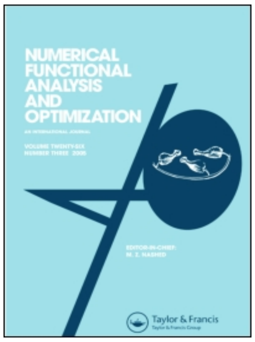

Numerical Functional Analysis and Optimization

ISSN: 0163-0563 (Print) 1532-2467 (Online) Journal homepage: http://www.tandfonline.com/loi/lnfa20

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To cite this article: Boualem Alleche & Vicențiu D. Rădulescu (2017) Inverse of the Sum of Set-Valued Mappings and Applications, Numerical Functional Analysis and Optimization, 38:2, 139-159, DOI: 10.1080/01630563.2016.1277741

To link to this article: http://dx.doi.org/10.1080/01630563.2016.1277741

Accepted author version posted online: 05 Jan 2017.
Published online: 05 Jan 2017.

NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
2017, VOL. 38, NO. 2, 139–159
http://dx.doi.org/10.1080/01630563.2016.1277741

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Inverse of the Sum of Set-Valued Mappings and Applications

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ABSTRACT

In this article, we develop results on the behavior of fixed points sets of set-valued pseudo-contraction mappings. Then, we investigate the notions related to the Aubin property and make use of connections between the two involved set-valued nonnecessarily Lipschitzian mappings to obtain results on the inverse of their sum similar to those in the literature generalizing Lyusternik and Graves theorems. By proximal convergence, we apply our results to the sensitivity analysis of variational inclusions.

ARTICLE HISTORY
Received 26 December 2015
Revised 23 December 2016
Accepted 27 December 2016

KEYWORDS
Aubin property; fixed point; proximal convergence; pseudo-Lipschitzian; set-valued mapping

2010 MATHEMATICS SUBJECT CLASSIFICATION
49J53; 47J22; 49J40; 49K40

1. Introduction

Studies about the inverse of sum of set-valued mappings have drawn in the last years the attention of many authors and constitute today an important and active research field. One of the principal motivations of such studies is related to the existence of solutions of variational inclusions. Recall that a *variational inclusion* (or a *generalized equation*) is a problem of the form

$$\text{find } x \in X \text{ such that } y \in A(x), \tag{VI}$$

where A is a set-valued mapping acting between two Banach spaces X and Y, and $y \in Y$ is a given point. In many cases, the point y could be of the form $f(x)$ where f is a single-valued function from X to Y or of the form $f(p,x)$ with p a parameter leading to an important class of variational inclusions called *parameterized generalized equations*.

It is well known that this problem serves as a general framework for describing in a unified manner various problems arising in nonlinear analysis and in other areas in mathematics including optimization problems and variational inequality problems; for more information on the subject with survey of old and recent developments, we refer to [20] and the references therein.

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In the simple case of a single-valued mapping A, problem (VI) reduces to a simply functional equation, and it is then related to the surjectivity of the involved single-valued mapping. From the same point of view, in the case of set-valued mappings, the problem is also related to the surjectivity of the involved set-valued mapping in the analog sense. The pioneering work in this direction is the well-known Banach open mapping theorem which guarantees that a continuous mapping acting between Banach spaces is open if and only if it is surjective.

Among various advancements in this area, there are also the famous works by Lyusternik [30] for nonlinear Fréchet differentiable functions and that of Graves [24] for nonlinear operators acting between Banach spaces. It should be emphasized that no differentiability assumption is made in the theorem of Graves. Also, many investigations about the solution mappings by classical differentiability or by the concepts of generalized differentiation have been performed and several results for variational inclusions have been obtained. This direction has given rise to the rich theory of what is known by the theory of implicit functions for parameterized generalized equations, see [20, 25, 34] and the references therein.

Another point of view having roots in the Milyutin's covering mapping theorem which, in turn, goes back to the theorem of Graves is what is known in the literature under the name of *openness with linear rate* or the *covering property*, see [15]. This approach makes use of a constant like that appearing in the Banach open mapping theorem for studying the regularity properties of the inverse of set-valued mappings and it has produced many results with applications to different kinds of variational inclusions in the infinite-dimensional settings. Recently, this direction has attracted a special attention of several authors, see, for instance, [5, 14, 16, 17, 21, 22] and the references therein. One can also consult, for instance, [6–9], to see the introduction of notion of *locally covering maps* and its applications to study the distance to the set of coincidence points of set-valued mappings. Many deep and important results are obtained and applied to different areas of mathematics, including stability and continuous dependence, system of differential inclusions, implicit function theorems, functional equations, and existence of double fixed points.

In this article, we investigate the necessary conditions to deal with the Lipschitzian property of the inverse of sum of two set-valued mappings. As the inverse of the inverse of a set-valued mapping is the set-valued mapping itself, and since the inverse of a Lipschitzian set-valued mapping need not be Lipschitzian, we wonder why always consider set-valued Lipschitzian mappings if we want to obtain that the inverse of their sum is Lipschitzian. This leads to say that nothing can prevent a set-valued non-Lipschitzian mapping to have an inverse set-valued mapping which is Lipschitzian. However, going back to the Banach open mapping theorem, we understand that this question has roots in the fact that the inverse of a surjective linear and continuous mapping acting between Banach spaces has some regularity properties, and by linearity, the mapping itself is Lipschitzian. Of course, the situation is different when dealing with set-valued mappings. Motivated by this question, we investigate here the property of being set-valued pseudo-Lipschitzian to study the Lipschitzian property of the inverse of sum of two set-valued mappings.

The article is structured as follows. In the next section, we give the necessary background to deal with set-valued mappings in the settings of metric spaces and introduce some notions defined from the properties of set-valued pseudo-Lipschitzian mappings. In Section 3, by new arguments, we obtain results on the behavior of fixed points sets of set-valued pseudo-contraction mappings. Under new conditions, our results are comparable with those obtained in the literature, and more recently in [3, 10, 11, 32]. In Section 4, we make use of our results on the behavior of fixed point sets of set-valued pseudo-contraction mappings to deal, following some techniques inspired from [14], with the inverse of sum of set-valued nonnecessarily Lipschitzian mappings. Under weakened conditions of the Lipschitzian property but with additional conditions on the existence of fixed points, we obtain that the inverse of sum of two set-valued mappings is Lipschitzian. In the last section, we make use of the proximal convergence to develop techniques and obtain results on the sensitivity analysis of variational inclusions.

2. Notations and preliminary results

Throughout this article, (X, d) stands for a metric space. Given $x \in X$ and $r > 0$, we denote by $B(x, r)$ (resp. $\overline{B}(x, r)$) the open (resp. closed) ball around x with radius r.

Let A be a nonempty subset of X. The distance from a point $x \in X$ is defined by $d(x, A) := \inf_{y \in A} d(x, y)$, and, as usual, $d(x, \emptyset) = +\infty$. The open ball around A with radius r is denoted by $B(A, r) := \bigcup_{u \in A} B(u, r)$.

For two subsets A and B of X, the *excess* of A over B (with respect to d) is denoted by $e(A, B)$ and is defined by $e(A, B) := \sup_{x \in A} d(x, B)$. In particular, we adopt the conventions $e(\emptyset, B) := 0$ and $e(A, \emptyset) := +\infty$ if $A \neq \emptyset$.

The distance between A and B (with respect to d) is denoted by $\mathrm{Haus}(A, B)$ and is defined by:

$$\mathrm{Haus}(A, B) := \max \{e(A, B), e(B, A)\}.$$

Restricted to the closed subsets, Haus is a (extended real-valued) metric the so-called *Pompeiu–Hausdorff metric*.

