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Partial Differential Equations

Multiple symmetric solutions for a Neumann problem with lack of compactness

Solutions symétriques multiples pour un problème de Neumann sans compacité

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A R T I C L E   I N F O

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A B S T R A C T

The existence of multiple cylindrically symmetric solutions for a class of non-autonomous elliptic Neumann problems in a strip-like domain of the Euclidean space is investigated. The proof combines a recent compactness result and the Palais symmetric critically principle. A concrete application illustrates the main abstract result of this Note.

R É S U M É

On étudie l'existence de solutions multiples à symétrie cylindrique pour une classe de problèmes elliptiques non autonomes de Neumann sans compacité. La preuve combine un résultat récent de compacité et le principe de Palais de symétrique critique. Une application met en évidence le résultat principal de cette Note.

Version française abrégée

Soit $\mathcal{O} \subset \mathbb{R}^m$ un domaine borné et régulier et soit le cylindre $\Omega := \mathcal{O} \times \mathbb{R}^n$. On considère l'espace des fonctions à symétrie cylindrique défini par

$$W_c^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega)\colon\ u(x,\cdot) \text{ a symétrie radiale, pour tout } x \in \mathcal{O}\}.$$

Dans cette Note on étudie le problème non linéaire

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda\alpha(x,y)f(u) & \text{dans } \Omega \\ \partial u/\partial\nu = 0 & \text{sur } \partial\Omega, \end{cases} \tag{P}$$

où $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $f:\mathbb{R}\to\mathbb{R}$ est une fonction continue, ν est la normale extérieure à $\partial\Omega$ et $p > m+n$. On suppose que $\alpha \in L^1(\Omega)$ est une fonction non négative à symétrie cylindrique telle que, pour un certain $\tau > 0$,

$$\operatorname{essinf}_{(x,y)\in\mathcal{O}\times B_n(0,\tau/2)} \alpha(x,y) > 0.$$

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Soit $B_{(1/2,1)}(n,p+1) := \int_{1/2}^{1} t^{n-1}(1-t)^p\,\mathrm{d}t$. On définit

$$\sigma(n,p) := 2^{p-n}\big(2^n-1\big); \qquad g(p,N) := \frac{1+2^{n+p}NB_{(1/2,1)}(n,p+1)}{2^n}; \qquad \omega_\tau^{(n)} := \tau^n \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})},$$

et

$$\kappa := \frac{\tau}{(\omega_\tau^{(n)}|\mathcal{O}|_m(\sigma(n,p)+\tau^p g(p,n)))^{1/p} C_p}.$$

Soit $\|\alpha\|_{(\mathcal{O},\tau/2)} := \int_{\mathcal{O}\times B_n(0,\tau/2)} \alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

Le résultat principal de cette Note est le suivant :

Théorème 0.1. *Supposons qu'il existe δ et γ avec $\delta > \kappa\gamma > 0$ et tels que les propriétés suivantes soient satisfaites* :

(h_0) $F(\xi) := \int_0^\xi f(s)\,\mathrm{d}s \geqslant 0$, *pour tout* $\xi \in [0,\delta]$;

(h_1) *si*

$$\tilde{\kappa} := \frac{\kappa^p \|\alpha\|_{(\mathcal{O},\tau/2)}}{\|\alpha\|_{L^1}},$$

alors

$$\frac{\max_{|\xi|\leqslant\gamma} F(\xi)}{\gamma^p} < \tilde{\kappa}\frac{F(\delta)}{\delta^p};$$

(h_2) *il existe $c_0 > 0$ et $0 < s < p-1$ tels que pour tout $t \in \mathbb{R}$,*

$$\big|f(t)\big| \leqslant c_0\big(1+|t|^s\big).$$

Alors, pour tout λ dans l'intervalle

$$\Lambda_{(\gamma,\delta)} := \left]\frac{\delta^p}{p\kappa^p C_p^p \|\alpha\|_{(\mathcal{O},\tau/2)} F(\delta)}, \frac{\gamma^p}{pC_p^p\|\alpha\|_{L^1}\max_{|\xi|\leqslant\gamma} F(\xi)}\right[,$$

le problème (P) admet au moins trois solutions faibles à symétrie cylindrique.

