Journal of Difference Equations and Applications
Vol. 00, No. 00, Month 200x, 1–9

RESEARCH ARTICLE

Discrete boundary value problems involving oscillatory nonlinearities: small and large solutions

Alexandru Kristály[a], Mihai Mihăilescu [b,c] and Vicenţiu Rădulescu [c,d,*]

[a] *Babeş-Bolyai University, Department of Economics, 400591 Cluj-Napoca, Romania*;
[b] *Department of Mathematics, Central European University, 1051 Budapest, Hungary*;
[c] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania*;
[d] *Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania*

(*Submitted: February 2009*)

We consider the discrete boundary value problem (P): $-\Delta(\Delta u(k-1)) = f(u(k)),\ k \in [1,T]$, $u(0) = u(T+1) = 0$, where the nonlinear term $f : [0,\infty) \to \mathbb{R}$ has an oscillatory behaviour near the origin or at infinity. By a direct variational method we show that (P) has a sequence of non-negative, distinct solutions which converges to 0 (resp. $+\infty$) in the sup-norm whenever f oscillates at the origin (resp. at infinity).

Keywords: difference equations; oscillatory nonlinearities; small solutions; large solutions

AMS Subject Classification: 46E39, 34K10, 35B38

1. Introduction and main results

In many cases a problem in a continuous framework can be handled by using a suitable method from discrete mathematics and conversely; a beautiful description of such phenomena can be found in Lovász [12]. The modeling/simulation of certain nonlinear problems from economics, biological neural networks, optimal control and others enforced in a natural manner the rapid development of the theory of difference equations. The reader may consult the comprehensive monographs of Agarwal [1], Kelley-Peterson [10], Lakshmikantham-Trigiante [11].

Within the theory of difference equations, a large class of problems is the nonlinear discrete boundary value problems. To be more precise, we consider the problem

$$\begin{cases} -\Delta(\Delta u(k-1)) = f(u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{P}$$

where $T \geq 2$ is an integer, $[1,T]$ is the discrete interval $\{1,...,T\}$, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, and f is a continuous nonlinearity. In order to establish existence/multiplicity of solutions for (P) under specific restrictions on f (sublinear or superlinear at infinity), the authors exploited various abstract methods as fixed point theorems, sub- and super-solution

*Corresponding author. Email: vicentiu.radulescu@math.cnrs.fr

ISSN: 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/1023619YYxxxxxxxx
http://www.informaworld.com

arguments, Brouwer degree and critical point theory. We refer the reader to the recent papers of Agarwal-Perera-O'Regan [2, 3], Bereanu-Mawhin [4, 5], Bereanu-Thompson [6], Bonanno-Candito [7], Cabada-Iannizzotto-Tersian [8], Cai-Yu [9], Mihăilescu-Rădulescu-Tersian [13], Yu-Guo [15], Tang-Luo-Li-Ma [14], Zhang-Liu [16], and references therein.

The main purpose of the present paper is to trait problem (P) when the nonlinear term $f : [0,\infty) \to \mathbf{R}$ has a suitable *oscillatory behaviour*. A direct variational argument provides two results (see Theorems 1.1 and 1.2), guaranteeing sequences of non-negative solutions with further asymptotic properties whenever f oscillates near the origin or at infinity. Before to state our results, we mention that solutions of (P) are going to be sought in the function space

$$X = \{u : [0, T+1] \to \mathbb{R};\ u(0) = u(T+1) = 0\}.$$

Clearly, X is a T-dimensional Hilbert space (see [2]) with the inner product

$$\langle u, v\rangle = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \quad \forall\, u, v \in X\,.$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{1/2}.$$

The space X being finite-dimensional, the sup-norm $\|\cdot\|_\infty$ is equivalent to $\|\cdot\|$; here, we denote $\|u\|_\infty = \max_{k\in[1,T]} |u(k)|$, $u \in X$.

In the sequel, we state our results. Let $F(s) = \int_0^s f(t)dt$, $s \in [0,\infty)$.

