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Nonlocal Kirchhoff Superlinear Equations with Indefinite Nonlinearity and Lack of Compactness

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Abstract: We study the following Kirchhoff equation:

$$-\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u),\quad x\in\mathbb{R}^3.$$

A feature of this paper is that the nonlinearity f and the potential V are indefinite, hence sign-changing. Under some appropriate assumptions on V and f, we prove the existence of two different solutions of the equation via the Ekeland variational principle and the mountain pass theorem.

Keywords: mountain pass, Ekeland variational principle, nonlocal Kirchhoff equation

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1 Introduction

In this paper we consider the following Kirchhoff equation

$$-\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u),\ x\in\mathbb{R}^3,\tag{K}$$

where b is a positive constant, the potential V and the nonlinearity f are allowed to be sign-changing.

Equation (K) is a modified version of the classical Kirchhoff equation, which has a strong physical meaning. Problem (K) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt}-\left(1+b\int_{\Omega}|\nabla_x u|^2dx\right)\Delta_x u=g(x,u)\tag{1}$$

which was proposed by Kirchhoff [1]. The early classical studies of the Kirchhoff equation were made by Bernstein [2] and Pohozaev [3]. However, eq. (1) received great attention only after Lions [4] proposed an abstract framework for the problem.

The Kirchhoff equation is a generalization of the d'Alembert wave equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=g(x,u)$$

for free vibrations of the elastic string. Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of its cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. It was pointed out in ref. [5] that eq. (1) models various physical phenomena, where u describes a process that depends on the average of itself. Nonlocal effects also arise in the description of biological systems. A parabolic version of problem (1) can be used to describe the growth and movement of some species. In this case, the integral term models the movement, which is assumed to be dependent on the energy of the entire system with the unknown u being its population density.

We focus on the Euclidean space 3-space with lack of compactness, since the Sobolev embedding is not compact for the whole space. A natural idea is study this equation on the radial space. Interested reader can consult the refs [6–11]. Recently, Wu [12] has studied this type of equations with positive coercive potential V. Four new existence results for nontrivial solutions and a sequence of high energy solutions for problem (K) were obtained by using a symmetric mountain pass theorem. Actually, coercive potential V was introduced by Rabinowitz [13] (see also [14]) to overcome the lack of compact Sobolev embedding. Later, many authors [15–26] used this type of potential. Very recently, the case when the potential V vanishes at some points has also been considered [27–31]. We also refer to the related papers [32–36] and the monograph [37], which deals with variational methods for nonlocal fractional equations.

In some of the aforementioned references, the potential V is always assumed to be positive or vanishing at infinity. The following technical Ambrosetti–Rabinowitz condition ((AR) for short) is usually required.

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(AR) There exists $\mu > 4$ such that

$$0 < \mu F(x,u) \le uf(x,u), \quad u \ne 0.$$

The role of (AR) is to ensure the boundedness of the Palais–Smale (PS) sequences of the energy functional, which is crucial in applying the critical point theory.

Motivated by the works [38, 39], we consider in this paper another case, namely that of f being superlinear, that is, $f(x,u)/u \to +\infty$ as $u \to \infty$. Furthermore, the potential V and the primitive of f are also allowed to be sign-changing, which is quite different from the previous results. Before stating our main results, we list the following assumption on $V(x)$.

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > -\infty$. Moreover, there exists a constant $d_0 > 0$ such that for any $M > 0$,

$$\lim_{|y|\to\infty} \mathrm{meas}\left(x \in \mathbb{R}^3 : |x-y| \le d_0, V(x) \le M\right) = 0,$$

where meas $(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^3$.

Inspired by Zhang and Xu [40], we can find a constant $V_0 > 0$ such that $\tilde{V}(x) := V(x) + V_0 \ge 1$ for all $x \in \mathbb{R}^3$ and let $\tilde{f}(x,u) := f(x,u) + V_0 u$, $\forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}$. Then it is easy to verify the following lemma.

Lemma 1.1: *Equation (K) is equivalent to the following problem*

$$-\left(1 + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + \tilde{V}(x)u = \tilde{f}(x,u),\ x \in \mathbb{R}^3. \qquad \text{(K')}$$

In what follows, we let $\mu > 4$ and impose some assumptions on $\tilde{f}$ and its primitive $\tilde{F}$ as follows:

(S1) $\tilde{f} \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and there exist constants c_1, $c_2 > 0$ and $q \in (4,6)$ such that

$$|\tilde{f}(x,u)| \le c_1|u|^3 + c_2|u|^{q-1}.$$

(S2) $\lim_{|u|\to\infty} \frac{|\tilde{F}(x,u)|}{|u|^4} = \infty$ a.e. $x \in \mathbb{R}^3$ and there exist constants $c_3 \ge 0$, $r_0 \ge 0$ and $\tau \in (0,2)$ such that

$$\inf_{x\in\mathbb{R}^3} \tilde{F}(x,u) \ge c_3|u|^\tau \ge 0, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R},\ |u| \ge r_0,$$

where (and in the sequel) $\tilde{F}(x,u) = \int_0^u \tilde{f}(x,s)ds$.

