manuscripta math. 142, 157–185(2013)

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Nonlinear elliptic problems on Riemannian manifolds and applications to Emden–Fowler type equations

Received: 7 June 2012 / Revised: 5 August 2012
Published online: 18 November 2012

Abstract. The existence of one non-trivial solution for a nonlinear problem on compact d-dimensional ($d \geq 3$) Riemannian manifolds without boundary, is established. More precisely, a recent critical point result for differentiable functionals is exploited, in order to prove the existence of a determined open interval of positive eigenvalues for which the considered problem admits at least one non-trivial weak solution. Moreover, as a consequence of our approach, a multiplicity result is presented, requiring the validity of the Ambrosetti–Rabinowitz hypothesis. Successively, the Cerami compactness condition is studied in order to obtain a similar multiplicity theorem in superlinear cases. Finally, applications to Emden-Fowler type equations are presented.

1. Introduction

The purpose of the present paper is to establish a new existence result associated with related energy estimates for elliptic problems defined on compact Riemannian manifolds.

Let $(\mathcal{M}, g)$ be a compact d-dimensional Riemannian manifold without boundary of dimension $d \geq 3$ and let Δ_g denote the Laplace–Beltrami operator whose expression, in local coordinates, is given by

$$\Delta_g w := g^{ij}\left(\frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial w}{\partial x^k}\right),$$

where Γ_{ij}^k are the usual Christoffel's symbols.

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Mathematics Subject Classification (2010): 35J60, 58J05.

DOI: 10.1007/s00229-012-0596-4

We are interested in the existence of one non-trivial weak solution of the following non-autonomous problem

$$- \Delta_g w + \alpha(\sigma) w = \lambda K(\sigma) f(w), \qquad (P_\lambda)$$

for every $\sigma \in \mathcal{M}$ and $w \in H_1^2(\mathcal{M})$.

We assume that the mappings $\alpha, K : \mathcal{M} \to \mathbb{R}$ satisfy

$$\alpha, K \in \Lambda_+(\mathcal{M}) := \left\{ \beta \in L^\infty(\mathcal{M}; \mathbb{R}) : \operatorname*{ess\,inf}_{\sigma \in \mathcal{M}} \beta(\sigma) > 0 \right\},$$

λ is a positive parameter, and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$(\mathrm{h}_\infty) \qquad |f(t)| \le a_1 + a_2 |t|^{q-1}, \quad \forall t \in \mathbb{R},$$

for some non-negative constants a_1, a_2, and $q \in]1, 2^*[$, where $2^* := 2d/(d-2)$.

A remarkable case of problem (P_λ) is

$$- \Delta_h w + s(1 - s - d) w = \lambda K(\sigma) f(w), \qquad \sigma \in \mathbb{S}^d, \ w \in H_1^2(\mathbb{S}^d), \quad (S_\lambda)$$

where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$, h is the standard metric induced by the embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$, s is a constant such that $1 - d < s < 0$, and Δ_h denotes the Laplace-Beltrami operator on $(\mathbb{S}^d, h)$.

For completeness we observe that the existence of a smooth positive solution for problem (S_λ), when $s = -d/2$ or $s = -d/2 + 1$, and $f(t) = |t|^{\frac{4}{d-2}} t$, can be viewed as an affirmative answer to the Yamabe problem [35] on $\mathbb{S}^d$ (see also the Nirenberg problem [29]); for these topics we refer to Aubin [3], Cotsiolis and Iliopoulos [13, 12], Hebey [15], Kazdan and Warner [18], Vázquez and Véron [34], and to the excellent survey by Lee and Parker [25]. In these cases the right hand-side of problem (S_λ) involves the critical Sobolev exponent. The Yamabe problem is often referred to in the literature on elliptic type equations with critical Sobolev growth and in terms of PDEs equations can be formulated as follows:

For any smooth compact Riemannian manifold $(\mathcal{M}, g)$ of dimension $d \ge 3$, there exists $w \in C^\infty(\mathcal{M})$, $w > 0$, and $\lambda \in \mathbb{R}$ such that

$$- \Delta_g w + \frac{d-2}{4(d-1)} S_g w = \lambda w^{\frac{d+2}{d-2}}, \qquad (Y_\lambda)$$

where S_g denotes the scalar curvature that, through the symmetric Ricci tensor R_{ij}, has the form $S_g = R_{ij} g^{ij}$.

Geometrically, the goal of this celebrated problem is to prove that, up to conformal changes of the metric, there always exists a metric of constant scalar curvature. This was announced to be true by Yamabe [35] in 1960. Roughly eight years later, Trudinger [33] discovered a serious difficulty in the original Yamabe's proof. He repaired the proof when the conformal class of the reference metric is nonpositive. Eight years later after Trudinger, Aubin [1] improved Yamabe's approach and reduced the problem to the proof of some strict inequality involving some geometrical invariants of the manifold. Such an inequality was proved to be true by Aubin in some cases, and then by Schoen [32] in the remaining more difficult cases. In particular, in his remarkable work, Schoen discovered the unexpected relevance of the positive mass theorem. The Yamabe problem, whose origin goes back to the beginning of the 1960's, was solved something like twenty five years later.

Moreover, existence results for problem (S_λ), can be used in order to study the existence of solutions for the following parameterized Emden-Fowler equation

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\}, \tag{$\mathfrak{F}_\lambda$}$$

see, for instance, Sect. 4.

Problem (P_λ) has been studied for power-type nonlinearities, that is, provided that $f(t) = |t|^{p-1}t$, $p > 1$ see Cotsiolis and Iliopoulos in [12,13] for the case of the sphere, and Vázquez and Véron [34] for a general compact manifold. In the aforementioned papers the authors obtained existence and multiplicity results for (P_λ) by using variational arguments.

More recently, in Kristály and Rădulescu [23], the authors are interested on the existence of multiple solutions of problem (P_λ) in order to obtain solutions for parameterized Emden-Fowler equation $(\mathfrak{F}_\lambda)$ considering nonlinear terms of sublinear type at infinity.

In particular, in [23, Theorem 1.1], for λ sufficiently large, the existence of two nontrivial solutions for problem (P_λ) has been successfully obtained through a careful analysis of the standard mountain pass geometry.

Furthermore, in Kristály et al. [24, Theorem 9.4, p. 222], the existence of an open interval of positive parameters for which problem (P_λ) admits two distinct nontrivial solutions is established by using an abstract three critical points theorem contained in Bonanno [6].

Moreover in [9], through a novel approach developed by Bonanno and Molica Bisci [8], the existence of a well localized open interval of positive parameters for which problem (P_λ) admits at least three non-trivial solutions has been studied.

Finally, Kristály in [20], proved a bifurcation result for a perturbed sublinear elliptic problem $(P_{\lambda,\mu})$ defined on $\mathcal{M}$. We just observe that for $\mu = 0$ the cited problem coincides with (P_λ) and, in particular, from [20, Theorem 1.1], it follows that if the nonlinearity f belongs to the set

$$\mathcal{F} := \left\{ f \in C(\mathbb{R}_+; \mathbb{R}_+) \setminus \{0\} : \lim_{t\to 0^+} \frac{f(t)}{t} = \lim_{t\to\infty} \frac{f(t)}{t} = 0 \right\},$$

for λ sufficiently small, then problem (P_λ) admits only the identically zero solution. For completeness, we just mention that, by using similar variational arguments, the existence of multiple solutions for non-homogeneous Neumann problem on Riemannian manifolds with boundary have been studied by Kristály et al. in [22].

The main result of this paper (Theorem 3.1) ensures the existence of precise values of parameters λ for which (P_λ) admits at least one non-trivial solution. A special case is also pointed out (Corollary 3.1) and a meaningful consequence for a suitable class of functions with a certain asymptotic behaviour at the origin is presented; see, for more details, Theorem 3.2.

We observe that when the nonlinear term is sublinear at infinity, then the corresponding energy functional is coercive, hence the existence of one solution (possibly zero) is ensured from the direct methods theorem. It is worth noticing that, in our cases, the potential may be also not coercive; see, for completeness, Example 3.1 and Remark 3.6. On the contrary, if the potential is coercive, our results ensure the existence of at least one non-trivial solution; see Remark 3.3.

A basic tool used in the proofs is a recent critical point theorem obtained by Bonanno in [7, Theorem 5.1] for functionals of the form $J_\lambda := \Phi - \lambda\Psi$, where λ is a positive parameter; see Theorem 2.1 below.

We state in what follows a special case of our results, which establishes the existence of a nontrivial solution in case of lower perturbations (small values of the parameter). The following theorem also shows that the energies of the corresponding solutions become smaller and smaller as the parameter converges to zero.

Theorem 1.1 *Let $d \geq 3$. Set $\alpha, K \in C^\infty(\mathbb{S}^d)$ be two positive maps and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\sup_{t\in\mathbb{R}} \left(\frac{|f(t)|}{1 + |t|^{q-1}} \right) < +\infty,$$

for some $q \in]1, 2^*[$. Assume that

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty.$$

Then, there exists $\lambda^\star > 0$ such that for every $\lambda \in]0, \lambda^\star[$, the following problem

$$-\Delta_h w + \alpha(\sigma)w = \lambda K(\sigma) f(\omega), \qquad \sigma \in \mathbb{S}^d,\ w \in H_1^2(\mathbb{S}^d), \tag{S_λ^α}$$

admits at least one non-trivial weak solution $w_{0,\lambda} \in H_1^2(\mathbb{S}^d)$. Finally, $\|w_{0,\lambda}\|_{H_1^2} \to 0$ as $\lambda \to 0^+$ and the map

$$\lambda \mapsto \frac{1}{2}\left(\int_{\mathbb{S}^d} |\nabla w_{0,\lambda}(\sigma)|^2 d\sigma_h + \int_{\mathbb{S}^d} w_{0,\lambda}(\sigma)^2 d\sigma_h \right)$$
$$-\lambda \int_{\mathbb{S}^d} K(\sigma) \left(\int_0^{w_{0,\lambda}(\sigma)} f(t)dt \right) d\sigma_h,$$

is negative and strictly decreasing in $]0, \lambda^\star[$.