Our first result concerns the case when f has a certain type of oscillation near the origin. To be more precise, we assume

(H^0) $\liminf_{s\to 0^+} \frac{f(s)}{s} < 0$; $\limsup_{s\to 0^+} \frac{F(s)}{s^2} > \frac{1}{T}$.

THEOREM 1.1. *Let $f \in C^0([0,\infty);\mathbf{R})$ verifying (H^0). Then there exists a sequence $\{u_n^0\}_n \subset X$ of non-negative, distinct solutions of* (P) *such that*

$$\lim_{n\to\infty} \|u_n^0\|_\infty = \lim_{n\to\infty} \|u_n^0\| = 0. \tag{1}$$

A perfect counterpart of Theorem 1.1 can be stated when the nonlinear term oscillates at infinity. Instead of (H^0), we assume

(H^∞) $\liminf_{s\to\infty} \frac{f(s)}{s} < 0$; $\limsup_{s\to\infty} \frac{F(s)}{s^2} > \frac{1}{T}$.

THEOREM 1.2. *Let $f \in C^0([0,\infty);\mathbf{R})$ verifying (H^∞) and $f(0) = 0$. Then there exists a sequence $\{u_n^\infty\}_n \subset X$ of non-negative, distinct solutions of* (P) *such that*

$$\lim_{n\to\infty} \|u_n^\infty\|_\infty = \lim_{n\to\infty} \|u_n^\infty\| = \infty. \tag{2}$$

Example 1.3 (a) Let $\alpha, \beta, \gamma \in \mathbf{R}$ such that $0 < \alpha < 1 < \alpha + \beta$, and $\gamma \in (0,1)$. Then, the function $f : [0,\infty) \to \mathbf{R}$ defined by $f(0) = 0$ and $f(s) = s^\alpha(\gamma + \sin s^{-\beta})$, $s > 0$, verifies hypothesis (H^0).

(b) Let $\alpha, \beta, \gamma \in \mathbf{R}$ such that $1 < \alpha$, $|\alpha - \beta| < 1$, and $\gamma \in (0,1)$. Then, the function $f : [0,\infty) \to \mathbf{R}$ defined by $f(s) = s^\alpha(\gamma + \sin s^\beta)$ verifies (H^∞).

The paper is divided as follows. In the next section we consider a related difference equation to (P); the existence of a non-negative solution is proved under some generic assumptions. This result is used in Sections 3 and 4, where Theorems 1.1 and 1.2 are proved by a careful analysis of certain energy levels associated to (P).

2. A key result

For a fixed $c > 0$, we consider the problem

$$\begin{cases} -\Delta(\Delta u(k-1)) + cu(k) = g(u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{P$_c$}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, let $E_c : X \to \mathbb{R}$ be the energy functional associated to problem (P$_c$) defined by

$$E_c(u) = \frac{1}{2}\|u\|^2 + \frac{c}{2}\sum_{k=1}^{T}(u(k))^2 - \mathcal{G}(u), \quad u \in X,$$

where

$$\mathcal{G}(u) = \sum_{k=1}^{T} G(u(k)), \quad \text{and} \quad G(s) = \int_0^s g(t)dt, \quad s \in \mathbb{R}.$$

It is immediate to show that E_c is well-defined, it belongs to $C^1(X;\mathbb{R})$ and

$$E_c'(u)(v) = \langle u, v\rangle + c\sum_{k=1}^{T} u(k)v(k) - \sum_{k=1}^{T} g(u(k))v(k), \quad \forall u, v \in X.$$

Since we have

$$\langle u, v\rangle = -\sum_{k=1}^{T+1}\Delta(\Delta u(k-1))v(k),$$

an element $u \in X$ is a solution for (P$_c$) if $E_c'(u)(v) = 0$ for every $v \in X$, i.e., u is a critical point of E_c.

Let $d < 0 < a < b$ some fixed numbers. We introduce the set

$$N^b = \{u \in X : d \le u(k) \le b \ \text{ for every } \ k \in [1,T]\}. \tag{3}$$

We assume on $g : \mathbf{R} \to \mathbf{R}$ that

(H_g) $\quad g(s) = 0$ for $s \le 0$, and $g(s) \le 0$ for every $s \in [a, b]$.