Let (X, d_X) and (Y, d_Y) be two metric spaces. In the sequel, a set-valued mapping T from X to Y will be denoted by $T : X \rightrightarrows Y$. The domain of T is the set $\mathrm{dom}(T) := \{x \in X \mid T(x) \neq \emptyset\}$, and its graph is given by $\mathrm{grph}(T) := \{(x, y) \in X \times Y \mid y \in T(x)\}$. If the graph of T is closed, then T has closed values. The converse holds under additional conditions, in particular, if T is upper semicontinuous, see for example [2, 12].

Recall that a set-valued mapping $T : X \rightrightarrows Y$ is said to be *upper semicontinuous* at a point $x_0 \in X$ if for every open subset V of Y such that $T(x_0) \subset V$, there exists an open neighborhood U of x_0 such that $T(x) \subset V$, for every $x \in U$. The set-valued mapping T is said to be upper semicontinuous if it is upper semicontinuous at every point of X.

For a subset A of X, we denote by $T(A) := \cup_{x\in A} T(x)$, the image of A by T. For a subset B of Y, the inverse image of B by T is $T^{-1}(B) := \{x \in X \mid B \cap T(x) \neq \emptyset\}$, while $T^{-1}(y)$ stands for $T^{-1}(\{y\})$, if $y \in Y$. A set-valued mapping $T : X \rightrightarrows Y$ is upper semicontinuous if and only if $T^{-1}(B)$ is closed, for every closed subset B of Y.

In the sequel, the fixed point set of a set-valued mapping $T : X \rightrightarrows X$ will be denoted by $\text{Fix}(T)$, that is, $\text{Fix}(T) := \{x \in X \mid x \in T(x)\}$.

The Lipschitz continuity (with respect to the Pompeiu–Hausdorff metric) is one of the most popular properties of set-valued mappings. A set-valued mapping $T : X \rightrightarrows Y$ is said to be L-Lipschitzian on $M \subset \text{dom}(T)$ if it has closed values on M and there exists $L \geq 0$ such that

$$\text{Haus}(T(x_1), T(x_2)) \leq L d_X(x_1, x_2) \quad \forall x_1, x_2 \in M.$$

If $X = Y$ and $L \in [0, 1)$, then T is called L-contraction on M.

To deal with the properties of inverse of the sum of two set-valued mappings, it has been proved in [14, Lemma 2] the following result for set-valued Lipschitzian mappings. If $T : X \rightrightarrows Y$ is L-Lipschitzian on M, then for every two nonempty subsets A and B of M,

$$e(T(A), T(B)) \leq Le(A, B).$$

This property being not adapted to our techniques, we develop here some analog properties related to pseudo-Lipschitzian set-valued mappings.

Recall that a set-valued mapping $T : X \rightrightarrows Y$ is said to be pseudo-Lipschitzian around $(x, y) \in \text{grph}(T)$ if there exist a constant $L \geq 0$ and neighborhoods $M_x \subset \text{dom}(T)$ of x and M_y of y such that

$$e\left(T(x_1) \cap M_y, T(x_2)\right) \leq L d_X(x_1, x_2) \quad \forall x_1, x_2 \in M_x.$$

The notion of being pseudo-Lipschitzian around (x, y) is called the *Aubin property* when M_x and M_y are closed balls around x and y, respectively. It is well-known that the Aubin property of the set-valued mapping T turns out to be equivalent to the metric regularity of the set-valued mapping T^{-1}, see [19, 22, 25, 27, 28, 31, 35] for more details on the notion of metric regularity and its applications to variational problems.

We extend the above definition to any two nonempty subsets $M_x \subset \text{dom}(T)$ and $M_y \subset Y$, and we say that T is *L-pseudo-Lipschitzian on M_x with respect to*

M_y. When $X = Y$, $M_x = M_y = M$, and $L \in [0, 1)$, the set-valued mapping T is called L-pseudo-contraction with respect to M, see [10].

Let $M \subset \operatorname{dom}(T)$ and N be two nonempty subsets of X, and S a nonempty subset of Y. We say that T is *fully L-pseudo-Lipschitzian on M for N with respect to S* if for any two nonempty subsets A and B of M, we have

$$e(T(A) \cap S, T(B)) \leq Le(A \cap N, B).$$

It results immediately from the definition that any set-valued fully L-pseudo-Lipschitzian on M for N with respect to S is L-pseudo-Lipschitzian on M with respect to S. It is also fully L-pseudo-Lipschitzian on M for N' with respect to S, for any subset N' containing N.

Conversely, any set-valued L-Lipschitzian mapping $T : X \rightrightarrows Y$ on a subset M is fully L-pseudo-Lipschitzian on M for N with respect to any subset of Y, for any subset N of X containing M.

More generally, we have the following result for set-valued pseudo-Lipschitzian mappings which can be compared to [14, Lemma 2] where the proof is similar.

Proposition 2.1. *Let $T : X \rightrightarrows Y$ be a set-valued L-pseudo-Lipschitzian on M with respect to S. Then, for any nonempty subsets A and B of M, we have*

$$e(T(A) \cap S, T(B)) \leq Le(A, B).$$

In particular, T is fully L-pseudo-Lipschitzian on M for N with respect to S, for any subset N containing M.

Proof. Let A and B be nonempty and contained in M. To avoid any confusion, put $A' = \{x \in A \mid T(x) \cap S \neq \emptyset\}$. Then,

$$\begin{aligned}
e(T(A) \cap S, T(B)) &= \sup_{u \in T(A) \cap S} d_Y(u, T(B)) \\
&= \sup_{x_1 \in A'} \sup_{u \in T(x_1) \cap S} \inf_{x_2 \in B} d_Y(u, T(x_2)) \\
&\leq \sup_{x_1 \in A'} \inf_{x_2 \in B} \sup_{u \in T(x_1) \cap S} d_Y(u, T(x_2)) \\
&= \sup_{x_1 \in A'} \inf_{x_2 \in B} e(T(x_1) \cap S, T(x_2)) \\
&\leq L \sup_{x_1 \in A'} \inf_{x_2 \in B} d_X(x_1, x_2) \\
&\leq L \sup_{x_1 \in A} d_X(x_1, B) = Le(A, B).
\end{aligned}$$

Since $e(A, B) = e(A \cap N, B)$ whenever N contains M, then the set-valued mapping T is fully L-pseudo-Lipschitzian on M for N with respect to S. $\square$

Although the notion of being fully pseudo-Lipschitzian seems to fit very well with the other existing notions such as those of being Lipschitzian and pseudo-Lipschitzian, it may be interesting to look for conditions involving it for a subset N which does not necessarily contain M.

Proposition 2.2. *Let $T : X \rightrightarrows Y$ be a set-valued L-pseudo-Lipschitzian on M with respect to S, and let N be a subset of X such that $T(x) \cap S = \emptyset$, whenever $x \in M \backslash N$. Then, T is fully L-pseudo-Lipschitzian on M for N with respect to S.*

Proof. Let A and B be nonempty and contained in M. We remark that $T(A) \cap S = T(A \cap N) \cap S$. The proof then follows step by step that of Proposition 2.1. □

The following example provides us with a set-valued non-Lipschitzian mapping which is fully pseudo-Lipschitzian mapping, where M is not contained in N. We can also choose N in such a way that neither M is contained in N nor N is contained in M.