1. Introduction

Let $\mathcal{O} \subset \mathbb{R}^m$ be a bounded domain with smooth boundary and set $\Omega := \mathcal{O}\times\mathbb{R}^n$. Define the space of cylindrically symmetric functions by

$$W_c^{1,p}(\Omega) := \big\{u \in W^{1,p}(\Omega)\colon\ u(x,\cdot) \text{ is radially symmetric for all } x \in \mathcal{O}\big\}.$$

The aim of this Note is to establish the existence of multiple cylindrically symmetric weak solutions for the following non-autonomous elliptic Neumann problem:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda\alpha(x,y)f(u) & \text{in } \Omega \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Here ν denotes the outward unit normal to $\partial\Omega$, $p > m+n$ is a real number, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We assume that $f : \mathbb{R}\to\mathbb{R}$ is a continuous function, α is a nonnegative cylindrically symmetric function, and λ is a positive real parameter.

In the present Note, just requiring a suitable oscillating behaviour of the potential $F(\xi) := \int_0^\xi f(t)\,\mathrm{d}t$, we are able to find a precise interval of values of the parameter λ for which problem (P) admits at least three cylindrically symmetric weak solutions.

Assume $\alpha \in L^1(\Omega)$ is a nonnegative cylindrically symmetric function such that for some $\tau > 0$,

$$\operatorname{essinf}_{(x,y)\in\mathcal{O}\times B_n(0,\tau/2)} \alpha(x,y) > 0, \tag{1}$$

where $B_n(0,\tau/2)$ denotes the open ball in $\mathbb{R}^n$ centred in zero and radius $\tau/2$.

We say that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (P) if for all $v \in W^{1,p}(\Omega)$,

$$\int_\Omega \big|\nabla u(x,y)\big|^{p-2}\nabla u(x,y)\cdot\nabla v(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \big|u(x,y)\big|^{p-2}u(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \lambda\int_\Omega \alpha(x,y)f\big(u(x,y)\big)v(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Then the weak solutions of problem (P) are the critical points of the C^1-functional

$$J_\lambda(u) := \frac{\|u\|^p_{W^{1,p}}}{p} - \lambda \int_\Omega \alpha(x,y)F\big(u(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y, \quad u \in W^{1,p}(\Omega)$$

where

$$\|u\|_{W^{1,p}} := \left(\int_\Omega \big|\nabla u(x,y)\big|^p\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \big|u(x,y)\big|^p\,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Let $B_{(1/2,1)}(n, p+1)$ denote the generalized incomplete beta function, namely $B_{(1/2,1)}(n,p+1) := \int_{1/2}^1 t^{n-1}(1-t)^p\,\mathrm{d}t$. Set

$$\sigma(n,p) := 2^{p-n}\big(2^n - 1\big); \qquad g(p,N) := \frac{1 + 2^{n+p} N B_{(1/2,1)}(n,p+1)}{2^n}; \qquad \omega_\tau^{(n)} := \tau^n \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}.$$

Define

$$\kappa := \frac{\tau}{(\omega_\tau^{(n)} |\mathcal{O}|_m (\sigma(n,p) + \tau^p g(p,n)))^{1/p} C_p},$$

in which the symbol $|\mathcal{O}|_m$ stands for the Lebesgue measure of the domain $\mathcal{O}$ in $\mathbb{R}^m$.

Set $\|\alpha\|_{(\mathcal{O},\tau/2)} := \int_{\mathcal{O}\times B_n(0,\tau/2)} \alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

2. Main result

The main result in this Note is the following multiplicity property:

Theorem 2.1. *Assume there are constants δ and γ such that*

$$\delta > \kappa\gamma > 0$$

and the following properties are fulfilled:

(h$_0$) $F(\xi) \geqslant 0$, *for every* $\xi \in [0,\delta]$;

(h$_1$) *if*

$$\tilde{\kappa} := \frac{\kappa^p \|\alpha\|_{(\mathcal{O},\tau/2)}}{\|\alpha\|_{L^1}},$$

then

$$\frac{\max_{|\xi|\leqslant\gamma} F(\xi)}{\gamma^p} < \tilde{\kappa} \frac{F(\delta)}{\delta^p};$$

(h$_2$) *there exist* $c_0 > 0$ *and* $0 < s < p-1$ *such that for all* $t \in \mathbb{R}$,

$$\big|f(t)\big| \leqslant c_0\big(1 + |t|^s\big).$$

Then for all λ belonging to the interval

$$\Lambda_{(\gamma,\delta)} := \left] \frac{\delta^p}{p\kappa^p C_p^p \|\alpha\|_{(\mathcal{O},\tau/2)} F(\delta)}, \frac{\gamma^p}{p C_p^p \|\alpha\|_{L^1} \max_{|\xi|\leqslant\gamma} F(\xi)} \right[,$$

the problem (P) has at least three distinct cylindrically symmetric weak solutions.

