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NONCOERCIVE ELLIPTIC EQUATIONS WITH SUBCRITICAL GROWTH

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Abstract. We study a class of nonlinear elliptic equations with subcritical growth and Dirichlet boundary condition. Our purpose in the present paper is threefold: (i) to establish the effect of a small perturbation in a nonlinear coercive problem; (ii) to study a Dirichlet elliptic problem with lack of coercivity; and (iii) to consider the case of a monotone nonlinear term with subcritical growth. This last feature enables us to use a dual variational method introduced by Clarke and Ekeland in the framework of Hamiltonian systems associated with a convex Hamiltonian and applied by Brezis to the qualitative analysis of large classes of nonlinear partial differential equations. Connections with the mountain pass theorem are also made in the present paper.

1. **Introduction.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Let 2^* denote the critical Sobolev exponent, that is, $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = +\infty$ if $N \in \{1, 2\}$. Throughout this paper we denote by λ_1 the first eigenvalue of the Laplace operator $(-\Delta)$ in $H_0^1(\Omega)$.

In this paper we are concerned with the nonlinear elliptic problem

$$\begin{cases} -\Delta u - \lambda u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a real parameter. The function $f \in C^1(\mathbb{R}, \mathbb{R})$ is assumed to satisfy the following hypotheses:

$$f(0) = f'(0) = 0; \tag{2}$$

$$|f(u)| \leq C(1 + |u|^p) \qquad \text{for all } u \in \mathbb{R}, \text{ where } 1 < p < 2^* - 1; \tag{3}$$

and there exists $\mu > 2$ such that for all $u > 0$ large enough,

$$0 < \mu F(u) \leq u f(u), \qquad \text{where } F(u) := \textstyle\int_0^u f(t)dt. \tag{4}$$

Under these assumptions, Ambrosetti and Rabinowitz [1] proved that problem (1) has at least one *positive* solution, provided that $\lambda < \lambda_1$. We are looking for weak solutions of (1), that is, $u \in H_0^1(\Omega)$ with $u > 0$ a.e. in Ω and such that for all $\xi \in H_0^1(\Omega)$,

$$\int_\Omega (\nabla u \cdot \nabla \xi - \lambda u \xi) dx = \int_\Omega f(u) \xi dx \,.$$

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A standard bootstrap argument (see Gilbarg and Trudinger [12]) that combines Schauder and Hölder regularity shows that if u is a weak solution then u is a classical solution of problem (1). We refer to Rădulescu [22, pp. 4-5] for complete details of such a regularity argument.

Returning to problem (1), we point out that a solution of this problem exists provided that the linear operator $-\Delta - \lambda I$ is *coercive* in $H_0^1(\Omega)$, that is, there exists $c > 0$ such that for all $\xi \in H_0^1(\Omega)$,

$$\int_\Omega (|\nabla \xi|^2 - \lambda \xi^2) dx \geq c \, \|\xi\|^2_{H_0^1}.$$

The hypothesis $\lambda < \lambda_1$ corresponds to the existence of a mountain pass geometry near the origin for the energy functional associated to problem (1).

The same argument based on the mountain pass theorem can be extended to the case where f has an *almost critical growth*, more precisely if assumption (3) is replaced with

$$\lim_{u \to +\infty} \frac{f(u)}{u^{(N+2)/(N-2)}} = 0,$$

provided that $N \geq 3$.

We observe that the *technical* assumption (4) implies a super-linear behaviour of the nonlinear term. Indeed, a straightforward computation shows that, by (4), there are C_1, $C_2 > 0$ such that

$$f(u) \geq C_1 u^{\mu - 1} - C_2, \qquad \text{for all } u \geq 0. \tag{5}$$

To the best of our knowledge, it is not known whether the above existence result for problem (1) still remains true if assumption (4) is replaced under the weaker hypothesis (5).

In the limiting case $p = 1$ in hypothesis (3) that corresponds to a linear growth of f, we have provided in [23] a sufficient condition for the existence of a nontrivial solution to problem (1). This corresponds to a weak slope of f near the origin, combined with a faster (linear) growth of f in a neighbourhood of $+\infty$.

We refer to the seminal paper by Brezis and Nirenberg [5] for several related existence and nonexistence results.