Example 1. Let $T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be the set-valued mapping defined by:

$$T\left((x,y)\right) = \begin{cases} \{2x\} \times ([0, 2|x|] \cup [3, +\infty[) & \text{if} \quad \| \left(x,y\right) \| < 1, \\ \{2x\} \times \left]0, x^2\right] & \text{if} \quad \| \left(x,y\right) \| \geq 1. \end{cases}$$

Let $M = S = B((0,0), 1)$ and $N = \left\{(x,y) \in M \mid |x| < \frac{1}{2}\right\}$. In this example, we have $N \subset M$. Clearly, the set-valued mapping T is not Lipschitzian on $\mathbb{R}^2$. However, T is $2\sqrt{2}$-pseudo-Lipschitzian on M with respect to S. We remark that for any $x \in M \backslash N$, $T(x) \cap S = \emptyset$. Then, we conclude by Proposition 2.2 that T is fully $2\sqrt{2}$-pseudo-Lipschitzian on M for N with respect to S.

If we take $N' = N \cup N_1$ where $N_1 \backslash M \neq \emptyset$, then T is still fully $2\sqrt{2}$-pseudo-Lipschitzian on M for N' with respect to S. In this case, neither M is contained in N' nor N' is contained in M.

Finally, recall that if (Y, d) is a linear metric space, the distance d is said to be *shift-invariant metric* if

$$d\left(y + z, y' + z\right) = d\left(y, y'\right) \quad \text{for all } y, y', z \in X.$$

Let A and B be two subsets of a linear metric space (Y, d) with shift-invariant metric d, and $a, b, b' \in Y$. It is shown in [14] that

$$e(A + a, B + a) \leq e(A, B) \quad \text{and} \quad e\left(A + b, A + b'\right) \leq d\left(b, b'\right).$$

3. On the behaviors of fixed points sets of set-valued pseudo-contraction mappings

In this section, we will be concerned with the behaviors of fixed points sets of set-valued pseudo-contraction mappings.

Existence of fixed points is a subject which is not limited to set-valued contraction or pseudo-contraction mappings, and in this spirit, we will not make use here of classical conditions assuring existence of fixed points for set-valued mappings. More precisely, we will assume that the fixed points sets of the involved set-valued mappings are nonempty and linked in such a way that a result on the behaviors of their fixed points sets is derived. And instead of conditions on the distance between the images of the set-valued mappings as considered in some recent articles (see, for instance, [10, 32]), we impose conditions only on those for the fixed points.

We will not follow here classical procedures usually used when dealing with the behaviors of fixed points sets of set-valued mappings but make use of the following more precise version of the well-known lemma on existence of fixed points of set-valued pseudo-contraction mappings called in [13], Dontchev–Hager fixed-point theorem; see also [18]. This version is enhanced in the sense that not only the completeness is assumed only on the closed ball, but more particulary, only the values of restriction of the set-valued mapping on the closed ball are assumed to be nonempty and closed. Of course the proof follows, step by step, the arguments used in [18] which are based on techniques having roots in the Banach contraction principle. A proof using arguments based on a weak variant of the Ekeland variational principle has also been performed recently in [13].

Lemma 3.1. *Let (X, d) be a metric space. Let $\bar{x} \in X$ and $\alpha > 0$ be such that $\overline{B}(\bar{x}, \alpha)$ is a complete metric subspace. Let $\lambda \in [0, 1)$ and $T : X \rightrightarrows X$ be a set-valued mapping with nonempty closed values on $\overline{B}(\bar{x}, \alpha)$ such that*
(1) $d(\bar{x}, T(\bar{x})) < (1 - \lambda)\alpha$ and
(2) $e\left(T(x) \cap \overline{B}(\bar{x}, \alpha), T\left(x'\right)\right) \leq \lambda d\left(x, x'\right)\ \forall\, x, x' \in \overline{B}(\bar{x}, \alpha)$.
Then, T has a fixed point in $\overline{B}(\bar{x}, \alpha)$.

Now, we derive the following result on the behavior of fixed points sets of set-valued mappings which should be compared to [10, Proposition 2.4], [11, Proposition 2.4], [23, Proposition 4.5], and more recently to [3, Theorem 3.1]. It is worthwhile noticing that one of the deep and most general result obtained in this direction is [7, Theorem 4.1]. However, our conditions seem to be somewhat different. In any case, and at the current stage of advancement, it is not easy to see if it is possible to derive our result from it, see Remark 1 below for explanation. Using Lemma 3.1, we give here the proof for the convenience of the reader.

Theorem 3.2. *Let (X, d) be a metric space. Let $x_0 \in X$ and $r > 0$ be such that $\overline{B}(x_0, r)$ is a complete metric subspace. Let $\lambda \in (0, 1)$ and $0 < \beta < (1 - \lambda) r$ and let $T, S : X \rightrightarrows X$ be two set-valued mappings such that*

(1) *T is λ-pseudo-contraction with respect to $B(x_0, r)$ and has nonempty closed values on $B(x_0, r)$;*

(2) *S has nonempty fixed points set and for every $x \in Fix(S)$,*

$$d(x, x_0) < \beta \quad and \quad d(x, T(x)) < \lambda\beta.$$

Then, T has a nonempty fixed points set and

$$e(Fix(S), Fix(T)) \le \frac{1}{1-\lambda} \sup_{x \in B(x_0, r)} e(S(x), T(x)).$$

Proof. Fix $\varepsilon > 0$ and put

$$\alpha = \min\left\{\frac{1}{1-\lambda} \sup_{x \in B(x_0, r)} e(S(x), T(x)) + \varepsilon, \frac{\lambda\beta}{1-\lambda}\right\}.$$

Let $\bar{x} \in \mathrm{Fix}(S)$ be an arbitrary element.

Claim 1: We prove that $\overline{B}(\bar{x}, \alpha) \subset B(x_0, r)$. To do this, let $x \in \overline{B}(\bar{x}, \alpha)$. Then, from assumption (2), we have

$$\begin{aligned} d(x, x_0) &\le d(x, \bar{x}) + d(\bar{x}, x_0) \\ &< \alpha + \beta \le \frac{1}{1-\lambda}\lambda\beta + \beta < \lambda r + (1-\lambda) r = r. \end{aligned}$$

Claim 2: We have $d(\bar{x}, T(\bar{x})) < (1-\lambda)\alpha$. Indeed, since $\bar{x} \in \mathrm{Fix}(S)$, then by assumption (2), $d(\bar{x}, T(\bar{x})) < \lambda\beta$. Also,

$$d(\bar{x}, T(\bar{x})) \le e(S(\bar{x}), T(\bar{x})) \le \sup_{x \in B(x_0, r)} e(S(x), T(x)),$$

and, since $d(\bar{x}, T(\bar{x}))$ is finite, then

$$d(\bar{x}, T(\bar{x})) < \sup_{x \in B(x_0, r)} e(S(x), T(x)) + (1-\lambda)\varepsilon.$$

Thus, $d(\bar{x}, T(\bar{x})) < (1-\lambda)\alpha$.