An explicit estimation of the parameter $\lambda^\star$ that appears in the above result is given in Remark 3.2. For an exhaustive overview on elliptic equations with critical exponent defined on Riemannian manifolds we refer to [24, Chapter 11] as well as the exhaustive lecture notes [16] and the references therein.

Finally, in Theorem 3.3, we prove that, adding to hypotheses of Theorem 3.2 the classical (AR) Ambrosetti and Rabinowitz condition, a second non-trivial solution is achieved; see also Corollary 4.2.

Successively, in Theorem 3.4 we establish a multiplicity result (similar to Theorem 3.3) without assuming (AR) condition. In this setting the Euler-Lagrange functional J_λ may possess unbounded Palais-Smale (briefly (PS)) sequences. The key point in our proof is that, although J_λ possesses unbounded (PS) sequences, under the assumptions of Theorem 3.4, the functional J_λ satisfies the Cerami condition. Our result is achieved through a local condition, namely ($\mathrm{h}_{\infty,F}$), near infinity, previously adopted by Liu in [26] studying the existence of solutions for superlinear p-Laplacian Dirichlet problems on bounded Euclidean domains; see also the work of Jeanjean [17].

The paper is arranged as follows: in Sect. 2, we recall some basic definitions and our main tool, while Sect. 3 is devoted to our main results. In the last section we find existence results for singular elliptic problems of Emden-Fowler type as an application of our theoretical approach. We cite the recent monograph by Kristály et al. [24] as general reference on this subject.

2. Preliminaries

We start this section with a short list of notions in Riemmanian geometry. We refer to Aubin [3] and Hebey [15] for detailed derivations of the geometric quantities, their motivation and further applications; see also the work [2]. Let $(\mathcal{M}, g)$ be a smooth compact d-dimensional ($d \geq 3$) Riemannian manifold without boundary and let g_{ij} be the components of the metric g. As usual, we denote by $C^\infty(\mathcal{M})$ the space of smooth functions defined on $\mathcal{M}$. Let $\alpha \in \Lambda_+(\mathcal{M})$ and set $\|\alpha\|_\infty := \operatorname{ess\,sup}_{\sigma\in\mathcal{M}} \alpha(\sigma)$.

For every $w \in C^\infty(\mathcal{M})$, we denote

$$\|w\|^2_{H^2_\alpha} := \int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) w(\sigma)^2 d\sigma_g,$$

where ∇w is the covariant derivative of w, and $d\sigma_g$ is the Riemannian measure. In local coordinates $(x^1, \ldots, x^d)$, the components of ∇w are given by

$$(\nabla^2 w)_{ij} = \frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial w}{\partial x^k},$$

where

$$\Gamma^k_{ij} := \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) g^{lk},$$

are the usual Christoffel's symbols and g^{lk} are the elements of the inverse matrix of g. Here, and in the sequel, the Einstein's summation convention is adopted. Moreover, the measure element $d\sigma_g$ assume the form $d\sigma_g = \sqrt{\det g}\, dx$, where dx stands for the Lebesgue's volume element of $\mathbb{R}^d$. Hence, let

$$\mathrm{Vol}_g(\mathcal{M}) := \int_{\mathcal{M}} d\sigma_g.$$

In particular, if $(\mathcal{M}, g) = (\mathbb{S}^d, h)$, where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$ and h is the standard metric induced by the embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$, we set

$$\omega_d := \mathrm{Vol}_h(\mathbb{S}^d) := \int_{\mathbb{S}^d} d\sigma_h.$$

The Sobolev space $H^2_\alpha(\mathcal{M})$ is defined as the completion of $C^\infty(\mathcal{M})$ with respect to the norm $\|\cdot\|_{H^2_\alpha}$. Then $H^2_\alpha(\mathcal{M})$ is a Hilbert space endowed with the inner product

$$\langle w_1, w_2\rangle_{H^2_\alpha} = \int_{\mathcal{M}} \langle \nabla w_1(\sigma), \nabla w_2(\sigma)\rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)\langle w_1(\sigma), w_2(\sigma)\rangle_g d\sigma_g,$$

for every $w_1, w_2 \in H^2_\alpha(\mathcal{M})$, where $\langle\cdot,\cdot\rangle_g$ is the inner product on covariant tensor fields associated to g. Since α is positive, the norm $\|\cdot\|_{H^2_\alpha}$ is equivalent with the standard norm

$$\|w\|_{H^2_1} := \left(\int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} w(\sigma)^2 d\sigma_g\right)^{1/2}.$$

Moreover, if $w \in H^2_\alpha(\mathcal{M})$, the following inequalities hold

$$\min\left\{1, \operatorname*{ess\,inf}_{\sigma\in\mathcal{M}} \alpha(\sigma)^{1/2}\right\} \|w\|_{H^2_1} \le \|w\|_{H^2_\alpha} \le \max\left\{1, \|\alpha\|_\infty^{1/2}\right\} \|w\|_{H^2_1}. \tag{1}$$

By the Rellich-Kondrachov theorem for compact manifolds without boundary we have

$$H^2_\alpha(\mathcal{M}) \hookrightarrow L^q(\mathcal{M}),$$

for every $q \in [1, 2d/(d-2)]$. In particular, the embedding is compact whenever $q \in [1, 2d/(d-2))$. Hence, there exists a positive constant S_q such that

$$\|w\|_{L^q(\mathcal{M})} \le S_q \|w\|_{H^2_\alpha}, \qquad \forall\, w \in H^2_\alpha(\mathcal{M}), \tag{2}$$

where the norm of the Lebesgue spaces $L^q(\mathcal{M})$ are denoted by $\|\cdot\|_{L^q(\mathcal{M})}$ for all $q \in [1, \infty[$.

If $K \in \Lambda_+(\mathcal{M})$, we recall that a function $w \in H^2_1(\mathcal{M})$ is a *weak solution* of problem (P_λ) if

$$\begin{aligned}\int_{\mathcal{M}} \langle \nabla w(\sigma), \nabla v(\sigma)\rangle_g d\sigma_g &+ \int_{\mathcal{M}} \alpha(\sigma)\langle w(\sigma), v(\sigma)\rangle_g d\sigma_g \\ &= \lambda \int_{\mathcal{M}} K(\sigma) f(w(\sigma)) v(\sigma) d\sigma_g,\end{aligned}$$

for every $v \in H^2_1(\mathcal{M})$.

Let X be a real Banach space. We say that a continuously Gâteaux differentiable functional $J : X \to \mathbb{R}$ verifies the *Palais-Smale condition* (in short (PS)-condition) if any sequence $\{u_n\}$ such that

(j_1) $\{J(u_n)\}$ is bounded,
(j_2) $\lim_{n\to\infty} \|J'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

For an exhaustive treatment of these topics we refer to [28,30] and the references therein. Now, let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Set

$$J = \Phi - \Psi,$$

and fix $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$; we say that function J verifies the *Palais-Smale condition cut off lower at* r_1 *and upper at* r_2 (in short ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that (j_1), (j_2) hold and

(j_3) $r_1 < \Phi(u_n) < r_2, \ \forall n \in \mathbb{N}$,

has a convergent subsequence.

Clearly, if $r_1 = -\infty$ and $r_2 = +\infty$ it coincides with the classical (PS)-condition. Moreover, if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(\mathrm{PS})^{[r_2]}$, while if $r_1 \in \mathbb{R}$ and $r_2 = +\infty$ it is denoted by ${}^{[r_1]}(\mathrm{PS})$. Clearly, if J satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition, then it satisfies ${}^{[\rho_1]}(\mathrm{PS})^{[\rho_2]}$-condition for all $\rho_1, \rho_2 \in [-\infty, +\infty]$ such that $r_1 \le \rho_1 < \rho_2 \le r_2$.

In particular, we deduce that if J satisfies the classical (PS)-condition, then it satisfies ${}^{[\rho_1]}(\mathrm{PS})^{[\rho_2]}$-condition for all $\rho_1, \rho_2 \in [-\infty, +\infty]$ with $\rho_1 < \rho_2$. Set

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \tag{3}$$

and

$$\rho_2(r_1, r_2) := \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1}, \tag{4}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$.

Now, for a fixed $\lambda > 0$, the function $w_\lambda(\sigma) = c \in \mathbb{R} \setminus \{0\}$, is a solution of (P_λ) if and only if the mapping $\sigma \mapsto \lambda K(\sigma)/\alpha(\sigma)$ is constant. In this case, nontrivial constant solutions of (P_λ) appear as fixed points of the function $t \mapsto c_\lambda f(t)$, where c_λ denotes the constant value $\lambda K(\sigma)/\alpha(\sigma)$.

A crucial role in the existence proof of one non-trivial (and non-constant) weak solution for our problem is played by the following version of an abstract local minimum theorem obtained in [7, Theorem 5.1], which we recall here for convenience.

Theorem 2.1 *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Assume that there are r_1, $r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2),$$

where β and ρ_2 are given by (3) and (4), and for each

$$\lambda \in \left] \frac{1}{\rho_2(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right[,$$

the function $J_\lambda := \Phi - \lambda\Psi$ satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition.

Then, for each $\lambda \in \left]\dfrac{1}{\rho_2(r_1, r_2)}, \dfrac{1}{\beta(r_1, r_2)}\right[$ *there is* $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ *such that* $J_\lambda(u_{0,\lambda}) \leq J_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r_1, r_2[)$ *and* $J'_\lambda(u_{0,\lambda}) = 0$.