The main result of this section is as follows.

PROPOSITION 2.1. *Assume that $g : \mathbf{R} \to \mathbf{R}$ verifies (H_g). Then*

(a) E_c is bounded from below on N^b attaining its infimum at $\tilde{u} \in N^b$;
(b) $\tilde{u}(k) \in [0, a]$ for every $k \in [1,T]$;
(c) $\tilde{u}$ is a solution of (P$_c$).

Proof. (a) Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent in the finite-dimensional space X, the set N^b is compact in X. Combining this fact with the continuity of E_c, we infer that $E_c|_{N^b}$ attains its infimum at $\tilde{u} \in N^b$.

(b) Let $K = \{k \in [1,T] : \tilde{u}(k) \notin [0,a]\}$ and suppose that $K \neq \emptyset$. Define the truncation function $\gamma : \mathbf{R} \to \mathbf{R}$ by $\gamma(s) = \min(s_+, a)$, where $s_+ = \max(s,0)$. Now, set $w = \gamma \circ \tilde{u}$. Since $\gamma(0) = 0$ we have that $w(0) = w(T+1) = 0$, so $w \in X$. Moreover, $w(k) \in [0,a]$ for every $k \in [1,T]$; thus $w \in N^a \subset N^b$.

We introduce the sets

$$K_- = \{k \in K : \ \tilde{u}(k) < 0\} \qquad \text{and} \qquad K_+ = \{k \in K : \ \tilde{u}(k) > a\}.$$

Thus, $K = K_- \cup K_+$, and we have that $w(k) = \tilde{u}(k)$ for all $k \in [1,T] \setminus K$, $w(k) = 0$ for all $k \in K_-$, and $w(k) = a$ for all $k \in K_+$. Moreover, we have

$$\begin{aligned} E_c(w) - E_c(\tilde{u}) &= \frac{1}{2}(\|w\|^2 - \|\tilde{u}\|^2) + \frac{c}{2}\sum_{k=1}^{T}[(w(k))^2 - (\tilde{u}(k))^2] - [\mathcal{G}(w) - \mathcal{G}(\tilde{u})] \\ &=: \frac{1}{2}I_1 + \frac{c}{2}I_2 - I_3. \end{aligned} \tag{4}$$

Since γ is a Lipschitz function with Lipschitz-constant 1, and $w = \gamma \circ \tilde{u}$, we have

$$\begin{aligned} I_1 = \|w\|^2 - \|\tilde{u}\|^2 &= \sum_{k=1}^{T+1}[|\Delta w(k-1)|^2 - |\Delta \tilde{u}(k-1)|^2] \\ &= \sum_{k=1}^{T+1}[|w(k) - w(k-1)|^2 - |\tilde{u}(k) - \tilde{u}(k-1)|^2] \\ &\leq 0. \end{aligned} \tag{5}$$

Moreover, we have

$$\begin{aligned} I_2 = \sum_{k=1}^{T}[(w(k))^2 - (\tilde{u}(k))^2] &= \sum_{k\in K}[(w(k))^2 - (\tilde{u}(k))^2] \\ &= \sum_{k\in K_-} -(\tilde{u}(k))^2 + \sum_{k\in K_+}[a^2 - (\tilde{u}(k))^2] \\ &\leq 0. \end{aligned} \tag{6}$$

Next, we estimate I_3. First, let us point out that $G(s) = 0$ for $s \leq 0$; thus, $\sum_{k\in K_-}[G(w(k)) - G(\tilde{u}(k))] = 0$. By the mean value theorem, for every $k \in K_+$, there exists $n_k \in [a, \tilde{u}(k)] \subset [a,b]$ such that $G(w(k)) - G(\tilde{u}(k)) = G(a) - G(\tilde{u}(k)) = g(n_k)(a - \tilde{u}(k))$. Taking into account hypothesis (H_g), we have that $G(w(k)) - G(\tilde{u}(k)) \geq 0$ for every $k \in K_+$. Consequently,

$$\begin{aligned} I_3 = \mathcal{G}(w) - \mathcal{G}(\tilde{u}) &= \sum_{k\in K}[G(w(k)) - G(\tilde{u}(k))] = \sum_{k\in K_+}[G(w(k)) - G(\tilde{u}(k))] \\ &\geq 0. \end{aligned} \tag{7}$$