2. **The main results.** In the present paper we have two objectives. We are first interested in the effect of a certain *perturbation* in problem (1). More precisely, we consider the problem

$$\begin{cases} -\Delta u - \lambda u = f(u) + h(x) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $h \in L^\infty(\Omega)$. We observe that if the perturbation h is *big* with respect to a suitable topology, then problem (6) does not have any solution. Indeed, let ϕ_1 be a positive eigenfunction of $(-\Delta)$ corresponding to λ_1. By multiplication in (6) with ϕ_1 and integration by parts we obtain

$$\int_\Omega h\phi_1 dx = \int_\Omega \left[(\lambda_1 - \lambda)u - f(u)\right] \phi_1 dx.$$

Thus, by (5),

$$\int_\Omega h\phi_1 dx \leq \int_\Omega \left[(\lambda_1 - \lambda)u - C_1 u^{\mu-1} + C_2\right] \phi_1 dx \leq M \int_\Omega \phi_1 dx, \tag{7}$$

where

$$M := \max\{(\lambda_1 - \lambda)t - C_1 t^{\mu-1} + C_2;\ t \in [0,\infty)\}.$$

Since $\mu > 2$ and $\lambda < \lambda_1$, it follows that $M \in (0,\infty)$. Returning to relation (7), we conclude that problem (6) cannot have any solution if h is *nonnegative* and $\|h\|_{L^\infty}$ is large enough. We refer to Zheng [25] for a related nonexistence result in the case where h is *negative* and $\|h\|_{L^\infty}$ is sufficiently large.

In contrast, our first main result establishes that if the perturbation is *small* with respect to the L^∞–topology, then problem (6) admits solution.

Theorem 2.1. *Assume $\lambda < \lambda_1$ and hypotheses (2)–(4) are fulfilled. Then there exists $\delta > 0$ such that for all $h \in L^\infty(\Omega)$ with $\|h\|_{L^\infty} < \delta$, problem (6) has at least one solution.*

The proof of Theorem 2.1 combines the mountain pass theorem with an argument by contradiction. For a somewhat related result we refer to Rădulescu [22, Theorem 5.8].

The second objective in this paper concerns the existence of solutions to problem (1) in the case where $\lambda \geq \lambda_1$, that is, provided that the operator $-\Delta - \lambda I$ is no longer coercive in $H^1_0(\Omega)$.

By multiplication with ϕ_1 in (1) we obtain

$$(\lambda_1 - \lambda)\int_\Omega u\phi_1 dx = \int_\Omega f(u)\phi_1 dx.$$

This shows that problem (1) cannot have *positive* solutions if $\lambda \geq \lambda_1$ and $f > 0$ on $(0,\infty)$. That is why we impose the following additional hypothesis:

$$f : \mathbb{R} \to \mathbb{R} \text{ is strictly increasing and onto.} \tag{8}$$

Under assumptions (2)–(4) and (8), problem (1) cannot have a *positive* solution. Indeed, by multiplication with ϕ_1 in (1), we obtain

$$(\lambda_1 - \lambda)\int_\Omega u\phi_1 dx = \int_\Omega f(u)\phi_1 dx\,,$$

which is not possible if $u > 0$ in Ω. This shows that a natural question is to see if, under the same assumptions, problem (1) admits a nontrivial *sign-changing* solution. The answer is *positive*, as shown by the next result.

Theorem 2.2. *Assume $\lambda \geq \lambda_1$ and hypotheses (2)–(4) and (8) are fulfilled. Then problem (1) has at least one nontrivial solution.*

The proof of this result uses a dual variational method of Clarke and Ekeland [7, 8, 9], which was initially introduced in the framework of Hamiltonian systems associated with a convex Hamiltonian. Toland [24] had introduced a related method for the study of variational problems involving the difference of two convex functions. We also refer to Brezis [4] for several applications of the dual variational principle to the qualitative analysis of nonlinear partial differential equations.

In the proofs of Theorems 2.1 and 2.2 we apply some methods developed in Brezis [4] and Brezis and Nirenberg [5]

3. **Proof of Theorem 2.1.** Arguing by contradiction, there is a sequence $(h_m)_{m\geq 1}$ in $L^\infty(\Omega)$ with $\|h_m\|_{L^\infty} \to 0$ such that for all $m \geq 1$, the problem

$$\begin{cases} -\Delta u_m - \lambda u_m = f(u_m) + h(u_m) & \text{in } \Omega \\ u_m > 0 & \text{in } \Omega \\ u_m = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

does not have any solution.

Set $F(t) := \int_0^t f(s)ds$. The energy functional associated to problem (9) is

$$\mathcal{E}_m(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx - \int_\Omega F(u^+)dx - \int_\Omega h_m u dx \qquad \text{for all } u \in H_0^1(\Omega),$$

where $u^+ := \max\{u, 0\}$.

We use in the proof the mountain pass theorem of Ambrosetti and Rabinowitz [1] in the following form.