Remark 2.1. We explicitly observe that Theorem 2.1 guarantees the existence of an open interval of parameters such that for any

$$\lambda \in \left]\frac{1}{\rho_2(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right[,$$

the functional J_λ has a minimizer $u_{0,\lambda}$ with respect to the open sublevel $\Phi^{-1}(]r_1, r_2[)$, that is,

$$J_\lambda(u_{0,\lambda}) \leq J_\lambda(u), \quad \forall\, u \in \Phi^{-1}(]r_1, r_2[).$$

Since $\Phi^{-1}(]-\infty, r[) \subset X$ is open, it follows that $u_{0,\lambda}$ must be a local minimizer of J_λ, hence it is a critical point of J_λ. Therefore, Theorem 2.1 may be considered also as a localization principle of critical points of J_λ that belong to the sublevel $\Phi^{-1}(]r_1, r_2[)$.

3. Main results

For every two nonnegative constants γ, δ, with $\gamma \neq \delta$, we set

$$a_\gamma(\delta) := \frac{A(\gamma) - qF(\delta)\|K\|_{L^1(\mathcal{M})}}{\|\alpha\|_{L^1(\mathcal{M})}(\gamma^2 - \delta^2)q}, \tag{5}$$

where

$$A(\gamma) := (q\|\alpha\|_{L^1(\mathcal{M})}^{1/2}\gamma S_1 a_1 + \|\alpha\|_{L^1(\mathcal{M})}^{q/2}\gamma^q S_q^q a_2)\|K\|_\infty,$$

and

$$F(\xi) := \int_0^\xi f(t)dt,$$

for every $\xi \in \mathbb{R}$.

With the above notations we are able to prove the following result.

Theorem 3.1 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that condition* (h_∞) *holds and assume that there exist three real constants* γ_1, γ_2 *and* δ, *with* $0 \leq \gamma_1 < \delta < \gamma_2$, *such that*

$$a_{\gamma_2}(\delta) < a_{\gamma_1}(\delta). \tag{6}$$

Then, for each parameter λ *belonging to*

$$\Lambda := \left]\frac{1}{2a_{\gamma_1}(\delta)}, \frac{1}{2a_{\gamma_2}(\delta)}\right[,$$

the problem (P_λ) admits at least one weak solution $w_{0,\lambda} \in H_1^2(\mathcal{M})$, such that

$$\frac{\|\alpha\|_{L^1(\mathcal{M})}^{1/2}\gamma_1}{\max\{1, \|\alpha\|_\infty^{1/2}\}} < \|w_{0,\lambda}\|_{H_1^2} \leq \frac{\|\alpha\|_{L^1(\mathcal{M})}^{1/2}\gamma_2}{\min\{1, \operatorname{ess\,inf}_{\sigma\in\mathcal{M}} \alpha(\sigma)^{1/2}\}}.$$

Proof. Our aim is to apply Theorem 3.1. Hence, let $X := H_1^2(\mathcal{M})$ and consider the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by

$$\Phi(w) := \frac{\|w\|_{H_\alpha^2}^2}{2}, \quad \Psi(w) := \int_{\mathcal{M}} K(\sigma)F(w(\sigma))d\sigma_g, \qquad \text{for all } w \in X.$$

Clearly $\Phi : X \to \mathbb{R}$ is a coercive, continuously Gâteaux differentiable. On the other hand, Ψ is well-defined and continuously Gâteaux differentiable. Moreover, we have

$$\Phi'(w)(v) = \int_{\mathcal{M}} \langle \nabla w(\sigma), \nabla v(\sigma)\rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)\langle w(\sigma), v(\sigma)\rangle_g d\sigma_g,$$

and

$$\Psi'(w)(v) = \int_{\mathcal{M}} K(\sigma)f(w(\sigma))v(\sigma)d\sigma_g,$$

for every $w, v \in X$. Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda\Psi$ is a function $w \in X$ such that

$$\Phi'(w)(v) - \lambda\Psi'(w)(v) = 0,$$

for every $v \in X$. Hence, the critical points of the functional J_λ are the weak solutions of problem (P_λ). Moreover, $\Phi(0_X) = \Psi(0_X) = 0$. Since condition (h_∞) holds, we have

$$F(\xi) \leq a_1|\xi| + a_2\frac{|\xi|^q}{q}, \tag{7}$$

for every $\xi \in \mathbb{R}$. Now, taking into account (7), it follows that

$$\Psi(w) = \int_{\mathcal{M}} K(\sigma)F(w(\sigma))d\sigma_g \leq \|K\|_\infty \left(a_1\|w\|_{L^1(\mathcal{M})} + \frac{a_2}{q}\|w\|_{L^q(\mathcal{M})}^q\right).$$

Then, for every $w \in X$ such that $\Phi(w) \leq r$, owing to (2), we get

$$\Psi(w) \leq \|K\|_\infty\left((2r)^{1/2}S_1a_1 + \frac{2^{q/2}S_q^q a_2}{q}r^{q/2}\right).$$

Therefore

$$\sup_{w\in\Phi^{-1}(]-\infty,r])} \Psi(w) \leq \|K\|_\infty\left((2r)^{1/2}S_1a_1 + \frac{2^{q/2}S_q^q a_2}{q}r^{q/2}\right). \tag{8}$$

Next, we set

$$r_1 = \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2}\gamma_1^2, \quad r_2 = \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2}\gamma_2^2, \quad \text{and} \quad w_\delta(\sigma) = \delta, \quad \text{for every} \quad \sigma \in \mathcal{M}.$$

Clearly $w_\delta \in X$ and we have

$$\Phi(w_\delta) = \frac{1}{2}\Big(\int_{\mathcal{M}} |\nabla w_\delta(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) w_\delta(\sigma)^2 d\sigma_g\Big) = \frac{\delta^2}{2}\|\alpha\|_{L^1(\mathcal{M})}. \tag{9}$$

Taking into account that $\gamma_1 < \delta < \gamma_2$, by a direct computation, we deduce that $r_1 < \Phi(w_\delta) < r_2$. Moreover,

$$\Psi(w_\delta) = \int_{\mathcal{M}} K(\sigma)F(w_\delta(\sigma))d\sigma_g = F(\delta)\|K\|_{L^1(\mathcal{M})}. \tag{10}$$

From (8) it follows that

$$\sup_{w \in \Phi^{-1}(]-\infty, r_2[)} \Psi(w) \le \|K\|_\infty \left((2r_2)^{1/2} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} r_2^{q/2}\right). \tag{11}$$

as well as

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \le \|K\|_\infty \left((2r_1)^{1/2} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} r_1^{q/2}\right). \tag{12}$$

Then $r_1 < \Phi(w_\delta) < r_2$ and

$$\begin{aligned}\beta(r_1, r_2) &:= \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{w \in \Phi^{-1}(]r_1, r_2[)} \Psi(w) - \Psi(v)}{r_2 - \Phi(v)} \\ &\le \frac{\sup_{w \in \Phi^{-1}(]-\infty, r_2[)} \Psi(w) - \Psi(w_\delta)}{r_2 - \Phi(w_\delta)},\end{aligned}$$

and

$$\begin{aligned}\rho_2(r_1, r_2) &:= \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{w \in \Phi^{-1}(]-\infty, r_1])} \Psi(w)}{\Phi(v) - r_1} \\ &\ge \frac{\Psi(w_\delta) - \sup_{w \in \Phi^{-1}(]-\infty, r_1])} \Psi(w)}{\Phi(w_\delta) - r_1}.\end{aligned}$$

Hence, by using the notation (5), and relations from (9) to (12) it follows that

$$\beta(r_1, r_2) \le 2a_{\gamma_2}(\delta), \quad \text{and} \quad \rho_2(r_1, r_2) \ge 2a_{\gamma_1}(\delta).$$

Finally, hypothesis (6) yields

$$\beta(r_1, r_2) < \rho_2(r_1, r_2).$$

Now, from [7, Proposition 2.1], the functional J_λ satisfies $^{[r_1]}$(PS)$^{[r_2]}$-condition for all r_1 and r_2 with $r_1 < r_2 < +\infty$. Therefore, owing to Theorem 2.1, for each

$\lambda \in \left]\frac{1}{2a_{\gamma_1}(\delta)}, \frac{1}{2a_{\gamma_2}(\delta)}\right[$, the functional J_λ admits at least one critical point $w_{0,\lambda}$ such that

$$r_1 < \Phi(w_{0,\lambda}) < r_2,$$

that is

$$\|\alpha\|_{L^1(\mathcal{M})}^{1/2}\gamma_1 < \|w_{0,\lambda}\|_{H^2_\alpha} < \|\alpha\|_{L^1(\mathcal{M})}^{1/2}\gamma_2.$$

The last part of our result is achieved from the above inequalities and taking into account relation (1). □

We now point out the following consequence of Theorem 3.1.

Corollary 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that condition* (h_∞) *holds and assume that there exist two positive constants γ and δ, with $\gamma > \delta$, for which*

$$\frac{F(\delta)}{\delta^2} > \frac{A(\gamma)}{q\gamma^2\|K\|_{L^1(\mathcal{M})}}. \tag{13}$$

Then, for each parameter λ belonging to

$$\left]\frac{\|\alpha\|_{L^1(\mathcal{M})}\delta^2}{2\|K\|_{L^1(\mathcal{M})}F(\delta)}, \frac{q\|\alpha\|_{L^1(\mathcal{M})}\gamma^2}{2A(\gamma)}\right[,$$

the problem (P_λ) admits at least one non-trivial weak solution $w_{0,\lambda} \in H^2_1(\mathcal{M})$, such that

$$\|w_{0,\lambda}\|_{H^2_1} < \frac{\|\alpha\|_{L^1(\mathcal{M})}^{1/2}\gamma}{\min\{1, \operatorname{ess\,inf}_{\sigma\in\mathcal{M}}\alpha(\sigma)^{1/2}\}}.$$

Proof. Our aim is to apply Theorem 3.1. To this end we pick $\gamma_1 = 0$ and $\gamma_2 := \gamma$. Bearing in mind (5), we obtain

$$a_\gamma(\delta) = \frac{A(\gamma) - q\|K\|_{L^1(\mathcal{M})}F(\delta)}{\|\alpha\|_{L^1(\mathcal{M})}(\gamma^2 - \delta^2)q},$$

as well as

$$a_0(\delta) = \frac{\|K\|_{L^1(\mathcal{M})}F(\delta)}{\delta^2\|\alpha\|_{L^1(\mathcal{M})}}.$$

Now, inequality (13), immediately yields

$$a_\gamma(\delta) < a_0(\delta).$$

Hence, Theorem 3.1 ensures the conclusion. □

Here is a direct result obtained by using Corollary 3.1.