Combining relations (5)-(7) with (4), we have that

$$E_c(w) - E_c(\tilde{u}) \le 0.$$

On the other hand, since $w \in N^b$, then $E_c(w) \ge E_c(\tilde{u}) = \inf_{N^b} E_c$. So, every term in $E_c(w) - E_c(\tilde{u})$ should be zero. In particular, from I_2, we have

$$\sum_{k\in K_-} (\tilde{u}(k))^2 = \sum_{k\in K_+} [a^2 - (\tilde{u}(k))^2] = 0,$$

which imply that $\tilde{u}(k) = 0$ for every $k \in K_-$ and $\tilde{u}(k) = a$ for every $k \in K_+$. By definition of the sets K_- and K_+, we must have $K_- = K_+ = \emptyset$, which contradicts $K_- \cup K_+ = K \neq \emptyset$.

(c) Let us fix $v \in X$ arbitrarily. Due to (b), it is clear that $\tilde{u} + \varepsilon v \in N^b$ for $|\varepsilon|$ small enough. Consequently, due to (a), the function $j(\varepsilon) = E_c(\tilde{u} + \varepsilon v)$ has its minimum at 0; being differentiable at 0, we have that $j'(0) = 0$, i.e., $E_c'(\tilde{u})(v) = 0$, which means that $\tilde{u}$ is a solution of (P_c). This completes the proof. □

3. Proof of Theorem 1.1

We assume hypothesis (H^0) holds. In particular, we have $f(0) = 0$. One may fix $c_0 > 0$ such that $\liminf_{s\to 0^+} \frac{f(s)}{s} < -c_0 < 0$. Consequently, there is a sequence $\{\overline{s}_n\}_n \subset (0,1)$ converging (decreasingly) to 0, such that

$$f(\overline{s}_n) < -c_0 \overline{s}_n. \tag{8}$$

Let us define the functions $g_0, G_0 : \mathbb{R} \to \mathbb{R}$ by

$$g_0(s) = f(s_+) + c_0 s_+ \quad \text{and} \quad G_0(s) = \int_0^s g_0(t)dt, \quad s \in \mathbb{R}, \tag{9}$$

where $s_+ = \max(s, 0)$. Due to (8), $g_0(\overline{s}_n) < 0$; so, there are two sequences $\{a_n\}_n$, $\{b_n\}_n \subset (0,1)$, both converging to 0, such that $b_{n+1} < a_n < \overline{s}_n < b_n$ for every $n \in \mathbb{N}$ and

$$g_0(s) \le 0 \text{ for all } s \in [a_n, b_n].$$

In this way, hypothesis (H_g) is verified for g_0 on every interval $[a_n, b_n]$, $n \in \mathbb{N}$. Applying Proposition 2.1 to every interval $[a_n, b_n], n \in \mathbb{N}$, the problem

$$\begin{cases} -\Delta(\Delta u(k-1)) + c_0 u(k) = g_0(u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{P_{c_0}}$$

has a sequence of non-negative solutions $\{u_n^0\}_n \subset X$, where u_n^0 is a relative minimum of the functional E_{c_0} associated to (P_{c_0}) on the set N^{b_n}, $n \in \mathbb{N}$. Furthermore, since $g_0(s) = f(s) + c_0 s$ on the interval $(0,1)$, the elements u_n^0 are also solutions of problem (P). Moreover, due to Proposition 2.1 (b), we also have

$$0 \le u_n^0(k) \le a_n \ \text{ for every } k \in [1,T], \ n \in \mathbb{N}. \tag{10}$$