It results by Claim 1 and assumption (1) that T has nonempty closed values on $\overline{B}(\bar{x}, \alpha)$ and for every $x, x' \in \overline{B}(\bar{x}, \alpha)$,

$$e\left(T(x) \cap \overline{B}(\bar{x}, \alpha), T\left(x'\right)\right) \le e\left(T(x) \cap B(x_0, r), T\left(x'\right)\right) \le \lambda d\left(x, x'\right).$$

Now, all the conditions of Lemma 3.1 are satisfied for T on $\overline{B}(\bar{x}, \alpha)$ and then, T has a fixed point $x^* \in \overline{B}(\bar{x}, \alpha)$. It results that

$$d(\bar{x}, \mathrm{Fix}(T)) \le d\left(\bar{x}, x^*\right) \le \alpha \le \frac{1}{1-\lambda} \sup_{x \in B(x_0, r)} e(S(x), T(x)) + \varepsilon.$$

This inequality being valid for any $\overline{x} \in \text{Fix}(S)$, we obtain

$$e(\text{Fix}(S), \text{Fix}(T)) \leq \frac{1}{1-\lambda} \sup_{x \in B(x_0,r)} e(S(x), T(x)) + \varepsilon.$$

Letting ε go to zero, we complete the proof. □

Remark 1. In [7, Theorem 4.1], the authors consider the set of coincidence points of two set-valued mappings Φ and Ψ which is exactly the fixed points set of Φ whenever Ψ is the embedding set-valued mapping Emb_X of X defined by $Emb_X(x) = \{x\}$, for every $x \in X$. Then, according to our notations, we take $\Phi = T$, $\tilde{\Phi} = S$, and $\Psi = \tilde{\Psi} = Emb_X$. In our assumptions, T is λ-pseudo-contraction with respect to $B(x_0, r)$ which is weaker than the property of being pseudo-Lipschitzian with Lipschitz constant λ considered in [7]. But to overcome this fact, we know that it is pseudo-Lipschitzian with Lipschitz constant $\lambda+\varepsilon$, for every $\varepsilon > 0$. According to the notations of [7], we take $x_0^* = y_0^*$ any point in $\text{Fix}(S)$ which plays the role of x_0 and y_0 in [7], respectively. But we can not take our x_0 because $d(x_0, T(x_0))$ is not known under our assumptions. Also, we take $R_1 = R_2 = \tilde{R} = \lambda r$, $\beta = \lambda + \varepsilon$, and $\alpha = 1$. As a conclusion, for any $r_1 > 0$ and $r_2 > 0$ verifying Condition (3.11) of page 821 [7], we obtain

$$e\left(\text{Fix}(S) \cap B\left(x_0^*, r_1\right), \text{Fix}(T)\right) \leq \frac{1}{1-\lambda-\varepsilon} \sup_{x \in B\left(x_0^*, r_1\right)} e(S(x), T(x)).$$

It is not clear how to choose, for every $\varepsilon > 0$, r_1 (which depends on ε) in such a way that $\text{Fix}(S) \subset B\left(x_0^*, r_1\right) \subset B(x_0, r)$, since the upper bound in the second term of inequality is taken on $B\left(x_0^*, r_1\right)$. Furthermore, neither X nor the graph of T are assumed to be complete in Theorem 3.2. This condition is required in [7, Theorem 4.1].

Now, we derive the following corollary.

Corollary 3.3. *Let (X, d) be a metric space. Let $x_0 \in X$ and $r > 0$ be such that $\overline{B}(x_0, r)$ is a complete metric subspace. Let $\lambda \in (0, 1)$ and $0 < \beta < (1-\lambda) r$ and let $T, S : X \rightrightarrows X$ be two set-valued mappings such that*

(1) *S and T are λ-pseudo-contractions with respect to $B(x_0, r)$ and have nonempty closed values on $B(x_0, r)$;*
(2) *S has nonempty fixed points set and for every $x \in Fix(S)$,*

$$d(x, x_0) < \beta \quad and \quad d(x, T(x)) < \lambda\beta;$$

(3) *T has nonempty fixed points set and for every $x \in Fix(T)$,*

$$d(x, x_0) < \beta \quad and \quad d(x, S(x)) < \lambda\beta.$$

Then,

$$Haus(Fix(S), Fix(T)) \leq \frac{1}{1-\lambda} \sup_{x \in B(x_0,r)} Haus(S(x), T(x)).$$

Remark 2. It is worthwhile emphasizing the importance of the above result which allows to replace the excess by the Pompeiu–Hausdorff metric in the conclusion of Theorem 3.2. To our knowledge, even if all the fixed points sets of the involved set-valued mappings are in $B(x_0, r)$, there does not seem to be any result in the literature dealing with set-valued pseudo-contraction mappings which provides such a conclusion, see, for comparison, [10, Proposition 2.4] and the recent generalization given in [1] of Lim's lemma, see [29].

In the following example, we give two set-valued mappings satisfying the conditions of Theorem 3.2 with respect to each other. Though some conditions are relaxed, this example provides us a situation where the Pompeiu–Hausdorff metric can be used in the conclusion of Theorem 3.2.

Example 2. According to Theorem 3.2, let $X = \mathbb{R}^2$, $x_0 = (0,0)$, $r = 1$, and $\lambda = \frac{1}{\sqrt{2}}$.

Let $T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be the set-valued mapping defined by:

$$T\left((x,y)\right) = \begin{cases} \left\{\dfrac{x}{2}\right\} \times \left(\left[0, \dfrac{|x|}{2}\right] \cup [3, +\infty[\right) & \text{if } \left\| (x,y) \right\| < 1, \\ \{2x\} \times \left]0, x^2\right] & \text{if } \left\| (x,y) \right\| \geq 1. \end{cases}$$

Clearly, T has nonempty closed values on $B(x_0, r)$ and the images of points of $B(x_0, r)$ are not necessarily included in $B(x_0, r)$. And since, for every (x_1, y_1) and (x_2, y_2) in $B(x_0, r)$, we have

$$e\left(T\left((x_1, y_1)\right) \cap B(x_0, r), T\left((x_2, y_2)\right)\right) \leq \frac{1}{\sqrt{2}} |x_1 - x_2|,$$

then, T is λ-pseudo-contraction with respect to $B(x_0, r)$. We note that T is not Lipschitzian on $\mathbb{R}^2$ and $\mathrm{Fix}(T) = \{(0,0)\}$.

Now, take any $\alpha \in \left]0, 2\sqrt{\frac{\lambda}{5}}\left(1 - \sqrt{\lambda}\right)\right[$ and define $S : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by:

$$S\left((x,y)\right) = \begin{cases} \left\{\dfrac{x+\alpha}{2}\right\} \times \left(\left[0, \dfrac{|x|}{2}\right] \cup [3, +\infty[\right) & \text{if } \left\| (x,y) \right\| < 1, \\ \{2x\} \times \left]0, x^3\right] & \text{if } \left\| (x,y) \right\| \geq 1. \end{cases}$$

The set-valued mapping S has nonempty closed values on $B(x_0, r)$ and the images of points of $B(x_0, r)$ are not necessarily included in $B(x_0, r)$. Also, it is λ-pseudo-contraction with respect to $B(x_0, r)$ and $\mathrm{Fix}(S) = \{\alpha\} \times \left[0, \frac{\alpha}{2}\right]$. Finally, S is not Lipschitzian on $\mathbb{R}^2$.

We put $\beta = \left(1 - \sqrt{\lambda}\right) < (1 - \lambda) r$ and we will verify the other conditions of Theorem 3.2.

(1) For the unique fixed point $(0,0)$ of T, we have

$$d\left((0,0), S\left((0,0)\right)\right) = \frac{\alpha}{2} \le \sqrt{\frac{\lambda}{5}}\left(1-\sqrt{\lambda}\right) < \lambda\left(1-\sqrt{\lambda}\right) = \lambda\beta.$$

(2) For any $(\alpha,\gamma) \in \mathrm{Fix}\,(S)$, we have

$$d\left((\alpha,\gamma),(0,0)\right) \le \sqrt{\alpha^2+\frac{\alpha^2}{4}} = \frac{\sqrt{5}}{2}\alpha = \sqrt{\lambda}\left(1-\sqrt{\lambda}\right) < \beta$$

and

$$d\left((\alpha,\gamma), T\left((\alpha,\gamma)\right)\right) \le d\left((\alpha,\gamma),\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)\right) = \sqrt{\frac{\alpha^2}{4}+\left(\gamma-\frac{\alpha}{2}\right)^2} \le \frac{\alpha}{\sqrt{2}} < \sqrt{2}\sqrt{\frac{\lambda}{5}}\left(1-\sqrt{\lambda}\right) < \lambda\left(1-\sqrt{\lambda}\right) = \lambda\beta.$$

We conclude this section by the following easily verified result. This corollary will be useful in the sequel.

