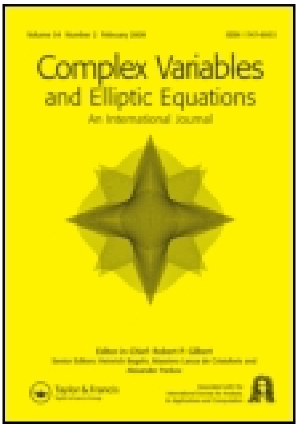

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Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent

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Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent†

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We establish the existence of an unbounded sequence of solutions for a class of quasilinear elliptic equations involving the anisotropic $\overrightarrow{p}(\cdot)$-Laplace operator, on a bounded domain with smooth boundary. We work on the anisotropic variable exponent Sobolev spaces and our main tool is the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz.

Keywords: quasilinear elliptic equations; multiple weak solutions; anisotropic variable exponent Sobolev spaces; symmetric mountain-pass theorem

AMS Subject Classifications: 35J25; 35J62; 35D30; 46E35; 35J20

1. Introduction

We are interested in the existence of multiple solutions for the nonhomogeneous anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left(\left|\partial_{x_i}u\right|^{p_i(x)-2}\partial_{x_i}u\right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the potential

$$F(x,t) = \int_0^t f(x,s)\mathrm{d}s,$$

and p_i are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$ for any $x \in \overline{\Omega}$ and $i \in \{1, \ldots, N\}$.

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Our study may be considered at the intersection of two directions in PDEs: the anisotropic Sobolev spaces theory developed by [1–5], and the variable exponent Sobolev spaces theory developed by [6–13]. Let us note that only a few results concerning the critical anisotropic problems can be found in the mathematical literature. On the other hand, many papers involving the variable exponent Sobolev spaces appeared in the past decades, mostly because the interest regarding such spaces and their applicability has recently grown. Materials requiring this theory have been studied experimentally since the middle of the last century, when the preoccupation for the electrorheological fluids (sometimes referred to as smart fluids) arose. The first major discovery in electrorheological fluids was due to Willis Winslow who obtained a US patent on the effect in 1947 [14] and wrote an article published in 1949 [15]. The above-mentioned fluids have a special property: when disposed to an electromagnetic field, their viscosity undergoes a significant change. Winslow was the first who noticed that in an electrical field, the viscosity of such fluids is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For more information on properties, modelling and the application of variable exponent spaces to these fluids we refer to [16–25]. Also, we must underline the fact that the variable exponent spaces have other major applications, for example in elastic mechanics [26], or in image restoration [27].

Consequently, we work on the so-called *anisotropic variable exponent Sobolev spaces* which were introduced for the first time by Mihăilescu et al. [28,29]. Also, one of the first contributions in this direction is due to Fragalà et al. [30]. The need for such theory comes naturally every time we want to consider materials with inhomogeneities that have different behaviour on different space directions. Since the concern for this topic is relatively new, only few papers have been published. To give some examples that were not previously mentioned, we refer the reader to [31–35]. By the nature of the conditions imposed on function f, our article is related to [32]. In [32], in addition to the discussion of the main problem, the following problem is brought to our attention:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left(\left|\partial_{x_i}u\right|^{p_i(x)-2}\partial_{x_i}u\right) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, f is a continuous function verifying some adequate conditions, λ is a positive parameter and p_i are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$ for any $x \in \overline{\Omega}$, $i \in \{1, \ldots, N\}$ and $\sum_{i=1}^{N} 1/\inf_{x\in\Omega} p_i(x) > 1$. In Remark 2, the author asserts the existence of a nontrivial solution to problem (1.2) for all $\lambda > 0$. His arguments are based on the mountain-pass theorem of Ambrosetti and Rabinowitz [36]. We impose similar conditions to our function f, but we consider f to be odd. This simple fact will allow us to show the existence of an unbounded sequence of weak solutions using the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz [37, Theorem 11.5]. The statement of this theorem and some mathematical details on the properties of variable exponent Sobolev spaces and anisotropic variable exponent Sobolev spaces will be reminded in Section 2.

2. Abstract framework

This section is dedicated to a general overview on the above mentioned spaces.