(S3) $\tilde{\mathcal{F}}(x,u) := \frac{1}{4}u\tilde{f}(x,u) - \tilde{F}(x,u) \ge 0$, and there exist $c_4 > 0$ and $\kappa > 1$ such that

$$|\tilde{F}(x,u)|^\kappa \le c_4|u|^{2\kappa}\tilde{\mathcal{F}}(x,u), \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R},\ |u| \ge r_0.$$

Now we state our main result as follows.

Theorem 1.2: *Suppose that conditions (V1), (S1), (S2) and (S3) are satisfied. Then problem (K) has at least two different solutions.*

Remark 1.3: There are some functions not satisfying the condition (AR). For example, the superlinear function $f(x,u) = \sin x \ln(1+|u|)u^2$ does not satisfy condition (AR). In our theorems, $\tilde{F}(x,u)$ is allowed to be sign-changing. Even if $\tilde{F}(x,u) \ge 0$, the assumptions (S2) and (S3) seem to be weaker than the superlinear conditions obtained in the aforementioned references. By straightforward computation we check that the following nonlinearity $\tilde{f}$ satisfies (S2) and (S3):

$$\tilde{f}(x,u) = a(x)(4u^4 + 2u^2 \sin u - 4u\cos u)$$

where $a \in (\mathbb{R}^3, \mathbb{R})$ and $0 < \inf_{\mathbb{R}^3} a(x) \le \sup_{\mathbb{R}^3} a(x) < \infty$.

Remark 1.4: To the best of our knowledge, the condition (V1) was first stated in ref. [41], but $\inf_{x\in\mathbb{R}^3} V(x) > 0$ was required. From (V1), one can see that the potential $V(x)$ is allowed to be sign-changing. Therefore, the condition (V1) is weaker than those in [15–31, 42].

Remark 1.5: It is not difficult to find the functions V satisfying the above conditions. For example, let $V(x)$ be a zigzag function with respect to $|x|$ defined by

$$V(x) = \begin{cases} 2n|x| - 2n(n-1) + a_0, & n-1 \le |x| \le (2n-1)/2, \\ -2n|x| + 2n^2 + a_0, & (2n-1)/2 \le |x| \le n, \end{cases}$$

where $n \in \mathbb{N}$ and $a_0 \in \mathbb{R}$.

Remark 1.6: Zhang et al. [26] studied eq. (K) with sign-changing potential V. They obtained multiple solutions in the case of odd nonlinearity. Here we do not need that the nonlinearity is odd and we also get two solutions for problem (K). Bahrouni [43] obtained infinitely many solutions for eq. (K) with the potential and nonlinearity both sign-changing. However, he studied the sublinear case and with odd nonlinearity. Here our results can be regarded as an extension of the results of [43, 26].

2 Preliminaries and variational setting

Hereafter, we use the following notation:

- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev spaces endowed with the standard scalar product and norm

$$(u,v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)dx, \quad \| u \| = (u,u)^{1/2}.$$

- $D^{1,2}(\mathbb{R}^3)$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\| u \|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

- $H = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x)|u|^2) dx < \infty \right\}$ is the Sobolev space endowed with the norm

$$\| u \|^2_H (\mathbb{R}^3) = \int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x)|u|^2) dx.$$

- H^* denotes the dual space of H.
- $L^s(\mathbb{R}^3)$, $1 \le s < +\infty$ denotes a Lebesgue space with the usual norm $\| u\|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{1/s}$.
- For any $\rho > 0$ and for any $z \in \mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z.
- C and C_i denote various positive constants, which may vary from line to line.
- S_i denote the Sobolev constant for the embedding.
- $\to$ denotes the strong convergence and $\rightharpoonup$ denotes the weak convergence.

Throughout this section, we make the following assumption instead of (V1):

(V2) $\tilde{V} \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^3} \tilde{V}(x) > 0$. Moreover, there exists a constant $d_0 > 0$ such that for any $M > 0$,

$$\lim_{|y| \to \infty} \text{meas}\left\{ x \in \mathbb{R}^3 : |x - y| \le d_0, V(x) \le M \right\} = 0.$$

Remark 2.1: Under assumptions (V2), we know by Lemma 3.1 in [41] that the embedding $HL^s(\mathbb{R}^3)$ is compact for $s \in [2, 6)$.