Proposition 3.4. *Under assumptions of Theorem 3.2, we have*

$$e(\mathit{Fix}\,(S)\cap B, \mathit{Fix}\,(T)) \le \frac{1}{1-\lambda}\sup_{x\in B(x_0,r)} e\left(S\left(x\right)\cap B, T\left(x\right)\right),$$

for every subset B of X such that $B \cap \mathit{Fix}\,(S) \neq \emptyset$.

Proof. It suffices to replace S in Theorem 3.2 by the set-valued mapping $S \cap B$ defined on X by $(S\cap B)\,(x) = S\,(x) \cap B$. □

4. On the inverse of the sum of two set-valued mappings

In this section, we will be concerned with the properties of inverse of the sum of two set-valued mappings.

As in Theorem 3.2 of the last section, the two set-valued mappings involved in the following results will be connected between them by some additional conditions related to the existence of fixed points. We formulate this connection in the following definition which can be compared to the notion of sum-stable maps used in [21, Definition 4.2].

Let $F, G : X \rightrightarrows Y$ be two set-valued mappings, $x_0 \in X, y_0 \in Y, B \subset Y, \alpha > 0$, and $\beta > 0$. In the sequel, we say that F is *(α,β)-compatible with respect to G on B for x_0 and y_0* if the following conditions hold:

(FP1) for every $y \in B$, there exists $x_y \in X$ such that $\left(y - G\left(x_y\right)\right) \cap \left(F\left(x_y\right) - y_0\right) \neq \emptyset$;

(FP2) whenever x is such that $\left(y - G(x)\right) \cap \left(F(x) - y_0\right) \neq \emptyset$ for some $y \in B$, then $d_X(x, x_0) < \beta$ and $d_X\left(x, F^{-1}\left(y' + y_0 - G(x)\right)\right) < \alpha\beta$, for every $y' \in B$ with $y' \neq y$.

Example 3. Put $X = Y = \mathbb{R}^2$ and $x_0 = y_0 = (0,0) \in \mathbb{R}^2$. Choose $\lambda = \frac{1}{\sqrt{2}}$, $\beta = \left(1 - \sqrt{\lambda}\right)$, and $\delta = 2\sqrt{\frac{\lambda}{5}}\left(1 - \sqrt{\lambda}\right)$. Put $B = B\left(y_0, \delta\right)$ and define, for every $z \in B$, the set-valued mapping $T_z : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by:

$$T_z\left((x,y)\right) = \begin{cases} \left\{\dfrac{x + \|z\|}{2}\right\} \times \left(\left[0, \frac{|x|}{2}\right] \cup [3, +\infty[\right) & \text{if } \|\left(x,y\right)\| < 1, \\ \{2x\} \times \left]0, x^3\right] & \text{if } \|\left(x,y\right)\| \geq 1. \end{cases}$$

As in Example 2, the set-valued mapping T_z is not Lipschitzian but λ-pseudo-contraction with respect to $B(x_0, 1)$ and has nonempty closed values on $B(x_0, 1)$ and $\mathrm{Fix}\,(T_z) = \{\|z\|\} \times \left[0, \frac{\|z\|}{2}\right]$, for every $z \in B$.

For $z \in B$ and $(\alpha, \gamma) \in \mathrm{Fix}\,(T_z)$, we have

$$d\left((\alpha,\gamma), x_0\right) < \beta \quad \text{and} \quad d\left((\alpha,\gamma), T_{z'}\left((\alpha,\gamma)\right)\right) < \lambda\beta,$$

for every $z' \in B$ such that $z \neq z'$.

Now, if F and G are the two set-valued mappings defined on $\mathbb{R}^2$ to $\mathbb{R}^2$ in such a way that for any $z \in B$ and $\left(x,y\right) \in \mathbb{R}^2$, we have

$$T_z\left((x,y)\right) = F^{-1}\left(y - G\left((x,y)\right)\right),$$

then F is (λ, β)*-compatible with respect to G on B for x_0 and y_0*

We formulate here the following inverse set-valued mapping result for the sum of two set-valued mappings similar to [14, Theorem 3], where the condition of being Lipschtzian is replaced by some local conditions such as the condition of being pseudo-Lipschitzian.

From now on, the metric of the linear metric space Y will be always assumed to be shift invariant and (-1)-homogeneous. This condition of homogeneity is also needed in [14]. A metric d_Y on a linear space Y is called α-homogeneous, $\alpha \in \mathbb{R}$, if $d_Y\left(\alpha x, \alpha y\right) = |\alpha|\, d_Y\left(x,y\right)$, for every $x, y \in Y$. Every metric associated to a norm is α-homogeneous, for every $\alpha \in \mathbb{R}$. Thus, the metric d_Y is (-1)-homogeneous if $d_Y\left(-x, -y\right) = d_Y\left(x,y\right)$, for every $x, y \in Y$.

Theorem 4.1. *Let (X, d_X) be a metric space, (Y, d_Y) be a linear metric space, $r > 0$, $x_0 \in X$ and $y_0 \in Y$ be such that $\overline{B}(x_0, r)$ is a complete metric subspace. Let $F, G : X \rightrightarrows Y$ be two set-valued mappings satisfying the following assumptions*

(1) G has nonempty closed values on $B(x_0, r)$, $G(x_0)$ is a bounded set with diameter d_0, and there exist $\alpha > 0$, $\delta > 0$, and a nonempty subset N of Y

such that $G(B(x_0,r)) \subset \overline{B(G(x_0),\alpha r)}$, *and* G *is* α*-pseudo-Lipschitzian on* $B(x_0,r)$ *with respect to* $B(G(x_0),\delta)+y_0-N$*;*

(2) $B(x_0,r) \subset dom(F)$, $B(y_0,\delta+\alpha r+d_0) \subset F(B(x_0,r))$, *F is upper semicontinuous, and there exists* $K>0$ *such that* $\alpha K<1$ *and* F^{-1} *is fully* K*-pseudo-Lipschitzian on* $B(y_0,\delta+\alpha r+d_0)$ *for* N *with respect to* $B(x_0,r)$*;*

(3) *there exists* $\beta>0$ *such that* $\beta<(1-\alpha K)r$, *and* F *is* (α,β)*-compatible with respect to* G *on* $B(G(x_0),\delta)$ *for* x_0 *and* y_0.

Then, $(F+G)^{-1}$ *is* $\frac{K}{1-\alpha K}$*-Lipschitzian on* $B(G(x_0)+y_0,\delta)$.

Proof. Let $y \in B(G(x_0),\delta)$ be fixed, and consider the set-valued mapping $T_y: X \rightrightarrows X$ defined by:

$$T_y(x) := F^{-1}(y+y_0-G(x)) = \{t \in X \mid \exists z \in G(x), y+y_0-z \in F(t)\}.$$

