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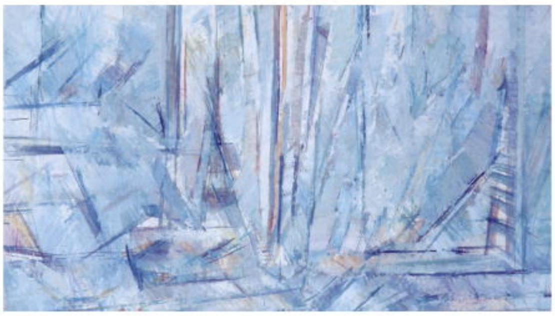

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Partial Differential Equations

# Nonhomogeneous Neumann problems in Orlicz–Sobolev spaces

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**Abstract**

We establish sufficient conditions for the existence of nontrivial solutions for a class of nonlinear Neumann boundary value problems involving nonhomogeneous differential operators. ***To cite this article: M. Mihăilescu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***


**Résumé**

**Problèmes de Neumann non homogènes dans les espaces d'Orlicz–Sobolev.** On établit des conditions suffisantes pour l'existence des solutions non triviales pour une classe de problèmes aux limites de Neumann avec des opérateurs différentiels non homogènes. ***Pour citer cet article : M. Mihăilescu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***


## Version française abrégée

Soit $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ un domaine borné et régulier. On considère le problème non linéaire

$$\begin{cases} -\operatorname{div}\big(a\big(x, |\nabla u(x)|\big)\nabla u(x)\big) + a\big(x, |u(x)|\big)u(x) = \lambda g\big(x, u(x)\big), & \text{pour } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = 0, & \text{pour } x \in \partial\Omega, \end{cases} \tag{1}$$

où $\nu$ est la normale extérieure à $\partial\Omega$. Soit $\phi(x,t) = a(x,|t|)t$ si $t \neq 0$ et $\phi(x,0) = 0$. On suppose qu'il existe deux constantes $\phi_0$ et $\phi^0$ telles que

$$1 < \phi_0 \leqslant \frac{t\phi(x,t)}{\Phi(x,t)} \leqslant \phi^0 < \infty, \quad \forall x \in \overline{\Omega},\ t \geqslant 0. \tag{2}$$

De plus, on suppose que la fonction $\Phi$ satisfait

$$M|t|^{p(x)} \leqslant \Phi(x,t), \quad \forall x \in \overline{\Omega},\ t \geqslant 0,$$

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où $p \in C(\overline{\Omega})$, $p(x) > 1$ pour chaque $x \in \overline{\Omega}$ et $M > 0$ est une constante. D'autre part, on suppose que la fonction $g$ satisfait les conditions

$$\left|g(x,t)\right| \leqslant C_0|t|^{q(x)-1}, \quad \forall x \in \Omega,\ t \in \mathbb{R},$$

et

$$C_1|t|^{q(x)} \leqslant G(x,t) := \int_0^t g(x,s)\,\mathrm{d}s \leqslant C_2|t|^{q(x)}, \quad \forall x \in \Omega,\ t \in \mathbb{R},$$

où $C_0$, $C_1$ et $C_2$ sont des constantes positives et la fonction $q \in C(\overline{\Omega})$ satisfait $1 < q(x) < \frac{N \min_{\overline{\Omega}} p}{N - \min_{\overline{\Omega}} p}$ pour tout $x \in \overline{\Omega}$.

Le résultat principal de cette Note est le suivant :

**Théorème 0.1.**

(i) *Si* $\min_{\overline{\Omega}} q < \phi_0$ *alors il existe* $\lambda^\star > 0$ *tel que pour chaque* $\lambda \in (0, \lambda^\star)$ *le problème* (1) *admet une solution faible non triviale.*
(ii) *Si* $\max_{\overline{\Omega}} q < \phi_0$ *alors il existe* $\lambda^\star > 0$ *et* $\lambda^{\star\star} > 0$ *tels que pour chaque* $\lambda \in (0, \lambda^\star) \cup (\lambda^{\star\star}, \infty)$ *le problème* (1) *admet une solution faible non triviale.*

## 1. The main result

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) be a bounded domain with smooth boundary. We consider the problem

$$\begin{cases} -\operatorname{div}\big(a\big(x, \left|\nabla u(x)\right|\big)\nabla u(x)\big) + a\big(x, \left|u(x)\right|\big)u(x) = \lambda g\big(x, u(x)\big), & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{3}$$

where $\nu$ is the outward unit normal to $\partial\Omega$. In the particular case when $a(x,t) = t^{p(x)-2}$, with $p$ a continuous function on $\overline{\Omega}$, we deal with problems involving variable growth conditions. The study of such problems has been stimulated by recent advances in fluid dynamics (see [3,5,12,13]), image processing (see [1]) and calculus of variations and differential equations with $p(x)$-growth conditions (see [4–7]).