We set $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\}$ and denote, for any $p \in C_+(\overline{\Omega})$,

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is introduced as

$$L^{p(\cdot)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(\cdot)} = \inf\left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ has some important properties. Firstly, it is a separable and reflexive Banach space [9, Theorem 2.5, Corollary 2.7]. Then, the inclusion between spaces generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous [9, Theorem 2.8]. Furthermore, the following Hölder–type inequality

$$\left| \int_\Omega uv\,\mathrm{d}x \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2.1}$$

holds true for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ [9, Theorem 2.1], where we denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise i.e. $1/p(x) + 1/p'(x) = 1$ [9, Corollary 2.7].

To continue describing the variable exponent Lebesgue spaces, let us introduce the important function $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

This application is called the $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space and plays a key role in handling these generalized Lebesgue spaces. Therefore we indicate some of its properties [9]: if $u \in L^{p(\cdot)}(\Omega)$, $(u_n) \subset L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then,

$$|u|_{p(\cdot)} < 1 \ (=1;\ >1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 \ (=1;\ >1), \tag{2.2}$$

$$|u|_{p(\cdot)} > 1 \Rightarrow |u|^{p^-}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p^+}_{p(\cdot)}, \tag{2.3}$$

$$|u|_{p(\cdot)} < 1 \Rightarrow |u|^{p^+}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p^-}_{p(\cdot)}, \tag{2.4}$$

$$|u|_{p(\cdot)} \to 0 \ (\to\infty) \Leftrightarrow \rho_{p(\cdot)}(u) \to 0 \ (\to\infty), \tag{2.5}$$

$$\lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0. \tag{2.6}$$

Next, for any $p \in C_+(\overline{\Omega})$, we give the definition of the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$,

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \ \partial_{x_i} u \in L^{p(\cdot)}(\Omega),\ i \in \{1, 2, \dots N\} \right\},$$

endowed with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}. \tag{2.7}$$

$(W^{1,p(\cdot)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. For the density of the smooth functions in $W^{1,p(\cdot)}(\Omega)$ we consider $p \in C_+(\overline{\Omega})$ to be logarithmic Hölder continuous, that is, there exists $M > 0$ such that $|p(x) - p(y)| \leq -M/\log(|x-y|)$ for all $x, y \in \Omega$ with $|x-y| \leq 1/2$. The Sobolev spaces with zero boundary values $W_0^{1,p(\cdot)}(\Omega)$, defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|$, have their importance [38,39]. Note that the norms

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)},$$

and

$$\|u\|_{p(\cdot)} = \sum_{i=1}^{N} \left|\partial_{x_i} u\right|_{p(\cdot)}$$

are equivalent to (2.7) in $W_0^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ is also a separable and reflexive Banach space. Moreover, if $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \geq N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous. For more details concerning the variable exponent spaces $W_0^{1,p(\cdot)}(\Omega)$ we refer to the papers [9,38–41].

Now we can present the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$, where $\vec{p} : \overline{\Omega} \to \mathbb{R}^N$ is the vectorial function

$$\vec{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot)),$$

where the components $p_i \in C_+(\overline{\Omega})$, $i \in \{1, \ldots, N\}$ are logarithmic Hölder continuous. The space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\vec{p}(\cdot)} = \sum_{i=1}^{N} \left|\partial_{x_i} u\right|_{p_i(\cdot)}.$$

Notice that we are dealing with a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ that allows the adequate treatment of the existence of the weak solutions for problem (1.1). But at the same time, $W_0^{1,\vec{p}(\cdot)}(\Omega)$ may be regarded as a generalization of the classical anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$, where $\vec{p}$ is the constant vector $(p_1, \ldots, p_N)$. $W_0^{1,\vec{p}}(\Omega)$ endowed with the norm

$$\|u\|_{1,\vec{p}} = \sum_{i=1}^{N} \left|\partial_{x_i} u\right|_{p_i}$$

is a reflexive Banach space for any $\vec{p} \in \mathbb{R}^N$ with $p_i > 1$ for all $i \in \{1, \ldots, N\}$ (obviously, for p constant, we have denoted by $|\cdot|_p$ the norm in L^p). This result can be easily extended to $W_0^{1,\vec{p}(\cdot)}(\Omega)$, see [29].