Clearly, $\mathrm{Fix}(T_y) = (F+G)^{-1}(y+y_0)$, and it follows by condition (3) that there exist $x_y \in X$, $y_G \in G(x_y)$, and $y_F \in F(x_y)$ such that

$$y-y_G = y_F-y_0.$$

Therefore, $y = y_F+y_G-y_0 \in (F+G)(x_y)-y_0$, and then $x_y \in (F+G)^{-1}(y+y_0)$. This proves in particular that

$$B(G(x_0)+y_0,\delta) \subset \mathrm{dom}(F+G)^{-1}.$$

To verify all the conditions of Theorem 3.2 to any couple of set-valued mappings T_y with $y \in B(G(x_0),\delta)$, we state first the following fact:

$$y+y_0-G(x) \subset B(y_0,\delta+\alpha r+d_0) \quad \forall x \in B(x_0,r).$$

Indeed, let $x \in B(x_0,r)$ and $z \in G(x)$. Since d_Y is a shift-invariant metric, it suffices to verify that $d_Y(y,z) < \delta+d_0+\alpha r$. Let $y_{x_0} \in G(x_0)$ be such that $d_Y(y,y_{x_0}) < \delta$ and put $\varepsilon = \delta - d_Y(y,y_{x_0}) > 0$. Let $u_{\varepsilon,z} \in G(x_0)$ be such that $d_Y(u_{\varepsilon,z},z) < \alpha r+\frac{\varepsilon}{2}$. Then, we obtain

$$\begin{aligned} d_Y(y,z) &\le d_Y(y,y_{x_0}) + d_Y(y_{x_0},u_{\varepsilon,z}) + d_Y(u_{\varepsilon,z},z) \\ &< d_Y(y,y_{x_0}) + d_0 + \alpha r + \frac{\varepsilon}{2} \\ &= \delta - \varepsilon + d_0 + \alpha r + \frac{\varepsilon}{2} < \delta + d_0 + \alpha r. \end{aligned}$$

The set-valued mapping T_y has nonempty closed values on $B(x_0,r)$. Indeed, let $x \in B(x_0,r)$. For every $z \in G(x)$, $y+y_0-z \in B(y_0,\delta+\alpha r+d_0)$, and then $F^{-1}(y+y_0-z) \neq \emptyset$. Thus, $T_y(x) \neq \emptyset$, for every $x \in B(x_0,r)$. Moreover, by the upper semicontinuity of F and since $y+y_0-G(x)$ is closed, then $T_y(x) = F^{-1}(y+y_0-G(x))$ is closed, for every $x \in B(x_0,r)$.

The set-valued mapping T_y is αK-pseudo-contraction with respect to $B(x_0,r)$. Indeed, for $x_1,x_2 \in B(x_0,r)$, we know from above that $y+y_0-G(x_1)$

and $y+y_0-G(x_2)$ are contained in $B\left(y_0,\delta+\alpha r+d_0\right)$. Then,

$$\begin{aligned}
&d_X\left(T_y(x_1)\cap B(x_0,r),T_y(x_2)\right)\\
&\quad=e\left(F^{-1}\left(y+y_0-G(x_1)\right)\cap B(x_0,r),F^{-1}\left(y+y_0-G(x_2)\right)\right)\\
&\quad\leq Ke\left(\left(y+y_0-G(x_1)\right)\cap N,y+y_0-G(x_2)\right).
\end{aligned}$$

Since d_Y is shift invariant and (-1)-homogeneous, then

$$\begin{aligned}
&Ke\left(\left(y+y_0-G(x_1)\right)\cap N,y+y_0-G(x_2)\right)\\
&\quad=Ke\left((G(x_1))\cap\left(y+y_0-N\right),G(x_2)\right)\leq\alpha Kd_X(x_1,x_2).
\end{aligned}$$

To verify the condition (2) of Theorem 3.2, take $y,y'\in B(G(x_0),\delta)$, $y\neq y'$ and suppose $x\in\operatorname{Fix}\left(T_y\right)$. Then, by condition (3), $d_X(x,x_0)<\beta$ and $d_X\left(x,F^{-1}\left(y'+y_0-G(x)\right)\right)<\alpha\beta$. That is, $d_X(x,x_0)<\beta$ and $d_X\left(x,T_y(x)\right)<\alpha\beta$ which are required.

It remains now to verify that the set-valued mapping $(F+G)^{-1}$ is $\frac{K}{1-\alpha K}$-Lipschitzian on $B\left(G(x_0)+y_0,\delta\right)$. Note first that we have $\operatorname{Fix}\left(T_y\right)\subset B(x_0,r)$, for every $y\in B(G(x_0),\delta)$.

For $z,z'\in B\left(G(x_0)+y_0,\delta\right)$, let $z=y+y_0$ and $z'=y'+y_0$ with $y,y'\in(G(x_0),\delta)$. We have

$$e\left((F+G)^{-1}(z),(F+G)^{-1}\left(z'\right)\right)=e\left(\operatorname{Fix}\left(T_y\right),\operatorname{Fix}\left(T_{y'}\right)\right)$$

and

$$\begin{aligned}
e\left(\operatorname{Fix}\left(T_y\right),\operatorname{Fix}\left(T_{y'}\right)\right)&=e\left(\operatorname{Fix}\left(T_y\right)\cap B(x_0,r),\operatorname{Fix}\left(T_{y'}\right)\right)\\
&\leq\frac{1}{1-\alpha K}\sup_{x\in B(x_0,r)}e\left(T_y(x)\cap B(x_0,r),T_{y'}(x)\right).
\end{aligned}$$

On the other hand, for every $x\in B(x_0,r)$, we have

$$\begin{aligned}
&e\left(T_y(x)\cap B(x_0,r),T_{y'}(x)\right)\\
&\quad=e\left(F^{-1}\left(y+y_0-G(x)\right)\cap B(x_0,r),F^{-1}\left(y'+y_0-G(x)\right)\right)\\
&\quad\leq Ke\left(\left(y+y_0-G(x)\right)\cap N,y'+y_0-G(x)\right)\\
&\quad\leq Ke\left(y+y_0-G(x),y'+y_0-G(x)\right)\leq Kd_Y\left(y,y'\right).
\end{aligned}$$

We conclude that

$$e\left((F+G)^{-1}(z),(F+G)^{-1}\left(z'\right)\right)\leq\frac{K}{1-\alpha K}d_Y\left(y,y'\right)=\frac{K}{1-\alpha K}d_Y\left(z,z'\right)$$

which, by interchanging z and z', completes the proof. □

Remark 3. We remark that in Condition (1) of the above theorem, the condition of G being α-pseudo-Lipschitzian on $B(x_0,r)$ with respect to $B(G(x_0),\delta)+y_0-B\left(y_0,\delta+\alpha r+d_0\right)$ can be replaced by the weak condition of G being

α-pseudo-Lipschitzian on $B(x_0, r)$ with respect to $y + y_0 - B\left(y_0, \delta + \alpha r + d_0\right)$, for every $y \in B(G(x_0), \delta)$.

Now, we are going to obtain a result similar to the classical result due to Graves on the inverse of continuous functions acting between Banach spaces. First, we state the following result.

Theorem 4.2. *Suppose that all the conditions of Theorem 4.1 are satisfied such that $\overline{B}\left(y_0, \delta + \alpha r + d_0\right) \subset F(\overline{B}(x_0, r))$, F^{-1} is fully K-pseudo-Lipschitzian on $\overline{B}\left(y_0, \delta + \alpha r + d_0\right)$ for N with respect to $B(x_0, r)$, and $B(G(x_0), \delta)$ is replaced by $\bigcup_{u \in G(x_0)} \overline{B}(u, \delta)$ in the corresponding conditions. Then, the set-valued mapping $(F+G)^{-1}$ is $\frac{K}{1-\alpha K}$-Lipschitzian on $\bigcup_{u \in G(x_0)} \overline{B}\left(u + y_0, \delta\right)$.*

Proof. The proof follows step by step the proof of Theorem 4.1 where instead of taking $y \in B(G(x_0), \delta)$, we take $y \in \bigcup_{u \in G(x_0)} \overline{B}(u, \delta)$. The unique fact which merits to be established is that for every $y \in \bigcup_{u \in G(x_0)} \overline{B}(u, \delta)$,

$$y + y_0 - G(x) \subset B\left(y_0, \delta + \alpha r + d_0\right) \quad \forall x \in B(x_0, r).$$