Let $I : H \to \mathbb{R}$ denote the energy functional defined by

$$\begin{aligned} I(u) = {} & \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x) u^2) dx \\ & + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \int_{\mathbb{R}^3} \tilde{F}(x, u) dx, \end{aligned} \tag{2}$$

for all $u \in H$. By condition (S1), we have

$$|\tilde{F}(x, u)| \le \frac{c_1}{4} |u|^4 + \frac{c_2}{q} |u|^q, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3}$$

Consequently, similar to the discussion in [12], under assumptions (V2) and eq. (3), the functional I is of class $C^1(H, \mathbb{R})$. Moreover,

$$\begin{aligned} \langle I'(u), v \rangle = {} & \left(1 + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx \\ & + \int_{\mathbb{R}^3} \tilde{V}(x) u v dx - \int_{\mathbb{R}^3} \tilde{f}(x, u) v dx. \end{aligned} \tag{4}$$

Hence, if $u \in H$ is a critical point of I, then u is a solution of eq. (K').

We now recall the mountain pass theorem of Ambrosetti and Rabinowitz [44] without the Palais–Smale condition (see also [45]). We also refer to Brezis and Nirenberg [46] for a simple proof of this result which uses the Ekeland variational principle in combination with a pseudo-gradient argument.

Lemma 2.2: *Let E be a real Banach space with its dual space $E^\star$, and suppose that $I \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \mu < \eta \le \inf_{\|u\| = \rho} I(u),$$

for some μ, η, $\rho > 0$ and $e \in E$ with $\| e \| > \rho$. Let $c \ge \eta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le \tau \le 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of all continuous paths joining 0 and e. Then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \ge \eta \text{ and } (1 + \| u_n \|) \| I'(u_n)\|_{E^\star} \to 0, \text{ as } n \to \infty.$$

This kind of sequence is usually called a Cerami sequence. Recall that a C^1 functional I satisfies the Cerami compactness condition at level c ($(C)_c$ condition for short) if any sequence $\{u_n\} \subset H$ such that $I(u_n) \to c$ and $(1 + \| u_n \|) \| I'(u_n)\|_{E^\star} \to 0$ has a convergent subsequence.

Here, we give the sketch of how to look for two distinct critical points of the functional I. First, we consider a minimization of I constrained to a neighborhood of zero via the Ekeland variational principle (see [47, 48]) and we can find a critical point of I which achieves the local minimum of I and the level of this local minimum is negative (see Step 1 of the proof of Theorem 1.2). Next, around the "zero" point, by using mountain pass theorem (see [44]), we obtain a second critical point of I with its positive level (see Step 2 of the proof of Theorem 1.2). Obviously, these two critical points do not coincide since they have different energy levels.

To prove Theorem 1.2, we cite the following auxiliary result, see [39].

Lemma 2.3: *Assume that $p_1, p_2 > 1$, $r, q \ge 1$ and $\Omega \subseteq \mathbb{R}^3$. Let $g(x, t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying*

$$|g(x, t)| \le a_1 |t|^{(p_1 - 1)/r} + a_2 |t|^{(p_2 - 1)/r}, \quad \forall (x, t) \in \Omega \times \mathbb{R},$$

where $a_1, a_2 \ge 0$. If $u_n \to u$ in $L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, and $u_n \to u$ a.e. $x \in \Omega$, then for any $v \in L^{p_1 q}(\Omega) \cap L^{p_2 q}(\Omega)$,

$$\lim_{n \to \infty} \int_\Omega |g(x, u_n) - g(x, u)|^r |v|^q dx \to 0. \tag{5}$$

3 Proof of the main result

Lemma 3.1: *Assume that the conditions (V2) and (S1) hold. Then there exist ρ, $\eta > 0$ such that $\inf\{I(u) : u \in H$ with $\| u\|_H = \rho\} > \eta$.*

Proof: By eq. (3) and the Sobolev inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \tilde{F}(x,u)dx\right| &\le \int_{\mathbb{R}^3}\left|\frac{c_1}{4}|u|^4 + \frac{c_2}{q}|u|^q\right|dx\\ &= \frac{c_1}{4}\| u \|_4^4 + \frac{c_2}{q}\| u \|_q^q \\ &\le S_4\frac{c_1}{4}\| u \|_H^4 + S_q\frac{c_2}{q}\| u \|_H^q ,\end{aligned} \tag{6}$$

for any $u \in H$. Combining eq. (2) with eq. (6), we obtain

$$\begin{aligned}I(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + \tilde{V}(x)u^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\\ &- \int_{\mathbb{R}^3}\tilde{F}(x,u)dx \ge \frac{1}{2}\| u \|_H^2 - \int_{\mathbb{R}^3}|\tilde{F}(x,u)|dx\\ &\ge \frac{1}{2}\| u \|_H^2 - S_4\frac{c_1}{4}\| u \|_H^4 - S_q\frac{c_2}{q}\| u \|_H^q\\ &= \frac{1}{2}\| u \|_H^2 - C_14\| u \|_H^4 - C_2\| u \|_H^q .\end{aligned} \tag{7}$$

Since $q \in (4,6)$, we deduce that there exists $\eta > 0$ such that this lemma holds if we let $\| u\|_H = \rho > 0$ be small enough.