We denote by $\vec{P}_+$, $\vec{P}_- \in \mathbb{R}^N$ the vectors

$$\vec{P}_+ = (p_1^+, \ldots, p_N^+), \quad \vec{P}_- = (p_1^-, \ldots, p_N^-),$$

and by P_+^+, P_-^+, $P_-^- \in \mathbb{R}^+$ the following:

$$P_+^+ = \max\{p_1^+, \ldots, p_N^+\}, \quad P_-^+ = \max\{p_1^-, \ldots, p_N^-\}, \quad P_-^- = \min\{p_1^-, \ldots, p_N^-\}.$$

Below we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{2.8}$$

and define $P_-^\star \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^\star = \frac{N}{\sum_{i=1}^{N} 1/p_i^- - 1}, \quad P_{-,\infty} = \max\{P_-^+, P_-^\star\}.$$

We end this section by recalling two important results: a theorem concerning the embedding between $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$, and the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz.

THEOREM 2.1 [29, Theorem 1] *Suppose that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary and relation* (2.8) *is fulfilled. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty} \quad \textit{for all } x \in \overline{\Omega}, \tag{2.9}$$

the embedding

$$W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous and compact.

THEOREM 2.2 [37, Theorem 11.5] *Let X be a real infinite-dimensional Banach space and $J \in C^1(X;\mathbb{R})$ a functional satisfying the Palais–Smale condition* (*i.e. any sequence $(u_n)_n \subset X$ such that $(J(u_n))_n$ is bounded and $J'(u_n) \to 0$ admits a convergent subsequence*). *Assume that J satisfies*:

(i) *$J(0) = 0$ and there are constants ρ, $\alpha > 0$ such that*

$$J_{|\partial B_\rho} \geq \alpha,$$

(ii) *J is even and*

(iii) *for all finite-dimensional subspaces $\widetilde{X} \subset X$ there exists $R = R(\widetilde{X}) > 0$ such that*

$$J(u) \leq 0 \quad \textit{for } u \in \widetilde{X} \setminus B_R(\widetilde{X}).$$

Then J possesses an unbounded sequence of critical values characterized by a minimax argument.

3. The main result

In this section we establish the existence of multiple weak solutions to problem (1.1). We start by giving the following definition.

Definition 3.1 By a *weak solution* to problem (1.1) we understand a function $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\int_\Omega \left[\sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \, \partial_{x_i} \varphi - f(x,u)\varphi \right] \mathrm{d}x = 0$$

for all $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

THEOREM 3.2 *Suppose that f satisfies the following*:

(f_1) *f is odd in t, i.e.* $f(x,-t)=-f(x,t)$ *for almost all* $x\in\Omega$ *and all* $t\in\mathbb{R}$

(f_2) *there exist a positive constant C and* $q\in C(\overline{\Omega})$, *with* $2\le P_-^- < P_+^+ < q^- < q^+ < P_-^\star$, *such that f satisfies the growth condition*

$$|f(x,t)|\le C|t|^{q(x)-1}$$

for almost all $x\in\Omega$ *and all* $t\in\mathbb{R}$;

(f_3) *there exists a constant* $\mu > P_+^+$ *such that for almost all* $x\in\Omega$ *and all* $t>0$

$$0<\mu F(x,t)\le tf(x,t).$$

Then problem (1.1) *admits an unbounded sequence of weak solutions.*

Remark 1 Since f is odd in t, so that F is even in t, the relation described by (f_3) remains valid for all $t\in\mathbb{R}$.

Throughout this article we work under the conditions (f_1)–(f_3) of Theorem 3.2. Moreover, for simplicity, we denote by E the anisotropic variable exponent space $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

The energy functional corresponding to problem (1.1) is defined as $I\colon E\to\mathbb{R}$,

$$I(u)=\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\mathrm{d}x-\int_\Omega F(x,u)\mathrm{d}x. \tag{3.1}$$

By standard arguments, $I\in C^1(E,\mathbb{R})$ and the Fréchet derivative is given by

$$\langle I'(u),v\rangle=\int_\Omega\left[\sum_{i=1}^N|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\,\partial_{x_i}v-f(x,u)v\right]\mathrm{d}x$$

for all $u,v\in E$. Thus the weak solutions of (1.1) coincide with the critical points of I.