Theorem 3.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that condition* (h_∞) *holds and assume that*

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = +\infty. \tag{$\mathrm{h}_{0,F}$}$$

Furthermore, for each $\gamma > 0$, set

$$\lambda_\gamma^\star := \frac{q\|\alpha\|_{L^1(\mathcal{M})}}{2} \frac{\gamma^2}{A(\gamma)}.$$

Then, for every $\lambda \in]0, \lambda_\gamma^\star[$, the problem (P_λ) admits at least one non-trivial weak solution $w_{0,\lambda} \in H_1^2(\mathcal{M})$. Moreover, we have

$$\lim_{\lambda \to 0^+} \|w_{0,\lambda}\|_{H_1^2} = 0,$$

and the map

$$\lambda \mapsto \frac{\|w_{0,\lambda}\|_{H_\alpha^2}^2}{2} - \lambda \int_{\mathcal{M}} K(\sigma) \left(\int_0^{w_{0,\lambda}(\sigma)} f(t)dt \right) d\sigma_g,$$

is negative and strictly decreasing in $]0, \lambda_\gamma^\star[$.

Proof. Fix $\gamma > 0$ and $\lambda \in]0, \lambda_\gamma^\star[$. From ($\mathrm{h}_{0,F}$) there exists a positive constant δ with $\delta < \gamma$ such that

$$\frac{\|\alpha\|_{L^1(\mathcal{M})}\delta^2}{2\|K\|_{L^1(\mathcal{M})}F(\delta)} < \lambda < \frac{q\|\alpha\|_{L^1(\mathcal{M})}\gamma^2}{2A(\gamma)}.$$

Hence, owing to Corollary 3.1, the problem (P_λ) admits at least one non-trivial weak solution $w_{0,\lambda} \in X$, such that

$$\|w_{0,\lambda}\|_{H_\alpha^2} < \|\alpha\|_{L^1(\mathcal{M})}^{1/2} \gamma.$$

Then, for every $\lambda \in]0, \lambda_\gamma^\star[$, there exists at least one non-trivial weak solution $w_{0,\lambda} \in \Phi^{-1}(]0, r_2[)$ of the problem (P_λ) and

$$\|w_{0,\lambda}\|_{H_1^2} < \frac{\|\alpha\|_{L^1(\mathcal{M})}^{1/2} \gamma}{\min\{1, \operatorname{ess\,inf}_{\sigma \in \mathcal{M}} \alpha(\sigma)^{1/2}\}}, \tag{14}$$

for every $\lambda \in]0, \lambda_\gamma^\star[$.

Therefore, from (h_∞), taking into account (2) and (14), it follows that there exists a positive constant $C_{a_1,a_2,q}^{\alpha,K}$ such that

$$\left| \int_{\mathcal{M}} K(\sigma) f(w_{0,\lambda}(\sigma)) w_{0,\lambda}(\sigma) d\sigma_g \right| \le C_{a_1,a_2,q}^{\alpha,K}, \tag{15}$$

for every $\lambda \in]0, \lambda_\gamma^\star[$.

Now, $J'_\lambda(w_{0,\lambda}) = 0$, for every $\lambda \in]0, \lambda^\star_\gamma[$ and in particular $J'_\lambda(w_{0,\lambda})(w_{0,\lambda}) = 0$, that is,

$$\|w_{0,\lambda}\|^2_{H^2_\alpha} = \lambda \int\limits_{\mathcal{M}} K(\sigma) f(w_{0,\lambda}(\sigma)) w_{0,\lambda}(\sigma) d\sigma_g,$$

for every $\lambda \in]0, \lambda^\star_\gamma[$.

Then, from (15), it follows that

$$\lim_{\lambda \to 0^+} \|w_{0,\lambda}\|^2_{H^2_\alpha} = \lim_{\lambda \to 0^+} \lambda \Psi'(w_{0,\lambda})(w_{0,\lambda}) = 0,$$

that implies $\lim_{\lambda \to 0^+} \|w_{0,\lambda}\|_{H^2_1} = 0$.

Finally, we claim that the mapping $\lambda \mapsto J_\lambda(w_{0,\lambda})$ is negative and strictly decreasing in $]0, \lambda^\star_\gamma[$. Indeed, the restriction of the functional J_λ to $\Phi^{-1}(]0, r_2[)$, where $r_2 := (\|\alpha\|_{L^1(\mathcal{M})}/2)\gamma_2^2$, admits a global minimum, which is a critical point (local minimum) of J_λ in X. Moreover, since $w_\delta := \delta \in \Phi^{-1}(]0, r_2[)$ and

$$\frac{\Phi(w_\delta)}{\Psi(w_\delta)} = \frac{\|\alpha\|_{L^1(\mathcal{M})}\delta^2}{2\|K\|_{L^1(\mathcal{M})}F(\delta)} < \lambda,$$

we have

$$J_\lambda(w_{0,\lambda}) \leq J_\lambda(w_\delta) = \Phi(w_\delta) - \lambda\Psi(w_\delta) < 0.$$

Next, we observe that

$$J_\lambda(w) = \lambda\left(\frac{\Phi(w)}{\lambda} - \Psi(w)\right),$$

for every $u \in X$ and fix $0 < \lambda_1 < \lambda_2 < \lambda^\star_\gamma$. Set

$$m_{\lambda_1} := \left(\frac{\Phi(w_{0,\lambda_1})}{\lambda_1} - \Psi(w_{0,\lambda_1})\right) = \inf_{w \in \Phi^{-1}(]0,r_2[)} \left(\frac{\Phi(w)}{\lambda_1} - \Psi(w)\right),$$

and

$$m_{\lambda_2} := \left(\frac{\Phi(w_{0,\lambda_2})}{\lambda_2} - \Psi(w_{0,\lambda_2})\right) = \inf_{w \in \Phi^{-1}(]0,r_2[)} \left(\frac{\Phi(w)}{\lambda_2} - \Psi(w)\right).$$

Clearly, as claimed before, $m_{\lambda_i} < 0$ (for $i = 1, 2$), and $m_{\lambda_2} \leq m_{\lambda_1}$ thanks to $\lambda_1 < \lambda_2$. Then the mapping $\lambda \mapsto J_\lambda(w_{0,\lambda})$ is strictly decreasing in $]0, \lambda^\star_\gamma[$ owing to

$$J_{\lambda_2}(w_{0,\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = J_{\lambda_1}(w_{0,\lambda_1}).$$

This concludes the proof. $\square$

Remark 3.1. A careful analysis of the proof of Theorem 3.2 ensures that the result still remains true after replacing condition $(\mathrm{h}_{0,F})$ with the more general assumption at zero

$$\limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}=+\infty. \qquad (\mathrm{h}''_{0,F})$$

Moreover, if f has the following asymptotic behaviour

$$\lim_{t\to 0^+}\frac{f(t)}{t}=+\infty, \qquad (\mathrm{h}_0)$$

then, obviously, hypothesis $(\mathrm{h}_{0,F})$ is trivially verified. Hence, it is easy to see that Theorem 1.1 in Introduction immediately follows from the above remark and Theorem 3.2.

Remark 3.2. In other words, Theorem 3.2 ensures that if f has the global growth given by (h_∞) and the asymptotic condition at zero $(\mathrm{h}_{0,F})$ is verified then, for every parameter λ belonging to the real interval $\Lambda_{\mathcal{M}}:=]0,\lambda^\star[$, where

$$\lambda^\star:=\frac{q\|\alpha\|_{L^1(\mathcal{M})}}{2}\sup_{\gamma>0}\frac{\gamma^2}{A(\gamma)},$$

problem (P_λ) admits at least one non-trivial solution $w_{0,\lambda}\in H_1^2(\mathcal{M})$. Moreover $\|w_{0,\lambda}\|_{H_1^2}\to 0$, as $\lambda\to 0^+$. Moreover, a straightforward computation shows that

$$\lambda^\star:=\begin{cases} +\infty & \text{if } \ 1<q<2\\ \frac{1}{\|K\|_\infty S_2^2 a_2} & \text{if } \ q=2\\ \frac{q\|\alpha\|^{1/2}_{L^1(\mathcal{M})}\overline{\gamma}_{\max}}{2\|K\|_\infty(qS_1a_1+\|\alpha\|^{(q-1)/2}_{L^1(\mathcal{M})}S_q^q a_2\overline{\gamma}^{q-1}_{\max})} & \text{if } \ q\in\left]2,\frac{2d}{d-2}\right[, \end{cases}$$

where

$$\overline{\gamma}_{\max}:=\frac{1}{\|\alpha\|^{1/2}_{L^1(\mathcal{M})}}\left(\frac{qS_1a_1}{(q-2)S_q^q a_2}\right)^{1/(q-1)}.$$

We also note that in the case $q\in\left]2,2^*\right[$, we have

$$\|u_{0,\lambda}\|_{H_1^2}<\left(\frac{qS_1a_1}{(q-2)S_q^q a_2}\right)^{1/(q-1)}\left(\frac{1}{\min\{1,\operatorname{ess\,inf}_{\sigma\in\mathcal{M}}\alpha(\sigma)^{1/2}\}}\right),$$

uniformly for every $\lambda\in\Lambda_{\mathcal{M}}$.

Remark 3.3. From the above observation, it follows that if f is a sublinear function at infinity, Theorem 3.2 ensures that, for each $\lambda>0$, the problem (P_λ) admits at least one non-zero weak solution. We explicitly observe that, in this case, also the classical direct methods theorem ensures the existence of at least one weak solution. However, in this case, it may be zero.