Lemma 3.2: *Assume that the conditions (V2) and (S2) hold. Then there exists $v \in H$ with $\| v\|_H = \rho$ such that $I(v) < 0$, where ρ is given in Lemma 3.1.*

Proof: By eq. (2), we have

$$\frac{I(tu)}{t^4} = \frac{1}{2t^2}\| u \|_H^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 - \frac{1}{t^4}\int_{\mathbb{R}^3}\tilde{F}(x,tu)dx.$$

Then, by (S2) and Fatou's lemma we can deduce that

$$\begin{aligned}\lim_{t\to\infty}\frac{I(tu)}{t^4} &= \lim_{t\to\infty}\left[\frac{1}{2t^2}\| u \|_H^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\right.\\ &\left.- \frac{1}{t^4}\int_{\mathbb{R}^3}\tilde{F}(x,tu)dx\right] = \limsup_{t\to\infty}\left[\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\right.\\ &\left.- \frac{1}{t^4}\int_{\mathbb{R}^3}\tilde{F}(x,tu)dx\right] \le \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\\ &- \liminf_{t\to\infty}\int_{\mathbb{R}^3}\frac{\tilde{F}(x,tu)}{t^4u^4}u^4dx\\ &\le C_3\| u \|_H^4 - \int_{\mathbb{R}^3}\liminf_{t\to\infty}\frac{\tilde{F}(x,tu)}{t^4u^4}u^4dx\\ &= -\infty \text{ as } t \to \infty.\end{aligned}$$

Thus the lemma is proved by taking $v = t_0u$ with large enough t_0. □

Based on Lemmata 3.1 and 3.2, Lemma 2.2 implies that there is a sequence $\{u_n\} \subset H$ such that

$$I(u_n) \to c > 0 \text{ and} (1 + \| u_n\|_H)\| I'(u_n)\|_{H^*} \to 0, \text{as } n \to \infty. \tag{8}$$

Lemma 3.3: *Assume that the conditions (V2), (S1), (S2) and (S3) hold. Then the sequence $\{u_n\}$ defined in eq. (8) is bounded in H.*

Proof: Arguing by contradiction, we can assume $\| u_n\|_H \to \infty$. Define $v_n := \frac{u_n}{\|u_n\|_H}$. Clearly, $\| v_n\|_H = 1$ and $\| v_n\|_s \le S_s\| v_n\|_H = S_s$, for $2 \le s < 6$. Observe that for large enough n, we can get from eq. (8) and (S3) that

$$\begin{aligned}c + 1 &\ge I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle\\ &= \frac{1}{4}\| u_n \|_H^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}\tilde{f}(x,u_n)u_n - \tilde{F}(x,u_n)\right)dx\\ &\ge \int_{\mathbb{R}^3}\tilde{F}(x,u_n)dx.\end{aligned} \tag{9}$$

In view of eqs (2) and (8), we have

$$\begin{aligned}\frac{1}{2} &= \frac{I(u_n)}{\| u_n \|_H^2} + \frac{1}{\| u_n \|_H^2}\int_{\mathbb{R}^3}\tilde{F}(x,u_n)dx\\ &- \frac{b}{4\| u_n \|_H^2}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\\ &\le \frac{I(u_n)}{\| u_n \|_H^2} + \frac{1}{\| u_n \|_H^2}\int_{\mathbb{R}^3}|\tilde{F}(x,u_n)|dx\\ &\le \limsup_{n\to\infty}\left[\frac{I(u_n)}{\| u_n \|_H^2} + \frac{1}{\| u_n \|_H^2}\int_{\mathbb{R}^3}|\tilde{F}(x,u_n)|dx\right]\\ &\le \limsup_{n\to\infty}\int_{\mathbb{R}^3}\frac{|\tilde{F}(x,u_n)|}{\| u_n \|_H^2}dx.\end{aligned} \tag{10}$$

For $0 \le a < b$, let $\Omega_n(a,b) := \{x \in \mathbb{R}^3 : a \le |u_n(x)| < b\}$. Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v$ in H. Then by Remark 2.1, we have $v_n \to v$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, and $v_n \to v$ a.e. on $\mathbb{R}^3$.

We now consider the following two possible cases concerning v.