We divide the proof of Theorem 3.2 in three auxiliary lemmas so that, at the end, by simply combining these lemmas with Theorem 2.2, we get the desired result.

LEMMA 3.3 *The energy functional I satisfies the Palais–Smale condition.*

Proof Let $(u_n)_n\subset E$ be a sequence such that

$$|I(u_n)|<M,\quad \forall n\ge 1, \tag{3.2}$$

where M is a positive constant, and

$$I'(u_n)\to 0\quad\text{as } n\to\infty. \tag{3.3}$$

We claim that $(u_n)_n$ is bounded. Arguing by contradiction, we assume that, passing eventually to a subsequence still denoted by $(u_n)_n$,

$$\|u_n\|_{\vec{p}(\cdot)}\to\infty\quad\text{as } n\to\infty. \tag{3.4}$$

Using relations (3.2)–(3.4) and (f_3), we obtain

$$\begin{aligned}1 + M + \|u_n\|_{\vec{p}(\cdot)} &\geq I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\mu}\right)\sum_{i=1}^{N}\int_\Omega \left|\partial_{x_i}u_n\right|^{p_i(x)}\mathrm{d}x \\ &\quad - \int_\Omega\left[F(x,u_n) - \frac{1}{\mu}u_n f(x,u_n)\right]\mathrm{d}x \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\mu}\right)\sum_{i=1}^{N}\int_\Omega \left|\partial_{x_i}u_n\right|^{p_i(x)}\mathrm{d}x.\end{aligned}$$

For each n, let us denote by $\mathcal{I}_{n_1}$, $\mathcal{I}_{n_2}$ the indices sets

$$\mathcal{I}_{n_1} = \{i \in \{1, 2, \ldots N\} : |\partial_{x_i}u_n|_{p_i(\cdot)} \leq 1\}$$

and

$$\mathcal{I}_{n_2} = \{i \in \{1, 2, \ldots N\} : |\partial_{x_i}u_n|_{p_i(\cdot)} > 1\}.$$

From (2.2)–(2.4) and the previous inequality we deduce

$$\begin{aligned}1 + M + \|u_n\|_{\vec{p}(\cdot)} &\geq \left(\frac{1}{P_+^+} - \frac{1}{\mu}\right)\left(\sum_{i\in\mathcal{I}_{n_1}}\left|\partial_{x_i}u_n\right|_{p_i(\cdot)}^{P_+^+} + \sum_{i\in\mathcal{I}_{n_2}}\left|\partial_{x_i}u_n\right|_{p_i(\cdot)}^{P_-^-}\right) \\ &= \left(\frac{1}{P_+^+} - \frac{1}{\mu}\right)\left[\sum_{i=1}^{N}\left|\partial_{x_i}u_n\right|_{p_i(\cdot)}^{P_-^-} - \sum_{i\in\mathcal{I}_{n_1}}\left(\left|\partial_{x_i}u_n\right|_{p_i(\cdot)}^{P_-^-} - \left|\partial_{x_i}u_n\right|_{p_i(\cdot)}^{P_+^+}\right)\right] \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\mu}\right)\left(\sum_{i=1}^{N}\left|\partial_{x_i}u_n\right|_{p_i(\cdot)}^{P_-^-} - N\right).\end{aligned}$$

Applying the Jensen inequality to the convex function $a : \mathbb{R}^+ \to \mathbb{R}^+$, $a(t) = t^{P_-^-}$, $P_-^- \geq 2$ (or using the generalized mean inequality), we find that

$$1 + M + \|u_n\|_{\vec{p}(\cdot)} \geq \left(\frac{1}{P_+^+} - \frac{1}{\mu}\right)\left(\frac{\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- - 1}} - N\right).$$

Dividing the above relation by $\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}$ and passing to the limit as $n \to \infty$ we obtain a contradiction.