Mountain Pass Theorem. *Assume Ψ is a C^1 function on a Banach space E and satisfies the Palais-Smale condition, that is,*

$$\begin{cases} \text{whenever a sequence } (v_m) \text{ in } E \text{ satisfies } |\Psi(v_m)| \le C \text{ and } \Psi'(v_m) \to 0 \text{ in } E^*, \\ \text{there exists a subsequence of } (v_m) \text{ that converges in } E. \end{cases}$$

Assume also the following geometric hypotheses are fulfilled:

$$\begin{cases} \text{there are constants } \delta > 0 \text{ and } c_0 > 0 \text{ such that } \Psi(v) \ge c_0 \\ \text{for every } v \in E \text{ with } \|v\| = \delta \end{cases} \tag{10}$$

and

$$\Psi(0) < c_0 \quad \text{and} \quad \Psi(v_0) < c_0 \quad \text{for some } v_0 \in E \text{ with } \|v_0\| > \delta. \tag{11}$$

Then there is a critical point v of Ψ such that $\Psi(v) \ge c_0$.

We check that $\mathcal{E}_m$ fulfills the geometric assumptions of the mountain pass theorem. We first observe that for all $u \in H_0^1(\Omega)$,

$$\mathcal{E}_m(u) \ge \frac{\lambda_1 - \lambda}{2\lambda_1}\|u\|^2 - \int_\Omega F(u^+)dx - \|h_m\|_{L^\infty}\int_\Omega |u|dx.$$

Fix $\varepsilon > 0$. Thus, by (2) and (3), there exists $C(\varepsilon) > 0$ such that for all $t \in \mathbb{R}$,

$$|f(t)| \le \varepsilon |t| + C(\varepsilon)|t|^p,$$

hence

$$|F(t)| \le \frac{\varepsilon}{2}t^2 + \frac{C(\varepsilon)}{p+1}|t|^{p+1}.$$

It follows that

$$\begin{aligned} \mathcal{E}_m(u) \quad &\ge \frac{\lambda_1 - \lambda}{2\lambda_1}\|u\|^2 - \frac{\varepsilon}{2}\int_\Omega (u^+)^2 dx - \frac{C(\varepsilon)}{p+1}\int_\Omega (u^+)^{p+1}dx - \|h_m\|_{L^\infty}\int_\Omega |u|dx \\ &\ge C_1\|u\|^2 - C_2\varepsilon\|u\|^2 - C_3(\varepsilon)\|u\|_{L^{p+1}}^{p+1} - C_4\|h_m\|_{L^\infty}\|u\|. \end{aligned}$$

Thus, if $\|u\| = \delta > 0$ is small enough, then

$$\mathcal{E}_m(u) \ge c_0 > 0.$$

In the above argument we have also used Sobolev embeddings and the assumption $\|h_m\|_{L^\infty} \to 0$ as $m \to \infty$.

Next, for all $m \ge 1$,

$$\begin{aligned} \mathcal{E}_m(t\phi_1) \quad &= \frac{(\lambda_1 - \lambda)t^2}{2}\int_\Omega \phi_1^2 dx - \int_\Omega F(t\phi_1)dx - t\int_\Omega h_m\phi_1 dx \\ &\le C_1 t^2 - \int_\Omega (C_2 t^\mu \phi_1^\mu - C_3)dx - t\int_\Omega h_m \phi_1 dx \\ &= -C_2 t^\mu \int_\Omega \phi_1^\mu dx + C_1 t^2 + C_3|\Omega| - O(t) \le 0, \end{aligned}$$

if $t > 0$ is large enough.

The verification of the Palais-Smale condition is standard for $\mathcal{E}_m$ and follows from the assumption $p < 2^* - 1$.

We now deduce from the mountain pass theorem that there exists $u_m \in H_0^1(\Omega) \setminus \{0\}$ such that $\mathcal{E}_m'(u_m) = 0$. The corresponding critical value is $\mathcal{E}_m(u_m) = c_m \geq c_0$. Next, by multiplication with u_m in (9) and using the assumption $\lambda < \lambda_1$ we deduce that $\sup_m \|u_m\| < +\infty$. Standard elliptic regularity arguments imply that (u_m) is bounded in $L^\infty(\Omega)$, hence in $L^p(\Omega)$ for all $1 < p < \infty$. Schauder estimates (see Brezis [3]) imply that (u_m) is bounded in $W^{2,p}(\Omega)$ for all $p < \infty$. By Sobolev embeddings, (u_m) is bounded in $C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$. Thus, up to a subsequence, $u_m \to u$ in $C^{1,\alpha}(\overline{\Omega})$. Taking $m \to \infty$ we obtain

$$\begin{cases} -\Delta u - \lambda u = f(u^+) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Moreover, the corresponding critical level is positive, that is,

$$\frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx - \int_\Omega F(u^+)dx \geq c_0 > 0.$$