Case 1: If $v = 0$, then $v_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, and $v_n \to 0$ a.e. on $\mathbb{R}^3$. Hence it follows from eq. (3) and $v_n := \frac{u_n}{\|u_n\|_H^2}$ that

$$\begin{aligned}\int_{\Omega_n(0,r_0)}\frac{|\tilde{F}(x,u_n)|}{\| u_n \|_H^2}dx &= \int_{\Omega_n(0,r_0)}\frac{|\tilde{F}(x,u_n)|}{|u_n|^2}|v_n|^2dx\\ &\le \left(\frac{c_1}{4}r_0^2 + \frac{c_2}{q}r_0^{q-2}\right)\int_{\Omega_n(0,r_0)}|v_n|^2dx\\ &\le C_4\int_{\mathbb{R}^3}|v_n|^2dx \to 0, \text{as } n \to \infty.\end{aligned} \tag{11}$$

By(S3), we know that $\kappa > 1$. Thus, if we set $\kappa' = \kappa/(\kappa - 1)$, then $2\kappa' \in (2,6)$. Hence it follows from (S3) and eq. (9) that

$$\begin{aligned}\int_{\Omega_n(r_0,\infty)} \frac{|\tilde{F}(x,u_n)|}{\| u_n \|_H^2} dx &= \int_{\Omega_n(r_0,\infty)} \frac{|\tilde{F}(x,u_n)|}{|u_n|^2} |v_n|^2 dx \\ &\le \left[\int_{\Omega_n(r_0,\infty)} \left(\frac{|\tilde{F}(x,u_n)|}{|u_n|^2}\right)^{\kappa} dx\right]^{1/\kappa} \left[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'} dx\right]^{1/\kappa'} \\ &\le c_4^{1/\kappa} \left[\int_{\Omega_n(r_0,\infty)} \tilde{\mathcal{F}}(x,u_n) dx\right]^{1/\kappa} \left[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'} dx\right]^{1/\kappa'} \\ &\le c_4^{1/\kappa}(c+1)^{1/\kappa} \left[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'} dx\right]^{1/\kappa'} \\ &\le C_5 \left[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'} dx\right]^{1/\kappa'} \to 0, \text{as } n \to \infty.\end{aligned} \tag{12}$$

Combining eq. (11) with eq. (12), we have

$$\begin{aligned}&\int_{\mathbb{R}^3} \frac{|\tilde{F}(x,u_n)|}{\| u_n \|_H^2} dx \\ &= \int_{\Omega_n(0,r_0)} \frac{|\tilde{F}(x,u_n)|}{\| u_n \|_H^2} dx \\ &+ \int_{\Omega_n(r_0,\infty)} \frac{|\tilde{F}(x,u_n)|}{\| u_n \|_H^2} dx \to 0 \text{ as } n \to \infty,\end{aligned}$$

which contradicts eq. (10).

Case 2: If $v \neq 0$, we set $A := \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Then $\text{meas}(A) > 0$. For a.e. $x \in A$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$. Hence $A \subset \Omega_n(r_0,\infty)$ for large enough $n \in N$. It follows from eqs (2), (3), (8) and Fatou's lemma that

$$\begin{aligned}0 &= \lim_{n\to\infty} \frac{c+o(1)}{\| u_n \|_H^4} = \lim_{n\to\infty} \frac{I(u_n)}{\| u_n \|_H^4} = \lim_{n\to\infty} \left[\frac{1}{2 \| u_n \|_H^2} + \frac{b}{4 \| u_n \|_H^4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} \frac{\tilde{F}(x,u_n)}{\| u_n \|_H^4} dx\right] \\ &= \left[\frac{b}{4 \| u_n \|_H^4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\Omega_n(0,r_0)} \frac{|\tilde{F}(x,u_n)|}{|u_n|^4} |v_n|^4 dx - \int_{\Omega_n(r_0,\infty)} \frac{|\tilde{F}(x,u_n)|}{|u_n|^4} |v_n|^4 dx\right] \\ &\le \frac{b}{4} + \limsup_{n\to\infty} \int_{\Omega_n(0,r_0)} \left(\frac{c_1}{4} + \frac{c_2}{q} |u_n|^{q-4}\right) |v_n|^4 dx - \liminf_{n\to\infty} \left[\int_{\Omega_n(r_0,\infty)} \frac{|\tilde{F}(x,u_n)|}{|u_n|^4} |v_n|^4 dx\right] \\ &\le \frac{b}{4} + \left(\frac{c_1}{4} + \frac{c_2}{q} |r_0|^{q-4}\right) \limsup_{n\to\infty} \int_{\Omega_n(0,r_0)} |v_n|^4 dx - \liminf_{n\to\infty} \left[\int_{\Omega_n(r_0,\infty)} \frac{|\tilde{F}(x,u_n)|}{|u_n|^4} |v_n|^4 dx\right] \\ &\le \frac{b}{4} + C_8 - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{|\tilde{F}(x,u_n)|}{|u_n|^4} |v_n|^4 dx = \frac{b}{4} + C_8 - \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{|\tilde{F}(x,u_n)|}{|u_n|^4} [\chi_{\Omega_n(r_0,\infty)}(x)] |v_n|^4 dx \\ &= C_9 - \int_{\mathbb{R}^3} \liminf_{n\to\infty} \frac{|\tilde{F}(x,u_n)|}{|u_n|^4} [\chi_{\Omega_n(r_0,\infty)}(x)] |v_n|^4 dx \to -\infty \text{ as } n \to \infty,\end{aligned} \tag{13}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in H. The proof is completed. □