In this Note we assume that the function $a : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ in (3) is such that the mapping $\phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $\phi(x,t) = a(x,|t|)t$ if $t \neq 0$ and $\phi(x,0) = 0$ satisfies:

($\phi$) for all $x \in \Omega$, $\phi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$; while the function $\Phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $\Phi(x,t) := \int_0^t \phi(x,s)\,\mathrm{d}s$, for all $x \in \overline{\Omega}$ and all $t \geqslant 0$ belongs to *class* $\Phi$ (see [9], p. 33), that is, $\Phi$ satisfies the following conditions:
($\Phi_1$) for all $x \in \Omega$, $\Phi(x,\cdot) : [0,\infty) \to \mathbb{R}$ is a nondecreasing continuous function, with $\Phi(x,0) = 0$ and $\Phi(x,t) > 0$ whenever $t > 0$; $\lim_{t\to\infty} \Phi(x,t) = \infty$;
($\Phi_2$) for every $t \geqslant 0$, $\Phi(\cdot,t) : \Omega \to \mathbb{R}$ is a measurable function.

**Remark 1.** Since $\phi(x,\cdot)$ satisfies condition ($\phi$) we deduce that $\Phi(x,\cdot)$ is convex and increasing from $\mathbb{R}^+$ to $\mathbb{R}^+$.

For the function $\Phi$ introduced above we define the *generalized Orlicz space* $L^\Phi(\Omega)$ as the Banach space of all measurable functions $u : \Omega \to \mathbb{R}$ for which the *Luxemburg norm*

$$|u|_\Phi = \inf\left\{\mu > 0;\ \int_\Omega \Phi\left(x, \frac{|u(x)|}{\mu}\right)\mathrm{d}x \leqslant 1\right\},$$

is finite.

In this Note we assume that there exist two positive constants $\phi_0$ and $\phi^0$ such that

$$1 < \phi_0 \leqslant \frac{t\phi(x,t)}{\Phi(x,t)} \leqslant \phi^0 < \infty, \quad \forall x \in \overline{\Omega},\ t \geqslant 0. \tag{4}$$

We point out that in the particular case when $\phi(x,t)=|t|^{p(x)-2}t$ with $p(x)\in C(\overline{\Omega})$ then we denote $\phi^0$ by $p^+:=\max_{\overline{\Omega}} p$ and $\phi_0$ by $p^-:=\max_{\overline{\Omega}} p$.

Furthermore, we assume that $\Phi$ satisfies the following condition:

$$\text{for each } x\in\overline{\Omega},\ \text{the function } [0,\infty)\ni t\to\Phi(x,\sqrt{t}) \text{ is convex.} \tag{5}$$

**Remark 2.** Relation (5) assures that $L^{\Phi}(\Omega)$ is an uniformly convex space and thus, a reflexive space.

On the other hand, we point out that assuming that $\Phi$ and $\Psi$ belong to class $\Phi$ and

$$\Psi(x,t)\leqslant K_1\cdot\Phi(x,K_2\cdot t)+h(x),\quad \forall x\in\overline{\Omega}, t\geqslant 0, \tag{6}$$

where $h\in L^1(\Omega)$, $h(x)\geqslant 0$ a.e. $x\in\Omega$ and $K_1$, $K_2$ are positive constants, then by Theorem 8.5 in [9] we have that there exists the continuous embedding $L^{\Phi}(\Omega)\subset L^{\Psi}(\Omega)$.