Proof. For all $u \in W_c^{1,p}(\Omega)$ we set

$$\Phi(u) := \frac{\|u\|^p_{W^{1,p}}}{p}, \quad \text{and} \quad \Psi(u) := \int_\Omega \alpha(x,y)F\big(u(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y.$$

Then Ψ is continuously Gâteaux differentiable and

$$\Psi'(u)(v) = \int_\Omega \alpha(x,y) f\big(u(x,y)\big) v(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

for every $v \in W_c^{1,p}(\Omega)$. Further, Φ' admits a continuous inverse on $W_c^{1,p}(\Omega)^*$ and Ψ' is a compact operator.

The energy functional associated to problem (P) is

$$J_\lambda(u) := \Phi(u) - \lambda\Psi(u), \quad \text{for all } u \in W_c^{1,p}(\Omega).$$

Set $r := \gamma^p/(pC_p^p)$, where C_p denotes the best constant of the continuous embedding $W_c^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ (see [2, Theorem 2.2]). Since

$$\bigl\{u \in W_c^{1,p}(\Omega)\colon \|u\|_{W^{1,p}} \leqslant (pr)^{1/p}\bigr\} \subseteq \bigl\{u \in W_c^{1,p}(\Omega)\colon \|u\|_\infty \leqslant \gamma\bigr\},$$

in which $\|u\|_{L^\infty} := \operatorname{esssup}_{(x,y)\in\Omega} |u(x,y)|$, we obtain

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} \leqslant pC_p^p\|\alpha\|_{L^1}\frac{\max_{|\xi|\leqslant\gamma}F(\xi)}{\gamma^p}.$$

Define $w_\delta \in W_c^{1,p}(\Omega)$ by $w_\delta(y) := \delta w(y)$, where

$$w(y) := \begin{cases} 0 & \text{if } |y| \geqslant \tau \\ \frac{2}{\tau}(\tau - |x|) & \text{if } \tau/2 < |y| < \tau \\ 1 & \text{if } |y| \leqslant \tau/2. \end{cases}$$

We have

$$\int_{\mathbb{R}^n} \bigl|\nabla w(y)\bigr|^p \,\mathrm{d}y + \int_{\mathbb{R}^n} \bigl|w(y)\bigr|^p \,\mathrm{d}y = \omega_\tau^{(n)}\biggl[\frac{\sigma(n,p)}{\tau^p} + g(p,n)\biggr].$$

Indeed

$$\int_{\mathbb{R}^n} \bigl|\nabla w(y)\bigr|^p \,\mathrm{d}y = \frac{2^{p-n}\omega_\tau^{(n)}}{\tau^p}\bigl(2^n - 1\bigr),$$

and

$$\int_{\mathbb{R}^n} \bigl|w(y)\bigr|^p \,\mathrm{d}y = \biggl(\int_{|y|\leqslant\tau/2} \mathrm{d}y + \frac{2^p}{\tau^p}\int_{\tau/2<|y|<\tau} \bigl(\tau - |y|\bigr)^p \,\mathrm{d}y\biggr) = \omega_\tau^{(n)} g(p,n).$$

Note that the last equality holds because

$$I_p := \int_{\tau/2<|y|<\tau} \bigl(\tau - |y|\bigr)^p \,\mathrm{d}y = n\omega_\tau^{(n)}\tau^p B_{(1/2,1)}(n, p+1).$$

Hence, since

$$\Phi(w_\delta) = \frac{\delta^p|\mathcal{O}|_m}{p}\biggl(\int_{\mathbb{R}^n} \bigl|\nabla w(y)\bigr|^p \,\mathrm{d}y + \int_{\mathbb{R}^n} \bigl|w(y)\bigr|^p \,\mathrm{d}y\biggr),$$

we deduce that

$$\Phi(w_\delta) = \frac{\delta^p\omega_\tau^{(n)}|\mathcal{O}|_m}{p}\biggl[\frac{\sigma(n,p)}{\tau^p} + g(p,n)\biggr].$$

Thus, by hypotheses (1) and (h_0), we deduce that

$$\Psi(w_\delta) \geqslant F(\delta)\int_{\mathcal{O}\times B_n(0,\tau/2)} \alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y = F(\delta)\|\alpha\|_{(\mathcal{O},\tau/2)}.$$

Therefore

$$\frac{\Psi(w_\delta)}{\Phi(w_\delta)} \geqslant p\kappa^p C_p^p\frac{F(\delta)\|\alpha\|_{(\mathcal{O},\tau/2)}}{\delta^p}.$$

Now, from (h_1), it follows that

$$\sup_{\Phi(u)\leqslant r}\frac{\Psi(u)}{r} < \frac{\Psi(w_\delta)}{\Phi(w_\delta)}.$$

On the other hand, by (h_2), there exists a positive constant c_1 such that

$$\int_\Omega \alpha(x,y)F\big(u(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant c_1\big(\|u\|_\infty + \|u\|_\infty^{s+1}\big),$$

for all $u \in W_c^{1,p}(\Omega)$.