Thus (u_n) is bounded in E, and, since E is reflexive, there exists a $u_0 \in E$ such that, up to a subsequence, (u_n) converges weakly to u_0 in E. Next, we show that (u_n) converges strongly to u_0 in E.

Note that q given in (f_2) fulfils (2.9), hence Theorem 2.1 yields that the embedding $E \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact. Thus (u_n) converges strongly to u_0 in $L^{q(\cdot)}(\Omega)$. In addition, (f_2) and the Hölder-type inequality (2.1) yield

$$\left|\int_\Omega f(x,u_n)(u_n - u_0)\mathrm{d}x\right| \leq 2C\left||u_n|^{q(x)-1}\right|_{\frac{q(\cdot)}{q(\cdot)-1}}|u_n - u_0|_{q(\cdot)}. \tag{3.5}$$

Since

$$\lim_{n\to\infty}|u_n|_{q(\cdot)} \neq \infty$$

and

$$\lim_{n\to\infty}\rho_{\frac{q(\cdot)}{q(\cdot)-1}}\left(|u_n|^{q(x)-1}\right)=\lim_{n\to\infty}\int_\Omega\left||u_n|^{q(x)-1}\right|^{\frac{q(x)}{q(x)-1}}\mathrm{d}x=\lim_{n\to\infty}\int_\Omega|u_n|^{q(x)}\mathrm{d}x$$
$$=\lim_{n\to\infty}\rho_{q(\cdot)}(u_n),$$

by (2.5) we get

$$\lim_{n\to\infty}\left||u_n|^{q(x)-1}\right|_{\frac{q(\cdot)}{q(\cdot)-1}}\neq\infty. \tag{3.6}$$

Taking into account (3.5), (3.6) and the strong convergence of (u_n) to u_0 in $L^{q(\cdot)}(\Omega)$, we arrive at

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u_0)\mathrm{d}x=0.$$

Keeping in mind the above relation and relying on the fact that, by (3.3),

$$\lim_{n\to\infty}\langle I'(u_n),u_n-u_0\rangle=0,$$

where $\langle I'(u_n),u_n-u_0\rangle=\int_\Omega\sum_{i=1}^N\left|\partial_{x_i}u_n\right|^{p_i(x)-2}\partial_{x_i}u_n(\partial_{x_i}u_n-\partial_{x_i}u_0)\mathrm{d}x-\int_\Omega f(x,u_n)(u_n-u_0)\mathrm{d}x$, we obtain

$$\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega\left|\partial_{x_i}u_n\right|^{p_i(x)-2}\partial_{x_i}u_n\left(\partial_{x_i}u_n-\partial_{x_i}u_0\right)\mathrm{d}x=0. \tag{3.7}$$

Furthermore, (u_n) converges weakly to u_0 in E, thus

$$\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega\left|\partial_{x_i}u_0\right|^{p_i(x)-2}\partial_{x_i}u_0\left(\partial_{x_i}u_n-\partial_{x_i}u_0\right)\mathrm{d}x=0. \tag{3.8}$$

From (3.7) and (3.8) we infer that

$$\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega\left(\left|\partial_{x_i}u_n\right|^{p_i(x)-2}\partial_{x_i}u_n-\left|\partial_{x_i}u_0\right|^{p_i(x)-2}\partial_{x_i}u_0\right)\left(\partial_{x_i}u_n-\partial_{x_i}u_0\right)\mathrm{d}x=0.$$

Let us recall now a very useful inequality [42, formula (2.2)]:

$$(|\xi_i|^{r_i-2}\xi_i-|\psi_i|^{r_i-2}\psi_i)\cdot(\xi_i-\psi_i)\geq 2^{-r_i}|\xi_i-\psi_i|^{r_i},\quad \xi_i,\psi_i\in\mathbb{R},$$

valid for all $r_i\geq 2$. We replace ξ_i by $\partial_{x_i}u_n$, ψ_i by $\partial_{x_i}u_0$ and r_i by $p_i(x)$ for each $i\in\{1,2,\ldots N\}$ and $x\in\Omega$. Combining the last two relations with (2.6), we find that (u_n) converges strongly to u_0 in E, in other words I satisfies the Palais–Smale condition. ■

