r\}$, $r<(Np-N+p)/(N-p)$. Alors le problème* (1) *admet une infinité de solutions dans l'espace $W_0^1L_\Phi(\Omega)$.*

Dans la démonstration du Théorème 0.1 on utilise une variante symmetrique du Lemme du Col (voir Rabinowitz [11, Theorem 9.12]). Des résultats d'existence dans l'étude des problèmes elliptiques dans les espaces d'Orlicz–Sobolev ont été établis dans les travaux récents [4,5,7–9].

1. The main result

Let $\Omega\subset\mathbb{R}^N$ ($N\geqslant 3$) be a bounded domain with smooth boundary. We assume that p, q and r are real numbers satisfying

$$p,q>1,\quad p+q<\min\{N,r\},\quad r<(Np-N+p)/(N-p). \tag{2}$$

Consider the nonlinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(\log(1+|\nabla u|^q)|\nabla u|^{p-2}\nabla u) = -\lambda|u|^{p-2}u+|u|^{r-2}u, & \text{for } x\in\Omega,\\ u=0, & \text{for } x\in\partial\Omega. \end{cases} \tag{3}$$

Problems of this type appear in image restoration or in the study of electrorheological (smart) fluids (see Ružička [13]).

Set $\Phi(t) := \int_0^t \log(1+|s|^q)\cdot|s|^{p-2}s\,\mathrm{d}s$. The Orlicz space $L_\Phi(\Omega)$ associated to Φ is the Banach space of those measurable functions $u:\Omega\to\mathbb{R}$ for which the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k>0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\mathrm{d}x\leqslant 1\right\}$$

is finite. The Orlicz–Sobolev space $W_0^1L_\Phi(\Omega)$ built upon $L_\Phi(\Omega)$ is the space of those weakly differentiable functions in Ω satisfying $\|\,|\nabla u|\,\|_\Phi := \|u\|_{W_0^1L_\Phi(\Omega)} < \infty$ and decaying to 0 on $\partial\Omega$, in the sense that the continuation of u outside Ω is a weakly differentiable function in $\mathbb{R}^N$ and $|\{x\in\Omega;\ |u(x)|>t\}|<\infty$ for every $t>0$ (see e.g. Adams [1], Cianchi [3], Donaldson and Trudinger [6], and Rao and Ren [12] for more details on the theory of Orlicz–Sobolev spaces).

In this Note, we establish the following multiplicity result:

Theorem 1.1. *Assume that condition* (2) *is fulfilled. Then problem* (3) *has infinitely many solutions in the Orlicz–Sobolev space $W_0^1L_\Phi(\Omega)$, for any $\lambda>0$.*

We point out that problem (3) has been studied by Ambrosetti and Rabinowitz in their pioneering paper [2] in the particular case

$$\begin{cases} -\Delta u = -\lambda u + |u|^{r-2}u, & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{4}$$

where $2 < r < 2^* = 2N/(N-2)$. They have showed that problem (4) has at least a solution in the classical Sobolev space $H_0^1(\Omega)$, for any $\lambda > 0$. The main feature of this paper is that our main result establishes the existence of infinitely many solutions for a related boundary value problem which involves a nonhomogeneous differential operator in the class of Orlicz–Sobolev spaces.

Define the Orlicz–Sobolev conjugate $\Phi_\star$ of Φ by $\Phi_\star^{-1}(t) := \int_0^t \Phi^{-1}(s)s^{-(N+1)/N}\,\mathrm{d}s$. A straightforward computation shows that since condition (2) is satisfied then

$$\lim_{t\to 0}\int_t^1 \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,\mathrm{d}s < \infty; \quad \lim_{t\to\infty}\int_1^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,\mathrm{d}s = \infty; \quad \lim_{t\to\infty}\frac{|t|^{\gamma+1}}{\Phi_\star(kt)} = 0, \quad \forall k > 0,\ \forall \gamma \in \left[1, \frac{Np-N+p}{N-p}\right).$$

Remark 1. The above relations enable us to apply Theorem 2.2 in [7] (see also [1] or [6]) in order to obtain that $W_0^1 L_\Phi(\Omega)$ is compactly embedded in $L^{\gamma+1}(\Omega)$ provided that $1 \leqslant \gamma < (Np-N+p)/(N-p)$.

An important role in our analysis will be played by

$$p^0 := \sup_{t>0}\frac{t\phi(t)}{\Phi(t)}.$$

Remark 2. By Example 2 on p. 243 in [5] it follows that

$$p^0 = p + q.$$

2. Proof of Theorem 1.1

Let E denote the Orlicz–Sobolev space $W_0^1 L_\Phi(\Omega)$. Fix arbitrarily $\lambda > 0$. The energy functional associated to problem (3) is $J_\lambda : E \to \mathbb{R}$ defined by

$$J_\lambda(u) := \int_\Omega \Phi\big(|\nabla u|\big)\,\mathrm{d}x + \frac{\lambda}{p}\int_\Omega |u|^p\,\mathrm{d}x - \frac{1}{r}\int_\Omega |u|^r\,\mathrm{d}x.$$

Then $J_\lambda \in C^1(E,\mathbb{R})$ and

$$\big\langle J_\lambda'(u), v\big\rangle = \int_\Omega \log\big(1+|\nabla u|^q\big)|\nabla u|^{p-2}\nabla u \nabla v\,\mathrm{d}x + \lambda\int_\Omega |u|^{p-2}uv\,\mathrm{d}x - \int_\Omega |u|^{r-2}uv\,\mathrm{d}x$$

for all $u, v \in E$. Thus, the weak solutions of (3) coincide with the critical points of J_λ.