Moreover, since $W_c^{1,p}(\Omega)$ is compactly embedded into $L^\infty(\Omega)$, it follows that there exists $c_2 > 0$ such that for all $u \in W_c^{1,p}(\Omega)$,

$$J_\lambda(u) := \Phi(u) - \lambda\Psi(u) \geqslant \|u\|^p_{W^{1,p}} - \lambda c_2\big(\|u\|_{W^{1,p}} + \|u\|^{s+1}_{W^{1,p}}\big).$$

Since $1 < s+1 < p$, it follows that for all $\lambda > 0$, $\lim_{\|u\|_{W^{1,p}}\to\infty} J_\lambda(u) = +\infty$. Thus, J_λ is coercive for every $\lambda > 0$, in particular for all

$$\lambda \in \Lambda_{(\gamma,\delta)} \subseteq \left]\frac{\Phi(w_\delta)}{\Psi(w_\delta)}, \frac{r}{\sup_{\Phi(u)\leqslant r}\Psi(u)}\right[.$$

Hence, by [1, Theorem 3.6] and Remark 3.3 of the cited paper, we deduce that for all $\lambda \in \Lambda_{(\gamma,\delta)}$, the functional J_λ has at least three distinct critical points.

Next, we prove that J_λ is an invariant functional with respect to the action of the compact group of linear isometries of $\mathbb{R}^n$. Indeed, let $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry. For any $u \in W^{1,p}(\Omega)$ define $u_\sigma \in W^{1,p}(\Omega)$ by putting, for any $(x,y) \in \Omega$,

$$u_\sigma(x,y) := u(x,\sigma y).$$

Since α is cylindrically symmetric, using the variable change $z = \sigma y$, we deduce that $J_\lambda(u_\sigma) = J_\lambda(u)$, for every $u \in W^{1,p}(\Omega)$.

Thus, by the Palais symmetric criticality principle [3] applied to the smooth and isometric invariant functional J_λ, we deduce that problem (P) admits at least three distinct cylindrically symmetric weak solutions. The proof is complete. □

A direct consequence of the previous result is the following multiplicity property:

Corollary 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative (not identically zero) continuous function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0. \tag{ℓ_0}$$

Further, assume that condition (h_2) *holds. Set*

$$S := \big\{\delta > 0:\ F(\delta) > 0\big\}.$$

Then, for all

$$\lambda > \frac{1}{p\kappa^p C_p^p \|\alpha\|_{(\mathcal{O},\tau/2)}} \inf_{\delta\in S} \frac{\delta^p}{F(\delta)},$$

problem (P) has at least two nontrivial cylindrically symmetric weak solutions.

Proof. Fix $\lambda > \frac{1}{p\kappa^p C_p^p \|\alpha\|_{(\mathcal{O},\tau/2)}} \inf_{\delta\in S} \frac{\delta^p}{F(\delta)}$. Then, there exists $\bar\delta$ such that $F(\bar\delta) > 0$ and $\lambda > \frac{\bar\delta^p}{p\kappa^p C_p^p \|\alpha\|_{(\mathcal{O},\tau/2)} F(\bar\delta)}$. By using condition (ℓ_0) we have

$$\lim_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} = 0.$$

Therefore, there exists $\bar\gamma > 0$ such that $\bar\gamma < \bar\delta/\kappa$ and

$$\frac{F(\bar\gamma)}{\bar\gamma^p} < \min\left\{\tilde\kappa \frac{F(\bar\delta)}{\bar\delta^p}, \frac{1}{pC_p^p\|\alpha\|_{L^1}\lambda}\right\}.$$

Hence

$$\lambda \in \Lambda_{(\bar\gamma,\bar\delta)} := \left]\frac{\bar\delta^p}{p\kappa^p C_p^p \|\alpha\|_{(\mathcal{O},\tau/2)} F(\bar\delta)}, \frac{\bar\gamma^p}{pC_p^p\|\alpha\|_{L^1} F(\bar\gamma)}\right[.$$

All the hypotheses of Theorem 2.1 are satisfied and problem (P) admits at least three (two nontrivial) distinct cylindrically symmetric weak solutions. The proof is complete. □

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