LEMMA 3.4 *There exist $\rho,\alpha>0$ such that $I(u)\geq\alpha>0$ for any $u\in E$, with $\|u\|_{\vec{p}(\cdot)}=\rho$.*

Proof Let $u\in E$ be such that $\|u\|_{\vec{p}(\cdot)}=\rho<1$, where ρ is a positive small number which will be conveniently chosen later. Hence $|\partial_{x_i}u|_{p_i(\cdot)}<1$ and, by (2.4),

$$\int_\Omega\sum_{i=1}^N\frac{\left|\partial_{x_i}u\right|^{p_i(x)}}{p_i(x)}\,\mathrm{d}x\geq\frac{1}{P_+^+}\sum_{i=1}^N\left|\partial_{x_i}u\right|_{p_i(\cdot)}^{P_+^+}$$

for all $u\in E$, with $\|u\|_{\vec{p}(\cdot)}<1$.

Using again the Jensen inequality (applied to the convex function $b:\mathbb{R}^+ \to \mathbb{R}^+$, $b(t) = t^{P_+^+}$, $P_+^+ \geq 2$) or the generalized mean inequality, we obtain

$$\sum_{i=1}^{N} \left|\partial_{x_i} u\right|_{p_i(\cdot)}^{P_+^+} \geq N \left(\frac{\sum_{i=1}^{N} \left|\partial_{x_i} u\right|_{p_i(\cdot)}}{N} \right)^{P_+^+}.$$

Combining the previous two relations, we get

$$\int_\Omega \sum_{i=1}^{N} \frac{\left|\partial_{x_i} u\right|^{p_i(x)}}{p_i(x)} \, \mathrm{d}x \geq \frac{\|u\|_{\vec{p}(\cdot)}^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} \tag{3.9}$$

for all $u \in E$, with $\|u\|_{\vec{p}(\cdot)} < 1$.

Let us fix now an arbitrary $x \in \overline{\Omega}$. If $u(x) \geq 0$, then, by (f_2),

$$\int_\Omega F(x, u(x)) \mathrm{d}x \leq \int_\Omega \int_0^{u(x)} \left(C t^{q(x)-1} \right) \mathrm{d}t \, \mathrm{d}x \leq \frac{C}{q^-} \int_\Omega |u(x)|^{q(x)} \mathrm{d}x.$$

If $u(x) \leq 0$, then, using the fact that F is even in t and (f_2),

$$\int_\Omega F(x, u(x)) \mathrm{d}x = \int_\Omega F(x, -u(x)) \mathrm{d}x \leq \int_\Omega \int_0^{-u(x)} \left(C t^{q(x)-1} \right) \mathrm{d}t \, \mathrm{d}x \\ \leq \frac{C}{q^-} \int_\Omega |u(x)|^{q(x)} \mathrm{d}x.$$

Therefore, we can conclude that, for any $x \in \overline{\Omega}$,

$$\int_\Omega F(x, u(x)) \mathrm{d}x \leq \frac{C}{q^-} \int_\Omega |u(x)|^{q(x)} \mathrm{d}x. \tag{3.10}$$

From

$$|u(x)|^{q^-} + |u(x)|^{q^+} \geq |u(x))|^{q(x)} \quad \forall x \in \overline{\Omega},$$

and (3.10) we deduce

$$\int_\Omega F(x, u) \mathrm{d}x \leq \frac{C}{q^-} \int_\Omega \left(|u|^{q^-} + |u|^{q^+} \right) \mathrm{d}x,$$

thus

$$\int_\Omega F(x, u) \mathrm{d}x \leq \frac{C}{q^-} \left(|u|_{q^-}^{q^-} + |u|_{q^+}^{q^+} \right) \quad \text{for all } u \in E. \tag{3.11}$$