Remark 3.4. In our context, a concrete upper bound for the constants S_q in Theorem 3.2 is essential for a concrete evaluation of the interval $\Lambda_{\mathcal{M}}$. In the case $(\mathcal{M}, g) = (\mathbb{S}^d, h)$, if $q \in [1, 2d/(d-2)[$, we have

$$S_q \leq S_q^\star := \frac{\kappa_q}{\min\left\{1, \operatorname{ess\,inf}_{\sigma \in \mathbb{S}^d} \alpha(\sigma)^{1/2}\right\}}, \tag{16}$$

where, we set

$$\kappa_q := \begin{cases} \omega_d^{\frac{2-q}{2q}} & \text{if} \quad q \in [1, 2[, \\ \max\left\{\left(\frac{q-2}{d\omega_d^{\frac{q-2}{q}}}\right)^{1/2}, \frac{1}{\omega_d^{\frac{q-2}{2q}}}\right\} & \text{if} \quad q \in \left[2, \frac{2d}{d-2}\right[. \end{cases}$$

Indeed, in Beckner [4], it is proved that for every $2 \leq q < 2d/(d-2)$ and any $w \in H_1^2(\mathbb{S}^d)$,

$$\left(\int_{\mathbb{S}^d} |w(\sigma)|^q d\sigma_h\right)^{2/q} \leq \frac{q-2}{d\omega_d^{1-2/q}} \int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \frac{1}{\omega_d^{1-2/q}} \int_{\mathbb{S}^d} w(\sigma)^2 d\sigma_h,$$

see also, for instance, Theorem 4.28 in Hebey [15]. Hence,

$$\begin{aligned} \|w\|_{L^q(\mathbb{S}^d)} \leq \max&\left\{\left(\frac{q-2}{d\omega_d^{\frac{q-2}{q}}}\right)^{1/2}, \frac{1}{\omega_d^{\frac{q-2}{2q}}}\right\} \\ &\times \left(\int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \int_{\mathbb{S}^d} w(\sigma)^2 d\sigma_h\right)^{1/2}, \end{aligned}$$

for every $w \in H_1^2(\mathbb{S}^d)$. Owing to (1) the desiderated statement follows. On the other hand, if $q \in [1, 2[$, as simple consequence of Hölder's inequality, it follows that

$$\|w\|_{L^q(\mathbb{S}^d)} \leq \omega_d^{\frac{2-q}{2q}} \|w\|_{L^2(\mathbb{S}^d)}, \qquad \text{for all } w \in L^2(\mathbb{S}^d).$$

The thesis is achieved taking into account that

$$\|w\|_{L^2(\mathbb{S}^d)} \leq \|w\|_{H_1^2} \leq \frac{\|w\|_{H_\alpha^2}}{\min\left\{1, \min_{\sigma \in \mathbb{S}^d} \alpha(\sigma)^{1/2}\right\}},$$

for every $w \in H_1^2(\mathbb{S}^d)$. Note also that if $d \geq 4$, it follows that

$$q - 2 < \frac{2d}{d-2} - 2 \leq d.$$

In this case, clearly

$$\frac{q-2}{\omega_d^{\frac{q-2}{q}}} < \frac{d}{\omega_d^{\frac{q-2}{q}}},$$

and we have $\kappa_q = \omega_d^{\frac{2-q}{2q}}$, for every $q \in [1, 2^*[$. Consequently, if $d \geq 4$, we obtain

$$S_q^\star := \frac{\omega_d^{\frac{2-q}{2q}}}{\min\left\{1, \operatorname{ess\,inf}_{\sigma \in \mathbb{S}^d} \alpha(\sigma)^{1/2}\right\}},$$

for every $q \in [1, 2^*[$.

Remark 3.5. We observe that if f is a non-negative function our results guarantee that the attained weak solution is non-negative. From our goal, let w_0 be a weak solution of problem (P_λ). Arguing by contradiction, assume that the set $\mathcal{M}_0 := \{\sigma \in \mathcal{M} : w_0(\sigma) < 0\}$ has positive Riemannian measure. Put $\overline{w}(\sigma) := \min\{0, w_0(\sigma)\}$ for all $\sigma \in \mathcal{M}$. Clearly, $\overline{w} \in H_1^2(\mathcal{M})$ and

$$\int_{\mathcal{M}} \langle \nabla w(\sigma), \nabla \overline{w}(\sigma) \rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) \langle w(\sigma), \overline{w}(\sigma) \rangle_g d\sigma_g$$

$$-\lambda \int_{\mathcal{M}} K(\sigma) f(w(\sigma)) \overline{w}(\sigma) d\sigma_g = 0,$$

that is,

$$\int_{\mathcal{M}_0} |\nabla w_0(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}_0} \alpha(\sigma) w_0(\sigma)^2 d\sigma_g = \lambda \int_{\mathcal{M}_0} K(\sigma) f(w(\sigma)) w_0(\sigma) d\sigma_g \leq 0.$$

Hence

$$\int_{\mathcal{M}_0} |\nabla w_0(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}_0} \alpha(\sigma) w_0(\sigma)^2 d\sigma_g = 0.$$

Then, $w_0 = 0$ almost everywhere in $\mathcal{M}_0$. This is not possible by the definition of $\mathcal{M}_0$, so it follows that w_0 is non-negative. It is also evident that the above assertion remains valid requiring only that f is non-negative in $[0, +\infty[$.

The following example deals with a nonlinear problem on the unit sphere endowed with the natural metric h and involving a nonlinearity with subcritical growth.

Example 3.1. Let $(\mathbb{S}^d, h)$ with $d \geq 3$ and $K \in \Lambda_+(\mathbb{S}^d)$. Further, consider the following equation

$$-\Delta_h w + w = \lambda K(\sigma)(|w|^{r-2}w + |w|^{s-2}w), \qquad (\widetilde{P}_\lambda)$$

for every $\sigma \in \mathbb{S}^d$ and $w \in H_1^2(\mathbb{S}^d)$, where $1 < r < 2$ and $2 < s < 2^*$. Then, for every

$$\lambda \in \left]0, \frac{s\omega_d^{1/2}\overline{\gamma}_{\max}}{4\|K\|_\infty(sS_1 + \omega_d^{(s-1)/2}S_s^s\overline{\gamma}_{\max}^{s-1})}\right[,$$

where

$$\overline{\gamma}_{\max} := \frac{1}{\omega_d^{1/2}}\left(\frac{S_1}{S_s^s}\left(\frac{s}{s-2}\right)\right)^{1/(s-1)},$$

the problem $(\widetilde{P}_\lambda)$ admits at least one non-negative (and non-trivial) weak solution $w_{0,\lambda} \in H_1^2(\mathbb{S}^d)$ such that

$$\|w_{0,\lambda}\|_{H_1^2} < \left(\frac{S_1}{S_s^s}\left(\frac{s}{s-2}\right)\right)^{1/(s-1)},$$

and $\lim_{\lambda\to 0^+}\|w_{0,\lambda}\|_{H_1^2} = 0$. To prove this, we can apply Theorem 3.2 with

$$f(t) := |t|^{r-2}t + |t|^{s-2}t,$$

for every $t \in \mathbb{R}$. Indeed, it is easy to verify that

$$|f(t)| \leq 2(1 + |t|^{s-1}), \quad \forall\, t \in \mathbb{R}.$$

Moreover, a direct computation shows that

$$\lim_{\xi\to 0^+}\frac{F(\xi)}{\xi^2} \geq \frac{1}{r}\left(\lim_{\xi\to 0^+}\frac{1}{\xi^{2-r}}\right) = +\infty.$$

Hence, all the assumptions of Theorem 3.2 are verified and the conclusion follows.

Remark 3.6. We point out that the energy functional J_λ associated to problem $(\widetilde{P}_\lambda)$ is unbounded from below. In fact, if we fix $v \in H_1^2(\mathbb{S}^d)$ and $\tau \in \mathbb{R}$, then

$$J_\lambda(\tau v) \leq \frac{\tau^2}{2}\|v\|_{H_\alpha^2}^2 - \lambda\left[\frac{\tau^r}{r}\|v\|_{L^r(\mathbb{S}^d)}^r + \frac{\tau^s}{s}\|v\|_{L^s(\mathbb{S}^d)}^s\right]\operatorname*{ess\,inf}_{\sigma\in\mathbb{S}^d} K(\sigma).$$

So, as $r < 2 < s$, it follows that

$$\lim_{\tau\to+\infty} J_\lambda(\tau v) = -\infty.$$

Hence, as consequence, the functional J_λ is not coercive. Hence, the classical Tonelli's method cannot be applied to the above case.

Example 3.2. Let $\alpha, K \in \Lambda_+(\mathbb{S}^d)$. Owing to Theorem 3.2 and taking into account Remark 3.2, the following elliptic parametric problem

$$-\Delta_h w + \alpha(\sigma)w = \lambda K(\sigma)\sqrt{|w|}, \qquad \sigma \in \mathbb{S}^d,\ w \in H_1^2(\mathbb{S}^d), \qquad (S_\lambda^\alpha)$$

admits at least one non-trivial weak solution for all $\lambda > 0$. In this case Theorem 1.1 obtained in [20] cannot be applied just because the real function defined by $f(t) = \sqrt{|t|}$, for every $t \in \mathbb{R}$, does not belong to $\mathcal{F}$.

Remark 3.7. We also point out that contributions on the existence of multiple solutions for elliptic problems on the spherical case are contained in Kristály [19]; see also the related paper Kristály and Marzantowicz [21].

In the sequel we prove how the previous results can be used in order to pass from the existence of at least one nontrivial solution to the existence of at least two nontrivial solutions. This goal will be achieved making use of the particular nature of the first solution, namely the local minimum of the associated energy functional. This information will be used to assure the existence of a second solution as a critical point of mountain pass type. In this direction, we begin with the following theorem, where the celebrated Ambrosetti–Rabinowitz condition is required. As usual, this assumption plays a crucial role in proving that every Palais-Smale sequence is bounded, as well as that the so called 'mountain pass geometry' is satisfied.