In the sequel, carrying out an energy-level analysis, we prove that there are infinitely many distinct elements in the sequence $\{u_n^0\}_n \subset X$. Due to (H^0) and (9), we have that $\limsup_{s\to 0^+} \frac{G_0(s)}{s^2} > \frac{1}{T} + \frac{c_0}{2}$. In particular, there exists a sequence $\{s_n\}_n$ with $0 < s_n \le a_n$, $n \in \mathbb{N}$, and

$$G_0(s_n) > \left(\frac{1}{T} + \frac{c_0}{2}\right) s_n^2.$$

Define the function $w_n \in X$ by $w_n(k) = s_n$ for every $k \in [1, T]$. Then, we have

$$\begin{aligned}
E_{c_0}(w_n) &= \frac{1}{2}\sum_{k=1}^{T+1} |\Delta w_n(k-1)|^2 + \frac{c_0}{2}\sum_{k=1}^{T}(w_n(k))^2 - \sum_{k=1}^{T} G_0(w_n(k)) \\
&= s_n^2 + \frac{c_0 T}{2} s_n^2 - T G_0(s_n) \\
&< s_n^2 + \frac{c_0 T}{2} s_n^2 - T\left(\frac{1}{T} + \frac{c_0}{2}\right) s_n^2 \\
&= 0.
\end{aligned}$$

The above estimation and $w_n \in N^{s_n} \subset N^{b_n}$ show that

$$E_{c_0}(u_n^0) = \min_{N^{b_n}} E_{c_0} \le E_{c_0}(w_n) < 0 \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

Once we prove that

$$\lim_{n\to\infty} E_{c_0}(u_n^0) = 0, \tag{12}$$

our claim holds. Indeed, (11) and (12) imply that there are infinitely many distinct elements in the sequence $\{u_n^0\}_n \subset X$. We clearly have

$$\begin{aligned}
E_{c_0}(u_n^0) &= \frac{1}{2}\sum_{k=1}^{T+1} |\Delta u_n^0(k-1)|^2 + \frac{c_0}{2}\sum_{k=1}^{T}(u_n^0(k))^2 - \sum_{k=1}^{T} G_0(u_n^0(k)) \\
&\ge -\sum_{k=1}^{T} G_0(u_n^0(k)) \ge -\sum_{k=1}^{T} u_n^0(k) \max_{s\in[0,u_n^0(k)]} |g_0(s)| \\
&\ge -\max_{s\in[0,a_n]} |g_0(s)| \sum_{k=1}^{T} u_n^0(k) \\
&\ge -a_n T \max_{s\in[0,1]} |g_0(s)|.
\end{aligned}$$

Since $\lim_{n\to\infty} a_n = 0$, the above estimate and (11) yield (12).

Relation (1) is an immediate consequence of (10), $\lim_{n\to\infty} a_n = 0$, and to the fact that the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent. The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. We assume hypothesis (H^∞) holds. We choose $c_\infty > 0$ such that $\liminf_{s\to\infty} \frac{f(s)}{s} < -c_\infty < 0$. Consequently, we may fix a sequence $\{\overline{s}_n\}_n \subset (0,\infty)$ such that $\lim_{n\to\infty} \overline{s}_n = \infty$ and

$$f(\overline{s}_n) < -c_\infty \overline{s}_n. \tag{13}$$

We define the functions $g_\infty, G_\infty : \mathbb{R} \to \mathbb{R}$ by

$$g_\infty(s) = f(s_+) + c_\infty s_+ \quad \text{and} \quad G_\infty(s) = \int_0^s g_\infty(t)dt, \quad s \in \mathbb{R}. \tag{14}$$

Due to the right hand side inequality of (H^∞) and (14), we have that $\limsup_{s\to\infty} \frac{G_\infty(s)}{s^2} > \frac{1}{T} + \frac{c_\infty}{2}$. In particular, for a small $\varepsilon_\infty > 0$, there exists a sequence $\{s_n\}_n$ tending to ∞ such that

$$G_\infty(s_n) > \left(\frac{1}{T} + \frac{c_\infty}{2} + \varepsilon_\infty\right) s_n^2. \tag{15}$$