Next, we build upon $L^{\Phi}(\Omega)$ the *generalized Orlicz–Sobolev space* $W^{1,\Phi}(\Omega)$ as the space of those weakly differentiable functions in $\Omega$ for which the weak derivatives belong to $L^{\Phi}(\Omega)$. This space endowed with the norm

$$\|u\|=\inf\left\{\mu>0;\ \int_\Omega\left[\Phi\left(x,\frac{|u(x)|}{\mu}\right)+\Phi\left(x,\frac{|\nabla u(x)|}{\mu}\right)\right]\mathrm{d}x\leqslant 1\right\},$$

is a reflexive Banach space. On $W^{1,\Phi}(\Omega)$ the following relations hold true:

$$\int_\Omega\left[\Phi\big(x,|u(x)|\big)+\Phi\big(x,|\nabla u(x)|\big)\right]\mathrm{d}x\geqslant\|u\|^{\phi_0},\quad \forall u\in W^{1,\Phi}(\Omega) \text{ with} \|u\|>1; \tag{7}$$

$$\int_\Omega\left[\Phi\big(x,|u(x)|\big)+\Phi\big(x,|\nabla u(x)|\big)\right]\mathrm{d}x\geqslant\|u\|^{\phi^0},\quad \forall u\in W^{1,\Phi}(\Omega) \text{ with } \|u\|<1. \tag{8}$$

We refer to Diening [2], Musielak [9], Musielak and Orlicz [10], Nakano [11] for further properties of generalized Orlicz–Sobolev spaces.

In this Note we study problem (3) in the particular case when $\Phi$ satisfies

$$M|t|^{p(x)}\leqslant\Phi(x,t),\quad \forall x\in\overline{\Omega}, t\geqslant 0, \tag{9}$$

where $p(x)\in C(\overline{\Omega})$ with $p(x)>1$ for all $x\in\overline{\Omega}$ and $M>0$ is a constant.

**Remark 3.** By relation (9) we deduce that $W^{1,\Phi}(\Omega)$ is continuously embedded in $W^{1,p(x)}(\Omega)$ (see relation (6) with $\Psi(x,t)=|t|^{p(x)}$). On the other hand, it is known (see [5]) that $W^{1,p(x)}(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for any $r(x)\in C(\overline{\Omega})$ with $1<r^-\leqslant r^+<\frac{Np^-}{N-p^-}$. Thus, we deduce that $W^{1,\Phi}(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for any $r(x)\in C(\overline{\Omega})$ with $1<r(x)<\frac{Np^-}{N-p^-}$ for all $x\in\overline{\Omega}$.

On the other hand, we assume that the function $g$ from problem (3) satisfies the hypotheses

$$\big|g(x,t)\big|\leqslant C_0|t|^{q(x)-1},\quad \forall x\in\Omega, t\in\mathbb{R} \tag{10}$$

and

$$C_1|t|^{q(x)}\leqslant G(x,t):=\int_0^t g(x,s)\,\mathrm{d}s\leqslant C_2|t|^{q(x)},\quad \forall x\in\Omega, t\in\mathbb{R}, \tag{11}$$

where $C_0$, $C_1$ and $C_2$ are positive constants and $q(x)\in C(\overline{\Omega})$ satisfies $1<q(x)<\frac{Np^-}{N-p^-}$ for all $x\in\overline{\Omega}$.

**Example.** (a) First, we point out certain examples of functions $g$ and $G$ which satisfy hypotheses (10) and (11).

(1) $g(x,t)=q(x)|t|^{q(x)-2}t$ and $G(x,t)=|t|^{q(x)}$, where $q(x)\in C(\overline{\Omega})$ satisfies $2\leqslant q(x)<\frac{Np^-}{N-p^-}$ for all $x\in\overline{\Omega}$;

(2) $g(x,t) = q(x)|t|^{q(x)-2}t + (q(x)-2)\cdot[\log(1+t^2)]|t|^{q(x)-4}t + \frac{t}{1+t^2}|t|^{q(x)-2}$ and $G(x,t) = |t|^{q(x)} + \log(1+t^2)\cdot|t|^{q(x)-2}$, where $q(x) \in C(\overline{\Omega})$ satisfies $4 \leqslant q(x) < \frac{Np^-}{N-p^-}$ for all $x \in \overline{\Omega}$.

(b) Second, we point out certain examples of functions $\phi(x,t)$ and $\Phi(x,t)$ for which the results of this paper can be applied.

(1) $\phi(x,t) = p(x)|t|^{p(x)-2}t$ and $\Phi(x,t) = |t|^{p(x)}$, with $p(x) \in C(\overline{\Omega})$ satisfying $2 \leqslant p(x) < N$, for all $x \in \overline{\Omega}$.