Theorem 3.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$|f(t)| \leq a_1 + a_2|t|^{q-1}, \quad \forall t \in \mathbb{R}, \qquad (\mathrm{h}'_\infty)$$

for some non-negative constants a_1, a_2, where $q \in]2, 2^*[$. Furthermore, assume that condition $(\mathrm{h}''_{0,F})$ holds in addition to:

(AR) there are constants $\mu > 2$ and $r > 0$ such that, for all $|\xi| \geq r$,

$$0 < \mu F(\xi) \leq \xi f(\xi).$$

Then, for each $\lambda \in \Lambda_{\mathcal{M}}$, the problem (P_λ) admits at least two weak solutions.

Proof. Fix $\lambda \in \Lambda_{\mathcal{M}}$. Owing to (h'_∞) and $(\mathrm{h}''_{0,F})$, Theorem 3.2 ensures that the problem (P_λ) admits at least one weak non-trivial solution w_1 which is a local minimum of the functional J_λ as defined in the proof of Theorem 3.1. In view of assumption (AR), reasoning in a standard way, it is possible to verify that every Palais-Smale sequence is bounded. This, leads to the fact that J_λ satisfies the classical Palais-Smale condition by the fact that Φ' is a linear isomorphism, while Ψ' is compact; see [14, Proposition 3.8]. Moreover, it is well known that, again by (AR), there exist suitable (positive) constants γ_1, γ_2 with

$$F(\xi) \geq \gamma_1|\xi|^\mu - \gamma_2, \quad \forall\, \xi \in \mathbb{R}. \qquad (17)$$

Let $\{\xi_n\}$ be a sequence in $\mathbb{R}$ such that $\xi_n \to +\infty$ and consider the related sequence of functions $w_n(\sigma) := \xi_n$ for every $\sigma \in \mathcal{M}$ which belongs to X. Obviously, in view of (17), we have

$$J_\lambda(w_n) \leq \frac{\xi_n^2}{2}\|\alpha\|_{L^1(\mathcal{M})} - \lambda\left(\gamma_1\xi_n^\mu - \gamma_2\right)\|K\|_{L^1(\mathcal{M})},$$

that is

$$\liminf_{\|w\|_X \to \infty} J_\lambda(w) = -\infty,$$

and there exists some $w_0 \in X$ such that

$$J_\lambda(w_0) < J_\lambda(w_1). \tag{18}$$

Now, we can assume that w_1 is a strict local minimum for J_λ in X, otherwise there exist infinitely many nontrivial critical points of J_λ. Hence, we can apply the classical mountain pass theorem and obtain a second critical point $w_2 \in X$ such that $J_\lambda(w_2) > J_\lambda(w_1)$, that is $w_1 \neq w_2$ and the proof is complete. $\square$

Remark 3.8. The existence of two solutions for the problem (P_λ) has been investigated by Kristály et al. in [24, Theorem 9.4, p. 222] and by Kristály and Rădulescu in [23, Theorem 1.1]. Moreover, very recently, Kristály in [20] studied bifurcation effects for a sublinear problem $(P_{\lambda,\mu})$ defined on a compact Riemannian manifold $\mathcal{M}$ without boundary. All the above cited results cannot be applied to our cases due to the asymptotic condition (h_0) that we assume at zero. Finally, we observe that assuming $f(0) \neq 0$, Corollary 3.3 ensures the existence of at least two non-zero solutions for problem (P_λ); see also Corollary 4.2 in the sequel.

As usual, condition (AR) plays a crucial role in proving that every Palais-Smale sequence is bounded, as well as that the so called 'mountain pass geometry' is satisfied. However, even dealing with different problems than ours, several authors studied more general or different assumptions that still allow to apply min-max methods in order to assure the existence of critical points. Here we show a result that moves in this direction. We recall that a C^1-functional $J_\lambda : X \to \mathbb{R}$ defined on a real Banach space X satisfies the Cerami condition (briefly (C)) if

(C) *Every sequence $\{x_n\}$ in X such that $\{J_\lambda(x_n)\}$ is bounded and*

$$(1 + \|x_n\|_X)\|J'_\lambda(x_n)\|_{X^*} \to 0,$$

admits a strongly convergent subsequence in X. A sequence $\{x_n\}$ that satisfies the above conditions is called a Cerami sequence.

We recall here, for reader's convenience, the following Lemma due to Liu; see [26, Lemma 2.5].

Lemma 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and set*

$$\widetilde{F}(\xi) := \xi f(\xi) - 2F(\xi), \tag{$\mathcal{F}$}$$

for every $\xi \in \mathbb{R}$. Assume that

(f_1) *there exists $\rho > 0$ such that the function*

$$g(t) := \frac{f(t)}{t},$$

is non-decreasing in $t \geq \rho$ and non-increasing in $t \leq -\rho$.

Then, the real mapping $\widetilde{F}$*, defined in* ($\mathcal{F}$)*, is non-decreasing in* $t \geq \rho$ *and non-increasing in* $t \leq -\rho$*. In particular, there is a constant* $C_1 > 0$ *such that*

$$\widetilde{F}(\xi_1) \leq \widetilde{F}(\xi_2) + C_1, \tag{19}$$

for $0 \leq \xi_1 \leq \xi_2$ *or* $\xi_2 \leq \xi_1 \leq 0$*.*

Proof. Let us consider only the case $\xi_1 \leq \xi_2 \leq t$. We have

$$\begin{aligned}
\widetilde{F}(\xi_2) - \widetilde{F}(\xi_1) &= 2\left[\frac{(f(\xi_2)t - f(\xi_1)\xi_1) - (F(\xi_2) - F(\xi_1))}{2}\right] \\
&= 2\Big[\int_{\rho}^{\xi_2} g(\xi_2)\tau d\tau - \int_{\rho}^{\xi_1} g(\xi_1)\tau d\tau - \int_{\xi_1}^{\xi_2} g(\tau)\tau d\tau \\
&\quad - \frac{g(\xi_2)\rho^2}{2} - \frac{g(\xi_1)\rho^2}{2}\Big] \\
&= 2\Big[\int_{\xi_1}^{\xi_2} (g(\xi_2) - g(\tau))\tau d\tau + \int_{\rho}^{\xi_1} (g(\xi_2) - g(\xi_1))\tau d\tau \\
&\quad + \frac{\rho^2}{2}(g(\xi_2) - g(\xi_1))\Big] \geq 0.
\end{aligned}$$

The case $\xi_2 \leq \xi_1 \leq -\rho$ is similar. Furthermore, condition implies that $\widetilde{F} \in C^0(\mathbb{R};\mathbb{R})$ and

$$C_1 := 1 + \max_{\xi\in[-\rho,\rho]} \widetilde{F}(\xi) - \min_{\xi\in[-\rho,\rho]} \widetilde{F}(\xi) < +\infty.$$

With this positive constant C_1 it is easy to see that condition (19) holds. □

As consequence of Theorem 3.2 and the above Lemma we obtain the following multiplicity result.

Theorem 3.4 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that conditions* (h'_∞) *and* (f_1) *hold in addition to*

$$\lim_{|\xi|\to\infty} \frac{F(\xi)}{\xi^2} = +\infty. \tag{$\mathrm{h}_{\infty,F}$}$$

Furthermore, assume that

$$\lim_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty. \tag{h'_0}$$

Then, for each $\lambda \in \Lambda_{\mathcal{M}}$*, the problem* ($P_\lambda$) *admits at least two weak solutions. If, in addition,* $f(0) \neq 0$ *the attained solutions are non-zero.*

Proof. Let $X := H_1^2(\mathcal{M})$. Fix $\lambda \in \Lambda_{\mathcal{M}}$ and argue as in the proof of Theorem 3.3 in order to assure the existence of a first non-trivial weak solution. Moreover, in view of assumption $(\mathrm{h}_{\infty,F})$, it is possible to verify that

$$\liminf_{\|w\|_X \to +\infty} J_\lambda(w) = -\infty. \tag{20}$$

Indeed, for

$$\eta > \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\lambda \|K\|_{L^1(\mathcal{M})}},$$

there exists $\tau > 0$ such that $F(\xi) > \eta \xi^2$ for every $|\xi| > \tau$. Hence, if $\{\xi_n\}$ is a sequence in $\mathbb{R}$ with $\xi_n \to +\infty$ and we consider the sequence in X defined by putting $w_n(\sigma) = \xi_n$ for every $\sigma \in \mathcal{M}$ we have

$$J_\lambda(w_n) = \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2}\xi_n^2 - \lambda \|K\|_{L^1(\mathcal{M})} F(\xi_n) < \left(\frac{\|\alpha\|_{L^1(\mathcal{M})}}{2} - \lambda \|K\|_{L^1(\mathcal{M})}\eta\right)\xi_n^2,$$

for every $n \in \mathbb{N}$ large enough and (20) follows from the choice of η. Hence, there exists $u_1 \in X$ such that (18) is true. To complete the proof, it is sufficient to prove that, from Lemma 3.1, the functional J_λ satisfies the (C)-condition. For our goal we argue following [26, Lemma 2.5]. Hence, let $\{w_n\}$ be a Cerami sequence of J_λ. We observe that it suffices to show that $\{w_n\}$ is bounded. Indeed, if $\{w_n\}$ is a bounded sequence in X such that $\{J_\lambda(w_n)\}$ is bounded and $\|J'_\lambda(w_n)\|_{X^*} \to 0$, since Φ' is a linear isomorphism and Ψ' is compact, passing to a subsequence if necessary, $\{w_n\}$ is strongly convergent in X; see [14, Proposition 3.8]. At this point, in order to prove the boundedness of $\{w_n\}$, we argue by contradiction. If $\{w_n\}$ is unbounded, up to a subsequence we may assume that there is $c \in \mathbb{R}$ such that:

$$\begin{aligned} J_\lambda(w_n) &\to c \\ \|w_n\|_X &\to +\infty \\ \|w_n\|_X \|J'_\lambda(w_n)\|_{X^*} &\to 0. \end{aligned}$$

In particular, since $J'_\lambda(w_n)(w_n) \to 0$ (as $n \to \infty$), we have

$$\lim_{n\to\infty} \int_{\mathcal{M}} K(\sigma) \frac{\widetilde{F}(w_n(\sigma))}{2} d\sigma_g = \frac{\lim_{n\to\infty}\left\{J_\lambda(w_n) - \frac{1}{2}J'_\lambda(w_n)(w_n)\right\}}{\lambda} = \frac{c}{\lambda}. \tag{21}$$

Let $x_n := w_n/\|w_n\|_{H^2_\alpha}$, for every $n \in \mathbb{N}$. Up to a subsequence, we have that

$$\begin{aligned} &x_n \rightharpoonup x \text{ in } X \\ &x_n \to x \text{ in } L^q(\mathcal{M}) \\ &x_n(\sigma) \to x(\sigma) \quad \text{a.e. } \sigma \in \mathcal{M}. \end{aligned}$$

If $x = 0$, we choose a sequence $\{t_n\} \subset [0, 1]$ such that

$$J_\lambda(t_n w_n) = \max_{t\in[0,1]} J_\lambda(t w_n).$$

For any $m > 0$, let $v_n := 2\sqrt{m}x_n$, for every $n \in \mathbb{N}$.