Since $\lim_{n\to\infty} \overline{s}_n = \infty$, one can fix a subsequence $\{\overline{s}_{m_n}\}_n$ of $\{\overline{s}_n\}_n$ such that $s_n \leq \overline{s}_{m_n}$ for every $n \in \mathbf{N}$. On account of (13), $g_\infty(\overline{s}_{m_n}) < 0$; thus, we may fix two sequences $\{a_n\}_n$, $\{b_n\}_n \subset (0,\infty)$ such that $a_n < \overline{s}_{m_n} < b_n < a_{n+1}$ for every $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$, and

$$g_\infty(s) \leq 0 \text{ for all } s \in [a_n, b_n].$$

Consequently, the function g_∞ fulfills (H_g) on every interval $[a_n, b_n]$, $n \in \mathbb{N}$. We apply Proposition 2.1 to every interval $[a_n, b_n], n \in \mathbb{N}$, obtaining that the problem

$$\begin{cases} -\Delta(\Delta u(k-1)) + c_\infty u(k) = g_\infty(u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{P_{c_∞}}$$

has a sequence of non-negative solutions $\{u_n^\infty\}_n \subset X$, where u_n^∞ is a relative minimum of the functional E_{c_∞} associated to (P_{c_∞}) on the set N^{b_n}, $n \in \mathbb{N}$. Since $g_\infty(s) = f(s) + c_\infty s$ on $[0,\infty)$, the elements u_n^∞ are solutions not only for (P_{c_∞}) but also for (P).

Now, we are going to prove that there are infinitely many distinct elements in the sequence $\{u_n^\infty\}_n \subset X$. To do this, it is enough to show that

$$\lim_{n\to\infty} E_{c_\infty}(u_n^\infty) = -\infty. \tag{16}$$

Define the function $w_n \in X$ by $w_n(k) = s_n$ for every $k \in [1,T]$. Then, by using

(15), we have

$$\begin{aligned} E_{c_\infty}(w_n) &= \frac{1}{2}\sum_{k=1}^{T+1}|\Delta w_n(k-1)|^2 + \frac{c_\infty}{2}\sum_{k=1}^{T}(w_n(k))^2 - \sum_{k=1}^{T} G_\infty(w_n(k)) \\ &= s_n^2 + \frac{c_\infty T}{2}s_n^2 - TG_\infty(s_n) \\ &< s_n^2 + \frac{c_\infty T}{2}s_n^2 - T\left(\frac{1}{T} + \frac{c_\infty}{2} + \varepsilon_\infty\right)s_n^2 \\ &= -T\varepsilon_\infty s_n^2. \end{aligned}$$

By construction, we know that $w_n \in N^{s_n} \subset N^{b_n}$, thus

$$E_{c_\infty}(u_n^\infty) = \min_{N^{b_n}} E_{c_\infty} \le E_{c_\infty}(w_n) < -T\varepsilon_\infty s_n^2 \quad \text{for all } n \in \mathbb{N}. \tag{17}$$

Since $\lim_{n\to\infty} s_n = \infty$, relation (17) implies (16).

It remains to prove (2). Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent, it is enough to prove the former limit, i.e., $\lim_{n\to\infty}\|u_n^\infty\|_\infty = \infty$. By contradiction, we assume that for a subsequence of $\{u_n^\infty\}_n$, still denoted by $\{u_n^\infty\}_n$, one can find a constant $C > 0$ such that $\|u_n^\infty\|_\infty \le C$ for every $n \in \mathbb{N}$. Therefore, we have

$$E_{c_\infty}(u_n^\infty) \ge -\sum_{k=1}^{T} G_\infty(u_n^\infty(k)) \ge -T \max_{s\in[0,C]}|G_\infty(s)| \quad \text{for every } n \in \mathbb{N}.$$

This inequality contradicts relation (16) which completes the proof of Theorem 1.2.