(2)

$$\phi(x,t) = p(x)\frac{|t|^{p(x)-2}t}{\log(1+|t|)} \quad \text{and} \quad \Phi(x,t) = \frac{|t|^{p(x)}}{\log(1+|t|)} + \int_0^{|t|} \frac{s^{p(x)}}{(1+s)(\log(1+s))^2}\,\mathrm{d}s$$

with $p(x) \in C(\overline{\Omega})$ satisfying $3 \leqslant p(x) < N$, for all $x \in \overline{\Omega}$.

(3) $\phi(x,t) = p(x)\log(1+\alpha+|t|)\cdot|t|^{p(x)-1}t$ and

$$\Phi(x,t) = \log\bigl(1+\alpha+|t|\bigr)\cdot|t|^{p(x)} - \int_0^{|t|} \frac{s^{p(x)}}{1+\alpha+s}\,\mathrm{d}x$$

where $\alpha > 0$ is a constant and $p(x) \in C(\overline{\Omega})$ satisfying $2 \leqslant p(x) < N$, for all $x \in \overline{\Omega}$.

We say that $u \in W^{1,\Phi}(\Omega)$ is a *weak solution* of problem (3) if

$$\int_\Omega a\bigl(x,|\nabla u|\bigr)\nabla u \nabla v\,\mathrm{d}x + \int_\Omega a\bigl(x,|u|\bigr)uv\,\mathrm{d}x - \lambda\int_\Omega g(x,u)v\,\mathrm{d}x = 0,$$

for all $v \in W^{1,\Phi}(\Omega)$.

The main result of this Note is given by the following theorem:

**Theorem 1.1.** *Assume $\phi$ and $\Phi$ verify conditions* ($\phi$), ($\Phi_1$), ($\Phi_2$), (4), (5) *and* (9) *and the functions $g$ and $G$ satisfy conditions* (10) *and* (11).

(i) *If $q^- < \phi_0$ then there exists $\lambda_\star > 0$ such that for any $\lambda \in (0,\lambda_\star)$ problem* (3) *has a nontrivial weak solution.*
(ii) *If $q^+ < \phi_0$ then there exists $\lambda_\star > 0$ and $\lambda^\star > 0$ such that for any $\lambda \in (0,\lambda_\star)\cup(\lambda^\star,\infty)$ problem* (3) *has a nontrivial weak solution.*

Let $E$ denote the generalized Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$.

For each $\lambda > 0$ we define the energy functional $J_\lambda : E \to \mathbb{R}$ by

$$J_\lambda(u) = \int_\Omega \bigl[\Phi\bigl(x,|\nabla u|\bigr) + \Phi\bigl(x,|u|\bigr)\bigr]\,\mathrm{d}x - \lambda\int_\Omega G(x,u)\,\mathrm{d}x, \quad \forall u \in E.$$

Standard arguments imply that $J_\lambda$ is well-defined on $E$, $J_\lambda \in C^1(E,\mathbb{R})$ and

$$\bigl\langle J_\lambda'(u), v\bigr\rangle = \int_\Omega a\bigl(x,|\nabla u|\bigr)\nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Omega a\bigl(x,|u|\bigr)uv\,\mathrm{d}x - \lambda\int_\Omega g(x,u)v\,\mathrm{d}x,$$

for all $u$, $v \in E$. Thus, we remark that the weak solutions of Eq. (3) are exactly the critical points of the energy functional $J_\lambda$.

The following auxiliary results will be useful in order to establish the result of Theorem 1.1(i):

**Lemma 1.2.** *Assume the hypotheses of Theorem* 1.1(i) *are fulfilled. Then there exists $\lambda_\star > 0$ such that for any $\lambda \in (0,\lambda_\star)$ there exist $\rho$, $\alpha > 0$ such that $J_\lambda(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\| = \rho$.*

**Lemma 1.3.** *Assume the hypotheses of Theorem* 1.1(i) *are fulfilled. Then there exists $\theta \in E$ such that $\theta \geqslant 0$, $\theta \neq 0$ and $J_\lambda(t\theta) < 0$, for $t > 0$ small enough.*

**Lemma 1.4.** *Assume that the sequence $\{u_n\}$ converges weakly to $u$ in $E$ and*

$$\limsup_{n\to\infty}\langle J_\lambda'(u_n), u_n - u\rangle \leqslant 0.$$

*Then $\{u_n\}$ converges strongly to $u$ in $E$.*

**Proof of Theorem 1.1(i).** Let $\lambda_\star > 0$ be given by Lemma 1.2 and $\lambda \in (0, \lambda_\star)$. By Lemma 1.2 it follows that on the boundary of the ball centered in the origin and of radius $\rho$ in $E$, denoted by $B_\rho(0)$, we have $\inf_{\partial B_\rho(0)} J_\lambda > 0$.