Let $y \in \bigcup_{u \in G(x_0)} \overline{B}(u, \delta)$ and take $u_y \in G(x_0)$ such that $y \in \overline{B}\left(u_y, \delta\right)$. Let $x \in B(x_0, r)$ and $z \in G(x)$. Since d_Y is a shift-invariant metric, it suffices to verify that $d_Y\left(y, z\right) \leq \delta + d_0 + \alpha r$. Since $d_X\left(y, u_y\right) \leq \delta$, let $(\varrho_n)_n$ be an increasing sequence of positive numbers such that $\lim\limits_{n \to +\infty} \varrho_n = 1$ and $\varepsilon_n = \delta - \varrho_n d_X\left(y, u_y\right) > 0$, for every n. Now, for every n, let $y_{n,z} \in G(x_0)$ be such that $d_Y\left(y_{n,z}, z\right) < \alpha r + \frac{\varepsilon_n}{2}$. Then, we obtain

$$d_Y\left(y, z\right) \leq d_Y\left(y, u_y\right) + d_Y\left(u_y, y_{n,z}\right) + d_Y\left(y_{n,z}, z\right) < d_Y\left(y, u_y\right) + d_0 + \alpha r + \frac{\varepsilon_n}{2},$$

and since $\lim_{n \to +\infty} \varepsilon_n = \delta - d_Y\left(y, u_y\right)$, we have

$$\begin{aligned} d_Y\left(y, z\right) &\leq d_Y\left(y, u_y\right) + d_0 + \alpha r + \frac{\delta - d_Y\left(y, u_y\right)}{2} \\ &= \frac{\delta + d_Y\left(y, u_y\right)}{2} + d_0 + \alpha r \leq \delta + d_0 + \alpha r \end{aligned}$$

which completes the proof. □

Remark 4. Theorems 4.1 and 4.2 provide us with the conclusion that the set-valued mapping $(F+G)^{-1}$ is Lipschitzian. In [7, Lemma 4.3], the authors obtain that the inverse of the considered set-valued mapping is pseudo-Lipschitzian, which is a property weaker than that of being Lipschitzian. It should be emphasized that this result has been used to derive sufficient conditions for the existence of double fixed points of set-valued mappings which, in particular, has applications to the problem of regularity of the composition of set-valued mappings, see [26].

Recall that the Banach open mapping theorem guarantees that a linear continuous mapping A from a Banach space X to a Banach space Y is surjective if and only if it is an open mapping. In particular, if A is surjective linear and continuous, then there exists $K > 0$ such that

$$B_Y(0,1) \subset A(B_X(0,K)).$$

Corollary 4.3. *Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two Banach spaces. Denote by $A : X \to Y$ a surjective, linear, and continuous mapping and let K be the constant arising from the Banach open mapping theorem. Let $r > 0$ and $x_0 \in X$. Let $g : X \to Y$ be a single-valued mapping and suppose that the following conditions are satisfied:*

(1) *there exist $\alpha > 0$ and a subset N of Y containing $A(x_0)$ such that $\alpha K < 1$, $g(B(x_0,r)) \subset \overline{B}\left(g(x_0), \alpha r\right)$, and g is α-pseudo-Lipschitzian on $B(x_0,r)$ with respect to $B\left(g(x_0), \frac{1-\alpha K}{K} r\right) + A(x_0) - N$;*

(2) *A^{-1} is fully K-pseudo-Lipschitzian on $\overline{B}\left(A(x_0), \frac{1-\alpha K}{K} r + \alpha r\right)$ for N with respect to $B(x_0,r)$;*

(3) *there exists $\beta > 0$ such that $\beta < (1-\alpha K) r$, and A is (α,β)-compatible with respect to g on $B\left(g(x_0), \frac{1-\alpha K}{K} r\right)$ for x_0 and y_0.*

Then, $(A+G)^{-1}$ is $\frac{K}{1-\alpha K}$-Lipschitzian on $\overline{B}\left(A(x_0) + g(x_0), \frac{1-\alpha K}{K} r\right)$.

Proof. Let $F = A$ and $G = g$. From the Banach open mapping theorem,

$$\overline{B}\left(A(x_0), \frac{r}{K}\right) \subset A\left(\overline{B}(x_0,r)\right) = F\left(\overline{B}(x_0,r)\right).$$

The proof then holds by applying Theorem 4.2 with $\delta = \frac{1-\alpha K}{K} r$, $y_0 = A(x_0)$ and $d_0 = 0$. □

We close this section by the following discussion about the conditions on mapping A which have been involved in the proof of Corollary 4.3. The continuity of A implies, by the Banach open mapping theorem, the openness of A. However, the linearity of A is not used in the proof. Instead of that, we need that A^{-1} is fully pseudo-Lipschitzian.

On the other hand, the openness of A can be involved without the linearity of A. In the literature and, especially in convex analysis without linearity, generalizations of some implicit function theorems and other questions of optimization have been obtained without linearity, see [33]. See also [36] where a notion denoted by PL weaker than that of the linearity has been recently defined and a generalization of the Banach open mapping theorem has been derived. It is shown, in particular, that every surjective continuous mapping acting between Banach spaces and satisfying the conditions of the notion PL is open.

5. Applications to variational inclusions

In this section, we deal with the sensitivity analysis of variational inclusions. Based on the proximal convergence, we make use here of our results developed above and develop techniques related to the existence of solutions of variational inclusions.

Let (P, d_P) be a metric space which is called the set of parameters, and let $A : P \times X \rightrightarrows Y$ be a set-valued mapping, where (X, d_X) is a metric space and (Y, d_Y) a linear metric space. For a fixed value of the parameter $p \in P$, we consider the parameterized generalized equation:

$$\text{find } x \in \operatorname{dom}\left(A\left(p, .\right)\right) \text{ such that } 0 \in A\left(p, x\right), \tag{5.1}$$

where its set of solutions is denoted by $S_A\left(p\right)$ which defines a set-valued mapping.

The regularity properties of solution mapping $p \mapsto S_A\left(p\right)$ has been the subject of study of many authors since it is related to the theory of implicit functions and its applications for variational inclusions, see for instance, [4, 20, 21] and the references therein.

We define a measure of the sensitivity of the solutions with respect to small changes in the problem's data to apply it to the problem of existence of solutions of variational inclusions. For any $p_0 \in P$, we define the full condition number of A at p_0 with respect to a subset W of X as the extended real-valued number by:

$$c_f^*\left(A \mid p_0, W\right) = \limsup_{\substack{Z,Z' \to \{p_0\} \\ Z \neq Z', Z \neq \emptyset}} \frac{e\left(S_A\left(Z\right) \cap W, S_A\left(Z'\right)\right)}{e\left(Z, Z'\right)}$$

where the convergence is taken in the sense of the upper proximal convergence. A net $\left(Z_\gamma\right)_\gamma$ is upper proximal convergent to Z if $\lim_\gamma e\left(Z_\gamma, Z\right) = 0$, see [3, 12]. Then, we have

$$c_f^*\left(A \mid p_0, W\right)$$
$$= \inf_{\varepsilon > 0} \sup \left\{ \frac{e\left(S_A\left(Z\right) \cap W, S_A\left(Z'\right)\right)}{e\left(Z, Z'\right)} \mid Z, Z' \subset B\left(p_0, \varepsilon\right), Z \neq Z', Z \neq \emptyset \right\}.$$

Also, the extended real number $K\left(A, \delta | p_0, W\right)$ is defined by:

$$K\left(A, \delta \mid p_0, W\right)$$
$$= \sup \left\{ \frac{e\left(S_A\left(Z\right) \cap W, S_A\left(Z'\right)\right)}{e\left(Z, Z'\right)} \mid Z, Z' \subset B\left(p_0, \delta\right), Z \neq Z', Z \neq \emptyset \right\}.$$

Clearly, the function $\delta \mapsto K\left(A, \delta \mid p_0, W\right)$ is decreasing and for every $p_0 \in P$, we have $\lim_{\delta \to 0} K\left(A, \delta \mid p_0, W\right) = c^*\left(A \mid p_0, W\right)$.