Lemma 3.4: *Assume that the conditions (V2) and (S1) hold. Then any bounded sequence $\{u_n\}$ satisfying eq. (8) has a convergent subsequence in H.*

Proof: Going if necessary to a subsequence, we may assume that $u_n \rightharpoonup u$ in H. Then by Remark 2.1, we have $v_n \to v$ in $L^s(\mathbb{R}^3)$, for $2 \le s < 6$. Let us take $r \equiv 1$ in Lemma 2.3 and combine with $u_n \to u$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, to get

$$\lim_{n\to\infty} |\tilde{f}(x,u_n) - \tilde{f}(x,u)||u_n - u| dx \to 0, \text{as } n \to \infty. \tag{14}$$

We observe that

$$\langle I'(u_n) - I'(u), u_n - u \rangle \to 0, \text{as } n \to \infty, \tag{15}$$

and we have

$$\begin{aligned}\langle I'(u_n) - I'(u), u_n - u \rangle &= \int_{\mathbb{R}^3} \tilde{V}(x) |u_n - u|^2 dx + \left(1 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n - u) dx \\ &- \left(1 + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_n - u) dx - \int_{\mathbb{R}^3} [f(x,u_n) - f(x,u)](u_n - u) dx \\ &= \| u_n - u \|_H^2 + \left(1 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 dx - \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right) \\ &\int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_n - u) dx - \int_{\mathbb{R}^3} [f(x,u_n) - f(x,u)](u_n - u) dx \\ &\ge \| u_n - u \|_H^2 - b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right) \\ &- \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_n - u) dx - \int_{\mathbb{R}^3} [f(x,u_n) - f(x,u)](u_n - u) dx.\end{aligned} \tag{16}$$

Then eq. (16) implies that

$$\| u_n - u \|_H^2 \le \langle I'(u_n) - I'(u), u_n - u\rangle + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_n - u) dx + \int_{\mathbb{R}^3} [f(x, u_n) - f(x, u)](u_n - u) dx. \tag{17}$$

Define the functional $h_u : H \to \mathbb{R}$ by

$$h_u(v) = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx, \forall v \in H.$$

Obviously, h_u is a linear functional on H. Furthermore,

$$|h_u(v)| \le \int_{\mathbb{R}^3} |\nabla u \cdot \nabla v| dx \le \| u\|_H \| v\|_H,$$

which implies that h_u is bounded on H. Hence $h_u \in H^*$. Since $u_n \rightharpoonup u$ in H, we have $\lim_{n\to\infty} h_u(u_n) = h_u(u)$, that is,

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_n - u) dx \to 0 \quad as\, n \to \infty.$$

Consequently, by $v_n \to v$ in $L^s(\mathbb{R}^3)$, for $2 \le s < 6$ and the boundedness of $\{u_n\}$, we obtain

$$b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_n - u) dx \to 0, \quad n \to +\infty. \tag{18}$$

Consequently, eqs (14), (15), (17), (18) imply that

$$u_n \to u \,\text{in}\, H \,\text{as}\, n \to \infty.$$

This completes the proof. □

Proof of Theorem 1.2: To complete the proof of the main result, we need to consider the following two steps.

Step 1: We first show that there exists a function $u_0 \in H$ such that $I'(u_0) = 0$ and $I(u_0) < 0$. Let $r_0 = 1$. For any $|u| \ge 1$, from (S2), we have

$$\tilde{F}(x, u_n) \ge c_3 |u|^\sigma > 0. \tag{19}$$

By (S1), for a.e. $x \in \mathbb{R}^3$ and $0 \le |u| \le 1$, there exists $M > 0$ such that

$$\left|\frac{\tilde{f}(x, u)u}{u^2}\right| \le \left|\frac{(c_1|u|^3 + c_2|u|^{q-1})|u|}{|u|^2}\right| \le M,$$

which implies that

$$\tilde{f}(x, u)u \ge -M|u|^2.$$

Using the equality $\tilde{F}(x, u) = \int_0^1 \tilde{f}(x, tu)dt$, for a.e. $x \in \mathbb{R}^3$ and $0 \le |u| \le 1$, we obtain

$$\tilde{F}(x, u) > -\frac{1}{2}M|u|^2. \tag{20}$$

In view of eqs (19) and (20), we have for a.e. $x \in \mathbb{R}^3$ and all $u \in \mathbb{R}$ that

$$\tilde{F}(x, u) \ge -\frac{1}{2}M|u|^2 + c_3|u|^\sigma.$$

Therefore we have

$$\tilde{F}(x, t\psi) \ge -\frac{1}{2}Mt^2|\psi|^2 + t^\sigma c_3|\psi|^\sigma. \tag{21}$$

Combing eq. (2) with eq. (21), we get

$$\begin{aligned} I(tu) &= \frac{t^2}{2} \| u \|_H^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} \tilde{F}(x, tu) dx \\ &\le \frac{t^2}{2} \| u \|_H^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \frac{t^2 M}{2}\int_{\mathbb{R}^3} |u|^2 dx \\ &\quad - t^\sigma c_3 \int_{\mathbb{R}^3} |u|^\sigma dx. \end{aligned}$$