On the other hand, by Lemma 1.3, there exists $\theta \in E$ such that $J_\lambda(t\theta) < 0$ for all $t > 0$ small enough. Moreover, relations (8) and (11) and the fact that $E$ is continuously embedded in $L^{q(x)}(\Omega)$ imply that for any $u \in B_\rho(0)$ we have

$$J_\lambda(u) \geqslant \|u\|^{\phi^0} - \lambda C_2 c_1^{q^-}\|u\|^{q^-},$$

where $c_1$ is a positive constant. It follows that $-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} J_\lambda < 0$.

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$J_\lambda(u_\epsilon) < \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \quad \text{and} \quad J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

Since

$$J_\lambda(u_\epsilon) \leqslant \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \leqslant \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon\|u - u_\epsilon\|$. It is clear that $u_\epsilon$ is a minimum point of $I_\lambda$ and thus for small $t > 0$ and any $v \in B_1(0)$ we have

$$\frac{I_\lambda(u_\epsilon + t\cdot v) - I_\lambda(u_\epsilon)}{t} \geqslant 0 \quad \text{or} \quad \frac{J_\lambda(u_\epsilon + t\cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon\|v\| \geqslant 0.$$

Letting $t \to 0$ it follows that $\langle J_\lambda'(u_\epsilon), v\rangle + \epsilon\|v\| > 0$ and we infer that $\|J_\lambda'(u_\epsilon)\| \leqslant \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \quad \text{and} \quad J_\lambda'(w_n) \to 0. \tag{12}$$

It is clear that $\{w_n\}$ is bounded in $E$. Thus, there exists $w \in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to $w$ in $E$. Using relation (12) we find

$$\lim_{n\to\infty}\langle J_\lambda'(w_n), w_n - w\rangle = 0.$$

Thus, by Lemma 1.4, we deduce that $\{w_n\}$ converges strongly to $w$ in $E$. So, by (12), $J_\lambda(w) = \underline{c} < 0$ and $J_\lambda'(w) = 0$. We conclude that $w$ is a nontrivial weak solution for problem (3) for any $\lambda \in (0, \lambda_\star)$. The proof of Theorem 1.1 (i) is complete. □

Next, we prove Theorem 1.1(ii).

**Proof of Theorem 1.1(ii).** Since $q^+ < \phi_0$ it follows that $q^- < \phi_0$ and thus, by Theorem 1.1(i) there exists $\lambda_\star > 0$ such that for any $\lambda \in (0, \lambda_\star)$ problem (3) has a nontrivial weak solution.

On the other hand, we point out that $J_\lambda$ is coercive and weakly lower semi-continuous in $E$, for all $\lambda > 0$. Then Theorem 1.2 in [14] implies that there exists $u_\lambda \in E$ a global minimizer of $I_\lambda$ and thus a weak solution of problem (3).

We show that $u_\lambda$ is not trivial for $\lambda$ large enough. Indeed, letting $t_0 > 1$ be a fixed real and $u_0(x) = t_0$, for all $x \in \Omega$ we have $u_0 \in E$ and

$$J_\lambda(u_0) = \Lambda(u_0) - \lambda\int_\Omega G(x, u_0)\,dx \leqslant \int_\Omega \Phi(x, t_0)\,dx - \lambda C_1\int_\Omega |t_0|^{q(x)}\,dx \leqslant L - \lambda C_1 t_0^{q^+}|\Omega_1|,$$

where $L$ is a positive constant. Thus, there exists $\lambda^\star > 0$ such that $J_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^\star, \infty)$. It follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda \geqslant \lambda^\star$ and thus $u_\lambda$ is a nontrivial weak solution of problem (3) for $\lambda$ large enough. The proof of Theorem 1.1(ii) is complete. □

We refer to [8] for complete proofs and additional results.

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