This shows that $u \not\equiv 0$. Applying now the maximum principle in (12), we deduce that $u > 0$ in Ω. Since $u_m \to u$ in $C^{1,\alpha}(\overline{\Omega})$, we obtain that $u_m > 0$ in Ω provided that m is large enough. Consequently, u_m is a solution of problem (9) corresponding to $h = h_m$. This contradicts our assumption that problem (9) does not have any solution. The proof of Theorem 2.1 is now complete. □

4. **Proof of Theorem 2.2.** The basic assumption $\lambda \geq \lambda_1$ implies that the geometric assumption (10) in the statement of the mountain pass theorem is no longer fulfilled. That is why we use an idea found in the proof of Theorem 4 in Brezis [4], which relies on the assumption that the nonlinear term f is one-to-one and onto. We introduce a new unknown $v = f(u)$ and we prove that the energy functional associated to v fulfills the hypotheses of the mountain pass theorem.

The solutions of problem (1) correspond to the critical points of the energy functional

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx - \int_\Omega F(u)dx, \qquad u \in H_0^1(\Omega).$$

We first consider the case where the linear operator $(-\Delta - \lambda I)$ is invertible, hence λ is not an eigenvalue of the Laplace operator. Thus, problem (1) can be written, equivalently,

$$u = (-\Delta - \lambda I)^{-1} f(u).$$

Set $T = (-\Delta - \lambda I)^{-1}$ and denote $f(u) = v$, hence $u = g(v)$ with $g = f^{-1}$. Conditions (3) and (5) imply that $g \in L^r(\Omega)$, with $(N-2)/(N+2) < r < 1$ if $N \geq 3$ and $0 < r < 1$ if $N \in \{1, 2\}$. Thus, T is a compact operator from $L^{r+1}(\Omega)$ into $L^{(r+1)'}(\Omega)$. This follows from the fact that T maps $L^{r+1}(\Omega)$ into $W^{2,r+1}(\Omega)$ combined with the compact embedding $W^{2,r+1}(\Omega) \subset L^{(r+1)'}(\Omega)$.

Using the dual formulation we have to find a nontrivial critical point for the functional

$$\mathcal{J}(v) := \int_\Omega G(v)dx - \frac{1}{2}\int_\Omega vTv dx, \qquad G(y) := \int_0^y g(t)dt\,.$$

The above remarks show that $\mathcal{J}$ is well-defined and of class C^1 on the space $L^{r+1}(\Omega)$. Moreover, for all $v,\, \xi \in L^{r+1}(\Omega)$,

$$\mathcal{J}'(v)(\xi) = \int_\Omega g(v)\xi dx - \int_\Omega \xi T v dx.$$

We prove in what follows that $\mathcal{J}$ satisfies the hypotheses of the mountain pass theorem. We first observe that

$$\mathcal{J}(v) \geq C_1 \|v\|_{L^{r+1}}^{r+1} - C_2 \|v\|_{L^1}^2 \geq C_1 \|v\|_{L^{r+1}}^{r+1} - C_3 \|v\|_{L^{r+1}}^2 .$$

Since $r+1<2$, this estimate shows that there are positive constants R and ρ such that $\mathcal{J}(v) \geq \rho$ for all $v \in L^{r+1}(\Omega)$ with $\|v\|_{L^{r+1}} = R$. This shows that condition (10) is fulfilled.

Let $\lambda_k > \lambda$ be an eigenvalue of $(-\Delta)$ and let φ be a corresponding eigenfunction. Choose $v_0 = t\varphi$, with $t>0$. Thus, since $r+1<2$,

$$\mathcal{J}(v_0) \leq C_1 t^{r+1} \int_\Omega |\varphi|^{r+1} dx + C_2 - t^2(\lambda_k - \lambda) \int_\Omega |T\varphi|^2 dx < 0,$$

provided that $t>0$ is large enough. This implies that condition (11) is fulfilled.