Since $\sigma \in (0, 2)$, for small enough t we infer that $I(tu) < 0$. Thus we obtain

$$c_0 = \inf\{I(u) : u \in \bar{B}_\rho\} < 0,$$

where $\rho > 0$ is given by Lemma 3.1 and $B_\rho = \{u \in H : \| u\|_H < \rho\}$. By the Ekeland variational principle, there exists a sequence $\{u_n\} \subset B_\rho$ such that

$$c_0 \le I(u_n) \le c_0 + \frac{1}{n},$$

and

$$I(w) \ge I(u_n) - \frac{1}{n} \| w - u_n\|_H,$$

for all $w \in B_\rho$. Then, following the idea of [48], we can show that $\{u_n\}$ is a bounded Cerami sequence of I. Therefore, Lemma 3.4 implies that there exists a function $u_0 \in H$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

Step 2: We now show that there exists a function $\tilde{u}_0 \in H$ such that $I'(\tilde{u}_0) = 0$ and $I(\tilde{u}_0) = \tilde{c}_0 > 0$. By Lemmata 3.1, 3.2 and 2.2, there is a sequence $\{u_n\} \in H$ satisfying eq. (8). Moreover, Lemma 3.3 and 3.4 shows that this sequence has a convergent subsequence and is bounded in H. So, we complete the Step 2.

Therefore, combining the above two steps and Lemma 1.1, we complete the proof of Theorem 1.2. □

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References

[1] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.
[2] S. Bernstein, Sur une classe d'équations fonctionnelles aux dérivées partielles, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4** (1940), 17–26.
[3] S. I. Pohožaev, A certain class of quasilinear hyperbolic equations, Mat. Sb. (N.S) **96** (1975), 152–166, 168.
[4] J. -L. Lions. On some questions in boundary value problems of mathematical physics. In Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), volume 30 of North-Holland Math. Stud., pages 284–346. North-Holland, Amsterdam, 1978.
[5] C. O. Alves, F. Corrêa, and T. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. **49** (2005), 85–93.
[6] A. Azzollini, The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity, Differ. Integral. Equ. **25** (2012), 543–554.
[7] A. Azzollini, A note on the elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity, Commun. Contemp. Math. **17** (2015), 1450039 (5 pages).
[8] A. Azzollini, P. d'Avenia, and A. Pomponio, Multiple critical points for a class of nonlinear functionals, Ann. Mat. Pure Appl. (4) **190** (2011), 507–523.
[9] C. Chen and Q. Zhu, Existence of positive solutions to ρ-Kirchhoff-type problem without compactness conditions, Appl. Math. Lett. **28** (2014), 82–87.
[10] J. Jin and X. Wu, Infinitely many radial solutions for Kirchhoff-type problems in $\mathbb{R}^N$, J. Math. Anal. Appl. **369** (2010), 564–574.
[11] Y. Li, F. Li, and J. Shi, Existence of a positive solution to Kirchhoff type problems without compactness conditions, J. Differ. Equ. **253** (2012), 2285–2294.
[12] X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in R^N, Nonlinear Anal. Real World Appl. **12** (2011), 1278–1287.
[13] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. **43** (1992), 270–291.
[14] F. Gazzola and V. Rădulescu, A nonsmooth critical point theory approach to some nonlinear elliptic equations in unbounded domains, Diff. Integral Equ. **13** (2000), 47–60.
[15] C. Chen, H. Song, and Z. Xiu, Multiple solutions for ρ-Kirchhoff equations in $\mathbb{R}^N$, Nonlinear Anal. **86** (2013), 146–156.
[16] S. -J. Chen and L. Li, Multiple solutions for the nonhomogeneous Kirchhoff equation on R^N, Nonlinear Anal. Real World Appl. **14** (2013), 1477–1486.
[17] B. Cheng, Nontrivial solutions for Schrödinger-Kirchhoff-type problem in R^N, Bound. Value Probl. **250** (2013), 2013.
[18] Y. Huang and Z. Liu, On a class of Kirchhoff type problems, Arch. Math. (Basel) **102** (2014), 127–139.
[19] Q. Li and X. Wu, A new result on high energy solutions for Schrödinger–Kirchhoff type equations in $\mathbb{R}^N$, Appl. Math. Lett. **30** (2014), 24–27.
[20] Q. Li and Z. Yang, Multiple solutions for N-Kirchhoff type problems with critical exponential growth in $\mathbb{R}^N$, Nonlinear Anal. **117** (2015), 159–168.
[21] W. Liu and X. He, Multiplicity of high energy solutions for superlinear Kirchhoff equations, J. Appl. Math. Comput. **39** (2012), 473–487.
[22] K. Wu and X. Wu, Infinitely many small energy solutions for a modified Kirchhoff-type equation in $\mathbb{R}^N$, Comput. Math. Appl. **70** (2015), 592–602.
[23] Z. Xiu, The existence of a nontrivial solution for a ρ-Kirchhoff type elliptic equation in $\mathbb{R}^N$, Abstr. Appl. Anal. (2013), 1–6. Art. ID 281949.
[24] H. Ye, The existence of least energy nodal solutions for some class of Kirchhoff equations and Choquard equations in $\mathbb{R}^N$, J. Math. Anal. Appl. **431** (2015), 935–954.
[25] Y. Ye and C. -L. Tang, Multiple solutions for Kirchhoff-type equations in $\mathbb{R}^N$, J. Math. Phys. **54** (2013), 081508, 16.
[26] J. Zhang, X. Tang, and W. Zhang, Existence of multiple solutions of Kirchhoff type equation with sign-changing potential, Appl. Math. Comput. **242** (2014), 491–499.
[27] Y. Guo and J. Nie, Existence and multiplicity of nontrivial solutions for ρ-Laplacian Schrödinger-Kirchhoff-type equations, J. Math. Anal. Appl. **428** (2015), 1054–1069.
[28] S. Liang and J. Zhang, Existence of solutions for Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^3$, Nonlinear Anal. Real World Appl. **17** (2014), 126–136.
[29] S. Liang and J. Zhang, Existence of solutions for Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^N$, Z. Angew. Math. Phys. **66** (2015), 547–562.
[30] J. Nie, Existence and multiplicity of nontrivial solutions for a class of Schrödinger-Kirchhoff-type equations, J. Math. Anal. Appl. **417** (2014), 65–79.
[31] Q. Xie and S. Ma, Existence and concentration of positive solutions for Kirchhoff-type problems with a steep well potential, J. Math. Anal. Appl. **431** (2015), 1210–1223.
[32] R. Agarwal, D. O'Regan, and V. Shakhmurov, B-Separable boundary value problems in Banach-valued function spaces, Appl. Math. Comput. **210** (2009), 48–63.
[33] G. Autuori, M. Salvatori, and P. Pucci, Asymptotic stability for nonlinear Kirchhoff systems, Nonlinear Anal. Real World Appl. **10** (2009), 889–909.
[34] G. Molica Bisci and D. Repovš, Existence and localization of solutions for nonlocal fractional equations, Asymptot. Anal. **90** (2014), 367–378.
[35] D. O'Regan and S. Staneǩ, The existence of positive solutions to a non-local singular boundary value problem, Math. Methods Appl. Sci. **29** (2006), 235–247.
[36] P. Pucci, M. Xiang, and B. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional ρ-Laplacian in $\mathbb{R}^N$, Calc. Var. Partial Differ. Equ. **54** (2015), 2785–2806.
[37] G. Molica Bisci, V. Rădulescu, and R. Servadei, Variational methods for nonlocal fractional problems, Encyclopedia of