Applying Theorem 2.1 we have

$$E \hookrightarrow L^{q^-}(\Omega), \quad E \hookrightarrow L^{q^+}(\Omega)$$

continuously. Then there exists a positive constant C_1 such that, by (3.11),

$$\int_\Omega F(x, u) \mathrm{d}x \leq C_1 \left(\|u\|_{\vec{p}(\cdot)}^{q^-} + \|u\|_{\vec{p}(\cdot)}^{q^+} \right) \leq 2C_1 \|u\|_{\vec{p}(\cdot)}^{q^-}$$

for all $u \in E$, with $\|u\|_{\vec{p}(\cdot)} < 1$. By the above relation and (3.9) we get

$$I(u) \geq C_0 \|u\|_{\vec{p}(\cdot)}^{P_+^+} - 2C_1 \|u\|_{\vec{p}(\cdot)}^{q^-} \quad \text{for all } u \in E, \text{ with } \|u\|_{\vec{p}(\cdot)} < 1,$$

where $C_0 = 1/P_+^+ N^{P_+^+ - 1}$. Thus,

$$I(u) \geq \|u\|_{\vec{p}(\cdot)}^{P_+^+} \left(C_0 - 2C_1 \|u\|_{\vec{p}(\cdot)}^{q^- - P_+^+} \right) \quad \text{for all } u \in E, \text{ with } \|u\|_{\vec{p}(\cdot)} < 1.$$

Let $g : [0, 1] \to \mathbb{R}$ be the function, defined by

$$g(t) = C_0 - 2C_1 t^{q^- - P_+^+}.$$

Clearly g is positive in a neighbourhood of the origin, so that the choice of $\rho \in (0, 1)$ is so small that $\alpha = \rho g(\rho) > 0$ and this completes the proof of the lemma. ■

Lemma 3.5 *For any finite-dimensional subspace $\widetilde{E} \subset E$ there exists $R = R(\widetilde{E}) > 0$ such that*

$$I(u) \leq 0 \quad \textit{for all } u \in \widetilde{E} \setminus B_R(\widetilde{E}). \tag{3.12}$$

Proof Let $\widetilde{E} \subset E$ be a finite-dimensional subspace, $u \in \widetilde{E}$ and $t > 1$. Then

$$I(tu) \leq \frac{t^{P_+^+}}{P_-^-} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)} \mathrm{d}x - \int_\Omega F(x, tu) \mathrm{d}x.$$

Keeping in mind Remark 1 we rewrite (f_3):

$$\frac{\mu}{s} \leq \frac{f(x, s)}{F(x, s)}, \quad \forall s \neq 0,$$

and, by integrating with respect to s, we obtain the existence of a positive constant C_2 such that

$$F(x, s) \geq C_2 |s|^\mu, \quad \forall s \in \mathbb{R}.$$

Using (3.12) and taking into account the assumption $\mu > P_+^+$, we find

$$I(tu) \leq \frac{t^{P_+^+}}{P_-^-} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)} \,\mathrm{d}x - C_2 t^\mu \int_\Omega |u|^\mu \,\mathrm{d}x \to -\infty \quad \text{as } t \to \infty.$$

On the other hand, for all $R > 0$,

$$\sup_{\|u\|_{\vec{p}(\cdot)} = R, u \in \tilde{E}} I(u) = \sup_{\|tu\|_{\vec{p}(\cdot)} = R, tu \in \tilde{E}} I(tu) = \sup_{\|tu\|_{\vec{p}(\cdot)} = R, u \in \tilde{E}} I(tu)$$

and combining the above two relations we infer

$$\sup_{\|u\|_{\vec{p}(\cdot)} = R, u \in \tilde{E}} I(u) \to -\infty \quad \text{as } R \to \infty.$$

Thus, we can choose $R_0 > 0$ so large that $I(u) \leq 0$ $\forall R \geq R_0$ and $\forall u \in \widetilde{E}$ with $\|u\|_{\vec{p}(\cdot)} = R$, that is

$$I(u) \leq 0 \quad \text{for all } u \in \widetilde{E} \setminus B_{R_0},$$

as desired. ■

Proof of Theorem 3.2 The fact that F is even in t implies that I is even. Since $I(0) = 0$, by Lemmas 3.3–3.5 and the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz we deduce the existence of an unbounded sequence of weak solutions to problem (1.1). ■

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