Lemma 2.1. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\| = \eta$.*

Proof. In order to prove Lemma 2.1 we first remember that (see, e.g., [4, p. 44])

$$\Phi(t) \geqslant \tau^{p^0}\Phi(t/\tau), \quad \forall t > 0 \text{ and } \tau \in (0,1]. \tag{5}$$

On the other hand, by Proposition 6, p. 77 in [12] we have

$$\int_\Omega \Phi\big(|\nabla u|\big)\,\mathrm{d}x \geqslant \|u\|^{p^0}, \quad \forall u \in E \text{ with } \|u\| < 1. \tag{6}$$

Since E is continuously embedded in $L^r(\Omega)$, there exists a positive constant C_1 such that

$$\int_\Omega |u|^r\,\mathrm{d}x \leqslant C_1 \cdot \|u\|^r, \quad \forall u \in E. \tag{7}$$

Using relations (6) and (7) we deduce that for all $u \in E$ with $\|u\| \leqslant 1$ we have

$$J_\lambda(u) \geqslant \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x - \frac{1}{r}\int_\Omega |u|^r\,\mathrm{d}x \geqslant \|u\|^{p^0} - \frac{C_1}{r}\|u\|^r = \left(1 - \frac{C_1}{r}\cdot\|u\|^{r-p^0}\right)\|u\|^{p^0}.$$

However, by Remark 2 and the hypotheses of Theorem 1.1, we have $p^0 = p+q < r$. We conclude that Lemma 2.1 holds true. □

Lemma 2.2. *Assume that E_1 is a finite dimensional subspace of E. Then the set $S = \{u \in E_1;\ J_\lambda(u) \geqslant 0\}$ is bounded.*

Proof. A straightforward computation shows that

$$\frac{\Phi(\sigma t)}{\Phi(t)} \leqslant \sigma^{p^0}, \quad \forall t > 0 \text{ and } \sigma > 1. \tag{8}$$

Then, for all $u \in E$ with $\|u\| > 1$, relation (8) implies

$$\int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x = \int_\Omega \Phi\left(\|u\|\frac{|\nabla u(x)|}{\|u\|}\right)\mathrm{d}x \leqslant \|u\|^{p^0}\int_\Omega \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right)\mathrm{d}x \leqslant \|u\|^{p^0}. \tag{9}$$

On the other hand, since E is continuously embedded in $L^p(\Omega)$, it follows that there exists a positive constant C_2 such that

$$\int_\Omega |u|^p\,\mathrm{d}x \leqslant C_2\|u\|^p, \quad \forall u \in E. \tag{10}$$

Relations (9) and (10) yield

$$J_\lambda(u) \leqslant \|u\|^{p^0} + \frac{\lambda}{p}C_2\|u\|^p - \frac{1}{r}\int_\Omega |u|^r\,\mathrm{d}x, \quad \forall u \in E \text{ with } \|u\| > 1. \tag{11}$$

We point out that the functional $|\cdot|_r : E \to \mathbb{R}$ defined by $|u|_r = (\int_\Omega |u|^r\,\mathrm{d}x)^{1/r}$ is a norm in E. In the finite dimensional subspace E_1 the norms $|\cdot|_r$ and $\|\cdot\|$ are equivalent, so there exists a positive constant $C_3 = C_3(E_1)$ such that $\|u\| \leqslant C_3 \cdot |u|_r$, for all $u \in E_1$. So, by (11),

$$J_\lambda(u) \leqslant \|u\|^{p^0} + \frac{\lambda}{p}C_2\|u\|^p - \frac{1}{r}C_3^{-1}\|u\|^r, \quad \forall u \in E_1 \text{ with } \|u\| > 1.$$

Hence

$$\|u\|^{p^0} + \frac{\lambda}{p}C_2\|u\|^p - \frac{1}{r}C_3^{-1}\|u\|^r \geqslant 0, \quad \forall u \in S \text{ with } \|u\| > 1. \tag{12}$$

Since, by Remark 2 and the hypotheses of Theorem 1.1 we have $r > p^0 > p$, the above relation implies that S is bounded in E. □

Lemma 2.3. *Assume that $\{u_n\} \subset E$ is a sequence which satisfies the properties*

$$|J_\lambda(u_n)| < M, \tag{13}$$

$$J'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty, \tag{14}$$

where M is a positive constant. Then $\{u_n\}$ possesses a convergent subsequence.