Mathematics and its Applications, Cambridge University Press, Cambridge, 2016.

[38] H. Liu, H. Chen, and X. Yang, Multiple solutions for superlinear Schrödinger-Poisson system with sign-changing potential and nonlinearity, Comput. Math. Appl. **68** (2014), 1982–1990.

[39] X. H. Tang, Infinitely many solutions for semilinear Schrodinger equations with sign-changing potential and nonlinearity, J. Math. Anal. Appl. **401** (2013), 407–415.

[40] Q. Zhang and B. Xu, Multiplicity of solutions for a class of semilinear Schrodinger equations with sign-changing potential, J. Math. Anal. Appl. **377** (2011), 834–840.

[41] T. Bartsch, Z. -Q. Wang, and M. Willem, The Dirichlet problem for superlinear elliptic equations, in: Stationary Partial Differential Equations. M. Chipot (eds.), Vol. II, Handb. Differ. Equ, 1–55, Elsevier/North-Holland, Amsterdam, 2005.

[42] G. Molica Bisci and V. Rădulescu, Applications of local linking to nonlocal Neumann problems, Commun. Contemp. Math. **17** (2015), 1–17.

[43] A. Bahrouni, Infinitely many solutions for sublinear Kirchhoff equations in $\mathbb{R}^N$ with sign-changing potentials, Electron. J. Differ. Equ. **98** (2013), 8.

[44] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. **14** (1973), 349–381.

[45] I. Ekeland, Convexity methods in Hamiltonian Mechanics, Volume 19 of Ergebnisse Der Mathematik Und Ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3), Springer-Verlag, Berlin, 1990.

[46] H. Brezis and L. Nirenberg, Remarks on finding critical points, Comm. Pure Appl. Math. **44** (1991), 939–963.

[47] I. Ekeland, On the variational principle, J. Math. Anal. Appl. **47** (1974), 324–353.

[48] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser Boston Inc., Boston, MA, 1996.