At this point, owing to $v_n \to 0$ in $L^q(\mathcal{M})$ (so $v_n \to 0$ in $L^1(\mathcal{M})$) and

$$\int_{\mathcal{M}} |K(\sigma)F(v_n(\sigma))|d\sigma_g \le \|K\|_\infty \left(a_1 \|v_n\|_{L^1(\mathcal{M})} + \frac{a_2}{q}\|v_n\|^q_{L^q(\mathcal{M})} \right) \to 0,$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} \left| \int_{\mathcal{M}} K(\sigma)F(v_n(\sigma))d\sigma_g \right| = 0.$$

So, for n sufficiently large, $2\sqrt{m}/\|w_n\|_{H^2_\alpha} \in]0,1[$, and

$$\|v_n\|^2_{H^2_\alpha} = \|2\sqrt{m}x_n\|^2_{H^2_\alpha} = \left\| 2\sqrt{m}\frac{w_n}{\|w_n\|_{H^2_\alpha}} \right\|^2_{H^2_\alpha} = 4m.$$

Hence, there exists $\nu \in \mathbb{N}$ such that for evert $n \ge \nu$, we can write

$$J_\lambda(t_n w_n) := \max_{t\in[0,1]} J_\lambda(t w_n) \ge J_\lambda(v_n) \ge 2m - \left| \int_{\mathcal{M}} K(\sigma) f(v_n(\sigma))d\sigma_g \right| \ge m.$$

Then, we have

$$J_\lambda(t_n w_n) \to +\infty, \tag{22}$$

as $n \to \infty$. Now, since $J_\lambda(0) = 0$ and $J_\lambda(w_n) \to c$, we deduce that $t_n \in]0,1[$ and

$$\begin{aligned} \|t_n w_n\|^2_{H^2_\alpha} - \lambda \int_{\mathcal{M}} K(\sigma) f(t_n w_n(\sigma)) t_n w_n(\sigma) d\sigma_g &= J'_\lambda(t_n w_n)(t_n w_n) \\ &= t_n \frac{d}{dt}|_{t=t_n} J_\lambda(t w_n) = 0. \end{aligned}$$

Therefore, using relation (19) in Lemma 3.1 and the above computation, we deduce that

$$\begin{aligned} \int_{\mathcal{M}} K(\sigma)\frac{\widetilde{F}(w_n(\sigma))}{2} d\sigma_g &\ge \int_{\mathcal{M}} K(\sigma)\frac{\widetilde{F}(t_n w_n(\sigma))}{2} d\sigma_g - \frac{C_1}{2}\|K\|_{L^1(\mathcal{M})} \\ &= \frac{1}{2\lambda}\|t_n w_n\|^2_{H^2_\alpha} - \int_{\mathcal{M}} K(\sigma)F(t_n w_n(\sigma))d\sigma_g \\ &- \frac{C_1}{2}\|K\|_{L^1(\mathcal{M})} \\ &= \frac{J_\lambda(t_n w_n)}{\lambda} - \frac{C_1}{2}\|K\|_{L^1(\mathcal{M})}. \end{aligned}$$

Therefore, since (22) holds, it follows that

$$\int\limits_{\mathcal{M}} K(\sigma)\frac{\widetilde{F}(w_n(\sigma))}{2}d\sigma_g \to +\infty.$$

This relation contradicts with (21). If $x \neq 0$ the sub-manifold

$$\mathcal{M}_\Theta := \{\sigma \in \mathcal{M} : x(\sigma) \neq 0\},$$

has positive Riemann measure. Since $|w_n(\sigma)| = |x_n(\sigma)|\|w_n\|_{H^2_\alpha}$ for every $\sigma \in \mathcal{M}$, for $\sigma \in \mathcal{M}_\Theta$, we have $|w_n(\sigma)| \to \infty$ and thanks to $(\mathrm{h}_{\infty,F})$, we obtain

$$\frac{F(w_n(\sigma))}{w_n(\sigma)^2}x_n(\sigma)^2 \to +\infty.$$

Hence, as $n \to \infty$, the Fatou's Lemma, implies

$$\int\limits_{\mathcal{M}_\Theta} \frac{F(w_n(\sigma))}{w_n(\sigma)^2}x_n(\sigma)^2 d\sigma_g \to +\infty. \tag{23}$$

Since $J_\lambda(w_n) \to c$, clearly

$$\frac{1}{2}\|w_n\|^2_{H^2_\alpha} - \lambda\int\limits_{\mathcal{M}} K(\sigma)F(w_n(\sigma))d\sigma_g - c = o(1),$$

where $o(1) \to 0$ as $n \to \infty$. Then

$$\frac{1}{2} - \frac{c + o(1)}{\|w_n\|^2_{H^2_\alpha}} = \lambda\int\limits_{\mathcal{M}} \frac{K(\sigma)F(w_n(\sigma))}{\|w_n\|^2_{H^2_\alpha}}d\sigma_g.$$

Consequently, we can write

$$\begin{aligned}\frac{1}{2} - \frac{c + o(1)}{\|w_n\|^2_{H^2_\alpha}} &= \lambda\int\limits_{\mathcal{M}} \frac{K(\sigma)F(w_n(\sigma))}{\|w_n\|^2_{H^2_\alpha}}d\sigma_g \\ &= \lambda\int\limits_{\mathcal{M}_\Theta} \frac{K(\sigma)F(w_n(\sigma))}{w_n(\sigma)^2}x_n(\sigma)^2 d\sigma_g \\ &+ \lambda\int\limits_{\mathcal{M}\setminus\mathcal{M}_\Theta} \frac{K(\sigma)F(w_n(\sigma))}{w_n(\sigma)^2}x_n(\sigma)^2 d\sigma_g\end{aligned} \tag{24}$$

On the other hand from hypothesis $(\mathrm{h}_{\infty,F})$ we have

$$\lim_{|\xi|\to\infty} F(\xi) = +\infty.$$

Hence, taking into account that $F \in C^0(\mathbb{R};\mathbb{R})$, there exists a positive constant ϱ_1 such that

$$F(\xi) \geq -\varrho_1,$$

for every $\xi \in \mathbb{R}$. Then, since $1/\|w_n\|_{H^2_\alpha} = x_n/w_n$, it follows that

$$\int_{\mathcal{M}\setminus\mathcal{M}_\Theta} \frac{K(\sigma)F(w_n(\sigma))x_n(\sigma)^2}{w_n(\sigma)^2} d\sigma_g \geq -\varrho_1 \frac{\|K\|_\infty}{\|w_n\|^2_{H^2_\alpha}} \mathrm{Vol}_g(\mathcal{M}\setminus\mathcal{M}_\Theta).$$

The above inequality together (23) and (24) yield

$$\frac{1}{2} - \frac{c+o(1)}{\|w_n\|^2_{H^2_\alpha}} \geq \lambda \operatorname*{ess\,inf}_{\sigma\in\mathcal{M}} K(\sigma) \int_{\mathcal{M}_\Theta} \frac{F(w_n(\sigma))}{w_n(\sigma)^2} x_n(\sigma)^2 d\sigma_g$$
$$- \varrho_1 \lambda \frac{\|K\|_\infty}{\|w_n\|^2_{H^2_\alpha}} \mathrm{Vol}_g(\mathcal{M}\setminus\mathcal{M}_\Theta).$$

This is clearly impossible. Hence, the sequence $\{w_n\}$ is bounded in X. In conclusion, applying the version of the mountain pass theorem, where (PS) is replaced by (C), the conclusion is achieved; see, for details, the work [11]. □

Remark 3.9. Condition ($\mathrm{h}_{\infty,F}$) is a consequence of the following

$$\lim_{|t|\to\infty} \frac{f(t)}{t} = +\infty,$$

that characterizes the problem (P_λ) as superlinear at infinity. As pointed out in precedence the boundedness of Palais-Smale sequences of the Euler-Lagrange functional J_λ can be obtained if the Ambrosetti–Rabinowitz condition is verified. However, hypothesis (AR) can not be useful treating some nonlinearities. Indeed, if a function f satisfies (AR) then there exist suitable (positive) constants γ_1, γ_2 with

$$F(\xi) \geq \gamma_1|\xi|^\mu - \gamma_2, \quad \forall\, \xi \in \mathbb{R}.$$

For this reason, in recent years, several authors studied superlinear problems trying to drop the condition (AR); see for instance the works [26,27] and the references therein.