Remark 1. When $T = 2$, the conclusions of Theorems 1.1 and 1.2 may be obtained in a very simple way. In this case, it is enough to solve the system

$$\begin{cases} 2a - b = f(a), \\ 2b - a = f(b), \\ a, b > 0. \end{cases} \tag{P$'$}$$

Indeed, a solution of (P) is any function $u : [0, 3] \to \mathbb{R}$ defined by $u(0) = u(3) = 0$, $u(1) = a$, $u(2) = b$. As one can observe, if there is a sequence of distinct fixed points for f, say $\{c_n\}_n \subset (0,\infty)$, we have infinitely many solutions for problem (P$'$) of the form $(a, b) = (c_n, c_n)$. Let us assume the contrary, i.e., there is at most finite number of distinct fixed points for f. Combining this assumption with the left hand side of (H^0), there exists a $\delta > 0$ such that $f(s) < s$ for every $s \in (0, \delta)$. After an integration we obtain that

$$\limsup_{s\to 0^+} \frac{F(s)}{s^2} \le \frac{1}{2} = \frac{1}{T}$$

which contradicts the right hand side of (H^0). In a similar manner, when (H^∞) holds, we can fix a compact set $L \subset [0,\infty)$ such that $f(s) < s$ for every $s \in (0,\infty)\setminus L$, which contradicts the right hand side of (H^∞).

The above arguments also suggest that the constant $\frac{1}{T}$ in (H^0) and (H^∞) is optimal.

Acknowledgments. A. Kristály is supported by Grant CNCSIS, IDEI 527/2007. M. Mihăilescu and V. Rădulescu are supported by Grant CNCSIS, IDEI 79/2007.

References

[1] R. P. Agarwal, *Difference equations and inequalities*, Marcel Dekker Inc., 2000.

[2] R. P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *Nonlinear Analysis* **58** (2004), 69-73.

[3] R. P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, *Advances in Difference Equations* **2** (2005) 93-99.

[4] C. Bereanu and J. Mawhin, Boundary value problems for second-order nonlinear difference equations with discrete ϕ-Laplacian and singular ϕ. *Journal of Difference Equations and Applications* **14** (2008), no. 10-11, 1099–1118.

[5] C. Bereanu and J. Mawhin, Existence and multiplicity results for periodic solutions of nonlinear difference equations. *Journal of Difference Equations and Applications* **12** (2006), no. 7, 677–695.

[6] C. Bereanu and H. B. Thompson, Periodic solutions of second order nonlinear difference equations with discrete ϕ-Laplacian. *J. Math. Anal. Appl.* **330** (2007), no. 2, 1002–1015.

[7] G. Bonanno and P. Candito, Nonlinear difference equations investigated via critical point methods, *Nonlinear Analysis* **70** (2009), 3180-3186.

[8] A. Cabada, A. Iannizzotto and S. Tersian, Multiple solutions for discrete boundary value problems, *J. Math. Anal. Appl.* **356** (2009), 418-428.

[9] X. Cai and J. Yu, Existence theorems for second-order discrete boundary value problems, *J. Math. Anal. Appl.* **320** (2006), 649-661.

[10] W. G. Kelley and A. C. Peterson, *Difference Equations. An Introduction with Applications*, Second edition. Harcourt/Academic Press, San Diego, CA, 2001.

[11] V. Lakshmikantham and D. Trigiante, *Theory of difference equations. Numerical methods and applications.* Mathematics in Science and Engineering, 181. Academic Press, Inc., Boston, MA, 1988.

[12] L. Lovász, Discrete and continuous: two sides of the same? GAFA 2000 (Tel Aviv, 1999). *Geom. Funct. Anal.* 2000, Special Volume, Part I, 359–382.

[13] M. Mihăilescu, V. Rădulescu and S. Tersian, Eigenvalue Problems for Anisotropic Discrete Boundary Value Problems, *Journal of Difference Equations and Applications* **15** (2009), 557-567.

[14] H. Tang, W. Luo, X. Li and M. Ma, Nontrivial solutions of discrete elliptic boundary value problems. *Comput. Math. Appl.* **55** (2008), no. 8, 1854–1860.

[15] J. Yu and Z. Guo, On boundary value problems for a discrete generalized Emden-Fowler equation, *J. Math. Anal. Appl.* **231** (2006), 18-31.

[16] G. Zhang and S. Liu, On a class of semipositone discrete boundary value problem, *J. Math. Anal. Appl.* **325** (2007), 175-182.