Proposition 5.1. *If $K\left(A,\delta \mid p_0, W\right) < +\infty$, then one of the following alternatives holds:*

(1)*there exists a neighborhood $V\left(p_0\right)$ of p_0 such that $S_A\left(p\right) = \emptyset$, for every $p \in U\left(p_0\right)$;*

(2)*there exists a neighborhood $V\left(p_0\right)$ of p_0 such that $S_A\left(p\right) \neq \emptyset$, for every $p \in U\left(p_0\right)$.*

In particular, if $0 < K\left(A,\delta \mid p_0, W\right) < +\infty$, then there exists a neighborhood $V\left(p_0\right)$ of p_0 such that the solution set of the parameterized generalized equation (5.1) is nonempty, for every $p \in V\left(p_0\right)$.

In the sequel, we focus on the special case where $P = Y$. We study the parameterized generalized equation associated to $A : Y \times X \rightrightarrows Y$ defined using a set-valued mapping $F : X \rightrightarrows Y$ as follows:

$$A\left(p,x\right) = \begin{cases} F\left(x\right) - p & \text{if } x \in B\left(x_0, r\right), \\ \emptyset & \text{otherwise.} \end{cases}$$

We remark that $S_A\left(Z\right) = F^{-1}\left(Z\right)$, for every subset Z of P and it results that in this framework, the full condition number given above takes the more explicit form

$$c_f^*\left(A \mid p_0, W\right) = \limsup_{\substack{Z,Z' \to \{p_0\} \\ Z \neq Z', Z \neq \emptyset}} \frac{e\left(F^{-1}\left(Z\right) \cap W, F^{-1}\left(Z'\right)\right)}{e\left(Z, Z'\right)}$$

In this setting, we will write $c_f^*\left(F \mid p_0, W\right)$ and $K\left(F,\delta \mid p_0, B\left(x_0,r\right)\right)$ instead of $c_f^*\left(A \mid p_0, W\right)$ and $K\left(A,\delta \mid p_0, B\left(x_0,r\right)\right)$, respectively.

Now, we obtain the following result on the existence of solutions of parameterized generalized equations.

Theorem 5.2. *Let $r > 0$, $x_0 \in X$, and $p_0 \in Y$ be such that $\overline{B}\left(x_0,r\right)$ is a complete metric subspace. Let $G : X \rightrightarrows Y$ be a set-valued mapping. Suppose that $0 < c^*\left(F \mid p_0, B\left(x_0,r\right)\right) < +\infty$ and choose $\overline{\delta}$ such that $K\left(F,\overline{\delta} \mid p_0, B\left(x_0,r\right)\right) < +\infty$. Suppose further that the following conditions are satisfied*

(1) *G has nonempty closed values on $B\left(x_0,r\right)$, $G\left(x_0\right)$ is a bounded set with diameter $d_0 < \overline{\delta}$, and there exist a subset N containing $B\left(p_0,\overline{\delta}\right)$ and $0 < \alpha < \min\left\{\frac{\overline{\delta}-d_0}{r}, \frac{1}{K\left(F,\overline{\delta} \mid p_0, B\left(x_0,r\right)\right)}\right\}$ such that $G\left(B\left(x_0,r\right)\right) \subset \overline{B\left(G\left(x_0\right), \alpha r\right)}$, and G is α-pseudo-Lipschitzian on $B\left(x_0,r\right)$ with respect to $B\left(G\left(x_0\right), \overline{\delta} - \alpha r - d_0\right) + p_0 - N$;*

(2) *$B\left(x_0,r\right) \subset \mathrm{dom}\left(F\right)$, $B\left(p_0,\overline{\delta}\right) \subset F\left(B\left(x_0,r\right)\right)$, and F is upper semicontinuous.*

(3) *there exists $\beta > 0$ such that $\beta < \left(1 - \alpha K\left(F,\overline{\delta} \mid p_0, B\left(x_0,r\right)\right)\right) r$, and F is $\left(\alpha,\beta\right)$-compatible with respect to G on $B\left(G\left(x_0\right), \overline{\delta} - \alpha r - d_0\right)$ for x_0 and p_0.*

Then, $c^*\left(F+G \mid p_0+y, B\left(x_0, r\right)\right) \leq \frac{K\left(F,\overline{\delta} \mid p_0, B\left(x_0, r\right)\right)}{1-\alpha K\left(F,\overline{\delta} \mid p_0, B\left(x_0, r\right)\right)}<+\infty$, *for every* $y \in G\left(x_0\right)$.

Proof. Put $\delta=\overline{\delta}-\alpha r-d_0>0$. We have $\alpha K\left(F, \overline{\delta} \mid p_0, B\left(x_0, r\right)\right)<1$. Also, for every subsets Z, Z' of $B\left(p_0, \overline{\delta}\right)$, we have

$$e\left(F^{-1}(Z) \cap B\left(x_0, r\right), F^{-1}\left(Z'\right)\right) \leq K\left(F, \overline{\delta} \mid p_0, B\left(x_0, r\right)\right) e\left(Z, Z'\right)$$

and then, F^{-1} is fully $K\left(F, \overline{\delta} \mid p_0, B\left(x_0, r\right)\right)$-pseudo-Lipschitzian on $B\left(p_0, \overline{\delta}\right)$ for N with respect to $B\left(x_0, r\right)$.

It results by applying Theorem 4.1 that the set-valued mapping $(F+G)^{-1}$ is $\frac{K\left(F,\overline{\delta} \mid p_0, B\left(x_0, r\right)\right)}{1-\alpha K\left(F,\overline{\delta} \mid p_0, B\left(x_0, r\right)\right)}$-Lipschitzian on $B\left(G\left(x_0\right)+p_0, \delta\right)$. Then, for every $y \in G\left(x_0\right)$, we have

$$\begin{aligned}
& c^*\left(F+G \mid y+p_0, B\left(x_0, r\right)\right) \\
& \quad=\limsup_{\substack{Z, Z' \rightarrow\left\{y+p_0\right\} \\ Z \neq Z', Z \neq \emptyset}} \frac{e\left((F+G)^{-1}(Z) \cap B\left(x_0, r\right),(F+G)^{-1}\left(Z'\right)\right)}{e\left(Z, Z'\right)} \\
& \quad \leq \sup_{\substack{Z, Z' \subset B\left(y+p_0, \delta\right), \\ Z \neq Z', Z \neq \emptyset}} \frac{e\left((F+G)^{-1}(Z),(F+G)^{-1}\left(Z'\right)\right)}{e\left(Z, Z'\right)} \\
& \quad \leq \frac{K\left(F, \overline{\delta} \mid p_0, B\left(x_0, r\right)\right)}{1-\alpha K\left(F, \overline{\delta} \mid p_0, B\left(x_0, r\right)\right)}<+\infty,
\end{aligned}$$

which completes the proof. □

In conclusion, we have obtained in this article the results on the behaviors of fixed points sets of set-valued mappings similar to the classical ones but with new conditions and different proofs. Then, we have highlighted the properties of set-valued pseudo-Lipschitzian mappings to deal with the Lipschitzian property of the inverse of sum of two set-valued mappings. In our approach, we have considered techniques based on handling subsets rather than points which are usually used in these studies.

Acknowledgement

We are very grateful to the reviewers for their valuable comments and suggestions, which greatly improved the article.

Funding

Author Rădulescu has been supported by a grant of Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-II-PT-PCCA-2013-4-0614.

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