Example 3.3. Set $f: \mathbb{R} \to \mathbb{R}$ be the continuous function given by

$$f(t) := t\log(1+|t|) + 1.$$

Hence, we deduce that

$$F(\xi) = \frac{|\xi|}{2} + \frac{(\xi^2-1)\log(|\xi|+1)}{2} - \frac{\xi^2}{4} + \xi, \quad \forall\, \xi \in \mathbb{R}$$

and it is easy to check that all the conditions of Theorem 3.4 hold. Then, with the usual notations, the following equation

$$-\Delta_g w + \alpha(\sigma)w = \lambda K(\sigma)(w\log(1+|w|)+1), \qquad (C_\lambda)$$

for every $\sigma \in \mathcal{M}$ and $w \in H^2_1(\mathcal{M})$, admits at least two non-negative (and non-trivial) weak solutions for every $\lambda \in \Lambda_{\mathcal{M}}$. Finally, since

$$\lim_{|\xi|\to+\infty} \frac{\xi f(\xi)}{F(\xi)} = \lim_{|\xi|\to+\infty} \frac{4\xi(\xi\log(|\xi|+1)+1)}{2|\xi|+2(\xi^2-1)\log(|\xi|+1)-\xi(\xi-4)} = 2,$$

the Ambrosetti–Rabinowitz hypothesis fails.

4. Applications to singular elliptic problems

In this section d and s denote two real fixed constants with $d \geq 3$ and $1-d < s < 0$, $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function (or more generally locally Hölder continuous) and K is a smooth and positive map on the unit sphere $\mathbb{S}^d$. Consider the following parameterized Emden-Fowler problem that arises in astrophysics, conformal Riemannian geometry, and in the theories of thermionic emission, isothermal stationary gas sphere, and gas combustion:

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\}. \tag{$\mathfrak{F}_\lambda$}$$

As pointed out in Introduction, equation ($\mathfrak{F}_\lambda$) has been studied by Cotsiolis–Iliopoulos [13], Vázquez-Véron [34] by using either minimization or minimax methods. More recently, in [23] and successively in [9], some existence results are achieved by variational methods. The solutions of ($\mathfrak{F}_\lambda$) are being sought in the particular form

$$u(x) = r^s w(\sigma), \tag{25}$$

where, $(r, \sigma) := (|x|, x/|x|) \in (0, \infty) \times \mathbb{S}^d$ are the spherical coordinates in $\mathbb{R}^{d+1} \setminus \{0\}$ and w be a smooth function defined on $\mathbb{S}^d$. This type of transformation is also used by Bidaut-Véron and Véron [5], where the asymptotic of a special form of ($\mathfrak{F}_\lambda$) has been studied. Throughout (25), taking into account that

$$\Delta u = r^{-d} \frac{\partial}{\partial r}\left(r^d \frac{\partial u}{\partial r}\right) + r^{-2} \Delta_h u,$$

equation ($\mathfrak{F}_\lambda$) reduces to

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma) f(w), \qquad \sigma \in \mathbb{S}^d, \; w \in H_1^2(\mathbb{S}^d).$$

Due to our regularity assumptions on the data, the weak solutions of our problem are also classical as observed by Kristály and Rădulescu in [23]. Now, for every two nonnegative constants γ, δ, with $\gamma \neq \delta$, let

$$a_\gamma^\star(\delta) := \frac{A^\star(\gamma) - qF(\delta)\|K\|_{L^1(\mathbb{S}^d)}}{s(1-s-d)\omega_d(\gamma^2 - \delta^2)q},$$

where

$$A^\star(\gamma) := (q(s(1-s-d)\omega_d)^{1/2} \gamma S_1 a_1 + (s(1-s-d)\omega_d)^{q/2} \gamma^q S_q^q a_2)\|K\|_\infty.$$

We have the following result.

Corollary 4.1 *Let $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function such that condition* (h_∞) *holds. Assume that there are three real constants γ_1, γ_2 and δ, with $0 \leq \gamma_1 < \delta < \gamma_2$, such that*

$$a_{\gamma_2}^\star(\delta) < a_{\gamma_1}^\star(\delta). \tag{26}$$

Then, for each parameter λ belonging to

$$\Lambda^{\star}_{\mathfrak{F}} := \left]\frac{1}{2a^{\star}_{\gamma_1}(\delta)}, \frac{1}{2a^{\star}_{\gamma_2}(\delta)}\right[,$$

the following problem

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\}, \tag{$\mathfrak{F}_\lambda$}$$

admits at least one classical solution.

Proof. Let us choose $(M, g) = (\mathbb{S}^d, h)$, and $\alpha(\sigma) := s(1-s-d)$ for every $\sigma \in \mathbb{S}^d$ in Theorem 3.1. Clearly $\alpha \in C^\infty(\mathbb{S}^d)$ and, thanks to $1-d < s < 0$, α to be positive on $\mathbb{S}^d$. Thus, for every $\Lambda^{\star}_{\mathfrak{F}} \subseteq \Lambda$, the problem

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma) f(w), \qquad \sigma \in \mathbb{S}^d, \ w \in H^2_1(\mathbb{S}^d),$$

has at least one non-trivial solution $w_\lambda \in H^2_1(\mathbb{S}^d)$. On account of (25), the element $u_\lambda(x) = |x|^s w_\lambda(x/|x|)$, is a non-trivial solution of $(\mathfrak{F}_\lambda)$. □

A special case of Theorem 3.3, reads as follows.

Corollary 4.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function such that conditions* (h'_∞) *and* $(\mathrm{h}''_{0,F})$ *hold. Then, there exists $\lambda^{\star}_{\mathfrak{F}} > 0$ such that for every $\lambda \in]0, \lambda^{\star}_{\mathfrak{F}}[$, the following problem*

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\}, \tag{$\mathfrak{F}_\lambda$}$$

admits at least one non-trivial classical solution. Moreover if, in addition, the function f satisfies

$$0 < \mu \int_0^t f(s)ds \leq t f(t),$$

for every $|t| \geq r$, for some $r > 0$ and $\mu > 2$, then, for each $\lambda \in]0, \lambda^{\star}_{\mathfrak{F}}[$, the problem $(\mathfrak{F}_\lambda)$ admits at least two non-trivial classical solutions.

Remark 4.1. Arguing as in Corollary 4.1, analogous of Theorems in Sect. 3 can be easily obtained for our new setting. For instance, the first part of Corollary 4.2 is an exhaustive version of Theorem 3.2 for Emden-Fowler type equations. In this case, for every parameter $\lambda \in]0, \lambda^{\star}_{\mathfrak{F}}[$, the problem $(\mathfrak{F}_\lambda)$ admits at least one non-trivial solution. Moreover, the existence of a second non-trivial solution is obtained arguing as in Theorem 3.3. Moreover, by using Remarks 3.2, a concrete expression for the value of $\lambda^{\star}_{\mathfrak{F}}$ is given by

$$\lambda^{\star}_{\mathfrak{F}} = \begin{cases} +\infty & \text{if} \quad 1 < q < 2 \\ \frac{1}{\|K\|_\infty S_2^2 a_2} & \text{if} \quad q = 2 \\ \frac{q(s(1-s-d)\omega_d)^{1/2}\overline{\gamma}_{\max}}{2\|K\|_\infty (q S_1 a_1 + (s(1-s-d)\omega_d)^{(q-1)/2} S_q^q a_2 \overline{\gamma}_{\max}^{q-1})} & \text{if} \quad q \in \left]2, \frac{2d}{d-2}\right[, \end{cases}$$

where

$$\overline{\gamma}_{\max} := \left(\frac{S_1 a_1}{(q-2)(s(1-s-d))^{q/2}\omega_d^{(q-1)/2} S_q^q a_2} \right)^{1/(q-1)}.$$

Finally, we give here a direct application of Corollary 4.2.

Example 4.1. Let $K \in C^\infty(\mathbb{S}^3)$ be a positive mapping and consider the parametric Emden-Fowler equation

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|)(1 + |x|^{-3s} u^3), \qquad x \in \mathbb{R}^4 \setminus \{0\}, \qquad (\mathfrak{F}'_\lambda)$$

where we fix $s \in]-2, 0[$. Then, for every λ belonging to the interval

$$\Lambda_{\mathfrak{F}'_\lambda} := \left] 0, \frac{1}{3\|K\|_\infty S_1} \left(\frac{2}{S_4^4} \right)^{1/3} \right[,$$

the problem $(\mathfrak{F}'_\lambda)$ admits at least two non-negative (and non-trivial) classical solutions. To prove this, we can apply Corollary 4.2 to the locally Lipschitz continuous function

$$f(t) := 1 + t^3, \qquad \forall t \in \mathbb{R},$$

bearing in mind Remark 4.1. Indeed, clearly the function f satisfies (h'_∞) and, since

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty,$$

also condition $(\mathrm{h}''_{0,F})$ holds true. Moreover, taking into account that

$$\lim_{|\xi| \to \infty} \frac{\xi f(\xi)}{F(\xi)} = 4 \lim_{|\xi| \to \infty} \frac{\xi^3 + 1}{\xi^3 + 4} = 4 > 2,$$

there exist $\mu > 2$ and $r > 0$ such that

$$0 < \mu F(\xi) \leq \xi f(\xi),$$

for every $|\xi| > r$. Hence, all the assumptions of Corollary 3.3 are verified and the conclusion follows.

Remark 4.2. It is well-known that sharp Sobolev inequalities are important in the study of partial differential equations. In our context, a concrete upper bound for the constants S_q in the above example is essential for an explicit evaluation of the interval of parameters. Now, we observe that in Example 4.1 a more precise information on the size of the interval $\Lambda_{\mathfrak{F}'_\lambda}$ can be easily obtained taking into account Remark 3.4. Indeed, if $q \in [1, 5]$ and $s \in]-2, 0[$, we have

$$S_q^\star = \frac{\omega_3^{\frac{2-q}{2q}}}{(-s(s+2))^{1/2}}.$$

Consequently, for λ sufficiently small, more precisely

$$0 < \lambda < \frac{(-s(s+2))^{7/6}}{3}\left(\frac{2^{1/3}}{\omega_3^{1/6}\|K\|_\infty}\right),$$

problem $(\mathfrak{F}'_\lambda)$ admits at least two non-negative (and non-trivial) classical solutions.

Acknowledgements. V. Rădulescu acknowledges the support through Grant CNCS PCE–47/2011.

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