We prove in what follows that $\mathcal{J}$ satisfies the Palais–Smale compactness condition. Let $(v_n) \subset L^{r+1}(\Omega)$ be an arbitrary Palais–Smale sequence for $\mathcal{J}$. Since $\mathcal{J}'(v_n) \to 0$ in $L^{(r+1)'}(\Omega)$ we have

$$g(v_n) - Tv_n = w_n \to 0 \qquad \text{in } L^{(r+1)'}(\Omega).$$

Therefore

$$\frac{1}{2}\int_\Omega g(v_n)v_n dx - \frac{1}{2}\int_\Omega v_n\, Tv_n dx = \frac{1}{2}\int_\Omega v_n w_n dx = o(1) \qquad \text{as } n \to \infty. \tag{13}$$

On the other hand, since $\sup_n |\mathcal{J}(v_n)| < +\infty$ we have

$$\int_\Omega G(v_n) dx - \frac{1}{2}\int_\Omega v_n\, Tv_n dx = O(1) \qquad \text{as } n \to \infty. \tag{14}$$

Since $G(y) = yg(y) - F(g(y))$, relations (13) and (14) yield

$$\int_\Omega \left[g(v_n)v_n - 2F(g(v_n))\right] dx = O(1) \qquad \text{as } n \to \infty.$$

Using now assumption (4) we deduce that

$$\int_\Omega g(v_n)v_n = O(1) \qquad \text{as } n \to \infty.$$

This implies that the sequence (v_n) is bounded in $L^{r+1}(\Omega)$. Thus, up to a subsequence, (v_n) converges weakly to v in $L^{r+1}(\Omega)$. Using now the compactness of the operator T we obtain $Tv_n \to Tv$ in $L^{(r+1)'}(\Omega)$, hence $v_n \to v$ in $L^{r+1}(\Omega)$. We conclude that $\mathcal{J}$ satisfies the Palais–Smale condition.

We now deduce from the mountain pass theorem that there exists $v \in L^{r+1}(\Omega)$, $v \neq 0$, such that $\mathcal{J}'(v) = 0$. Then $u = g(v)$ is a nontrivial solution of problem (1). A standard bootstrap argument implies that $u \in H_0^1(\Omega)$.

It remains to treat the case where the operator $(-\Delta - \lambda I)$ is not invertible. This corresponds to a resonant problem, according to Landesman and Lazer [14]. We reduce this framework to the previous one and we try to put into evidence the same operator T in a suitable function space.

Since $(-\Delta - \lambda I)$ is not invertible, then λ is an eigenvalue of the Laplace operator. Let E be the finite dimensional space of eigenfunctions corresponding to λ. Then $H_0^1(\Omega) = E \oplus E^\perp$, where

$$E^\perp := \left\{ w \in H_0^1(\Omega);\ \int_\Omega w\xi dx = 0 \text{ for all } \xi \in E \right\}.$$

We first observe that if u is a solution of (1) then for all $\xi \in E$,

$$\int_\Omega [(-\Delta u)\xi - \lambda u \xi]dx = \int_\Omega f(u)\xi dx.$$

Since $\xi \in E$ then $-\Delta\xi = \lambda\xi$ in Ω. Thus, by Green's formula, $\int_\Omega f(u)\xi dx = 0$, hence $v = f(u) \in E^\perp$. We now consider $T = (-\Delta - \lambda I)^{-1}$ as a linear continuous operator from $E^\perp$ into itself. This operator is well–defined and our problem (1) becomes

$$\begin{cases} v \in E^\perp \\ Tv - g(v) \in E. \end{cases} \tag{15}$$

Conversely, we observe that if v solves problem (15) then $u = g(v)$ is a solution of (1). This means that we can repeat the argument developed in the first part of the proof for the energy functional $\mathcal{J}$ but defined this time on $E^\perp$. This completes the proof of Theorem 2.2. □

Open problem. We do not know if the result established in Theorem 2.2 still remains true if assumption (8) is removed.

Further comments. The original proof of Ambrosetti and Rabinowitz [1] of the mountain pass theorem relies on some deep deformation techniques developed by Palais and Smale [16, 17], who put the main ideas of the Morse theory into the framework of differential topology on infinite dimensional manifolds. Brezis and Nirenberg provided in [5] a simpler proof which combines two major tools: Ekeland's variational principle and the pseudo-gradient lemma. Ekeland's variational principle is the nonlinear version of the Bishop–Phelps theorem and it may be also viewed as a generalization of Fermat's theorem. Relevant extensions of the mountain pass theorem are due to Pucci and Serrin [19, 20, 21] and Ghoussoub and Preiss [11]. For applications of the mountain pass theorem to nonlinear partial differential equations we refer to Bonanno and Marano [2], Carl and Motreanu [6], Filippucci, Pucci and Robert [10], Kristály, Rădulescu and Varga [13], and Marano and Motreanu [15]. We also point the recent paper by Pucci and Rădulescu [18] for a survey on the mountain pass theorem.

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