Proof. First, we show that $\{u_n\}$ is bounded in E. Assume by contradiction the contrary. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\| > 1$ for any integer n. By (14) we deduce that there exists a positive integer N_1 such that for any $n > N_1$ we have $\|J'_\lambda(u_n)\| \leqslant 1$. On the other hand, for any $n > N_1$ fixed, the application $E \ni v \to \langle J'_\lambda(u_n), v\rangle$ is linear and continuous. It follows that

$$|\langle J'_\lambda(u_n), v\rangle| \leqslant \|J'_\lambda(u_n)\|\,\|v\| \leqslant \|v\|, \quad \forall v \in E,\ n > N_1.$$

Setting $v = u_n$ we have

$$-\|u_n\| \leqslant \int_\Omega \log\bigl(1+|\nabla u_n|^q\bigr)|\nabla u_n|^p \,\mathrm{d}x + \lambda \int_\Omega |u_n|^p \,\mathrm{d}x - \int_\Omega |u_n|^r \,\mathrm{d}x \leqslant \|u_n\|, \quad \forall n > N_1.$$

We obtain

$$-\|u_n\| - \int_\Omega \log\bigl(1+|\nabla u_n|^q\bigr)|\nabla u_n|^p \,\mathrm{d}x - \lambda \int_\Omega |u_n|^p \,\mathrm{d}x \leqslant - \int_\Omega |u_n|^r \,\mathrm{d}x, \quad \forall n > N_1. \tag{15}$$

If $\|u_n\| > 1$, then relation (15) implies

$$\begin{aligned} J_\lambda(u_n) &= \int_\Omega \Phi\bigl(|\nabla u_n|\bigr)\,\mathrm{d}x + \frac{\lambda}{p}\int_\Omega |u_n|^p \,\mathrm{d}x - \frac{1}{r}\int_\Omega |u_n|^r \,\mathrm{d}x \\ &\geqslant \int_\Omega \Phi\bigl(|\nabla u_n|\bigr)\,\mathrm{d}x + \lambda\left(\frac{1}{p}-\frac{1}{r}\right)\int_\Omega |u_n|^p \,\mathrm{d}x - \frac{1}{r}\int_\Omega \log\bigl(1+|\nabla u_n|^q\bigr)|\nabla u_n|^p \,\mathrm{d}x - \frac{1}{r}\|u_n\| \\ &= \int_\Omega \Phi\bigl(|\nabla u_n|\bigr)\,\mathrm{d}x - \frac{1}{r}\int_\Omega \phi\bigl(|\nabla u_n|\bigr)|\nabla u_n| \,\mathrm{d}x + \lambda\left(\frac{1}{p}-\frac{1}{r}\right)\int_\Omega |u_n|^p \,\mathrm{d}x - \frac{1}{r}\|u_n\|. \end{aligned}$$

Using the definition of p^0 we find

$$\int_\Omega \Phi\bigl(|\nabla u_n|\bigr)\,\mathrm{d}x - \frac{1}{r}\int_\Omega \phi\bigl(|\nabla u_n|\bigr)|\nabla u_n| \,\mathrm{d}x \geqslant \left(1-\frac{p^0}{r}\right)\int_\Omega \Phi\bigl(|\nabla u_n|\bigr)\,\mathrm{d}x.$$

Using the above relations we deduce that for any $n > N_1$ such that $\|u_n\| > 1$ we have

$$M > \left(1-\frac{p^0}{r}\right)\int_\Omega \Phi\bigl(|\nabla u_n|\bigr)\,\mathrm{d}x - \frac{1}{r}\|u_n\|. \tag{16}$$

Since $\Phi(t) \leqslant (t\phi(t))/p$ for all $t \in \mathbb{R}$ we deduce by Lemma C.9 in [5] that

$$\int_\Omega \Phi\bigl(|\nabla u_n|\bigr)\,\mathrm{d}x \geqslant \|u_n\|^p, \quad \forall n > N_1 \text{ with } \|u_n\| > 1. \tag{17}$$

Relations (16) and (17) imply

$$M > \left(1-\frac{p^0}{r}\right)\|u_n\|^p - \frac{1}{r}\|u_n\|, \quad \forall n > N_1 \text{ with } \|u_n\| > 1.$$

Since $p^0 < r$, letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. Thus, there exists $u_0 \in E$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u_0 in E. Since E is compactly embedded in $L^p(\Omega)$ and $L^r(\Omega)$ it follows that $\{u_n\}$ converges strongly to u_0 in $L^p(\Omega)$ and $L^r(\Omega)$. Combining the above consideration with similar arguments as in the end of the proof of Lemma 4.1 in [4, pp. 49–50], we conclude that, in fact, u_n converges strongly to u_0 in E. The proof of Lemma 2.3 is complete. □

Proof of Theorem 1.1 completed. It is clear that the functional J_λ is even and verifies $J_\lambda(0) = 0$. On the other hand, Lemmas 2.1, 2.2 and 2.3 show that the hypotheses of the Symmetric Mountain Pass Lemma (see Theorem 9.12 in Rabinowitz [11]) are satisfied. We conclude that Eq. (3) has infinitely many weak solutions in E. The proof of Theorem 1.1 is complete. □

We refer to [10] for further results in the study of quasilinear nonhomogeneous problems in Orlicz–Sobolev spaces.

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