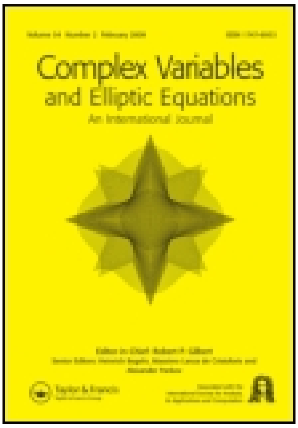

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A Caffarelli--Kohn--Nirenberg-type inequality with variable exponent and applications to PDEs

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A Caffarelli–Kohn–Nirenberg-type inequality with variable exponent and applications to PDEs†

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Given $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ a bounded smooth domain we establish a Caffarelli–Kohn–Nirenberg type inequality on Ω in the case when a variable exponent $p(x)$, of class C^1, is involved. Our main result is proved under the assumption that there exists a smooth vector function $\vec{a} : \Omega \to \mathbb{R}^N$, satisfying $\operatorname{div} \vec{a}(x) > 0$ and $\vec{a}(x) \cdot \nabla p(x) = 0$ for any $x \in \Omega$. Particularly, we supplement a result by Fan et al. [X. Fan, Q. Zhang, and D. Zhao, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. 302 (2005), pp. 306–317] regarding the positivity of the first eigenvalue of the $p(x)$-Laplace operator. Moreover, we provide an application of our result to the study of degenerate PDEs involving variable exponent growth conditions.

Keywords: Caffarelli–Kohn–Nirenberg-type inequality; eigenvalue problem; degenerate elliptic equation; variable exponent; critical point

AMS Subject Classifications: 35D05; 35J60; 35J70; 58E05; 35P05

1. Introduction

In [1], Caffarelli et al. proved in the context of some more general inequalities, the following result: given $p \in (1, N)$, for all $u \in C_c^1(\Omega)$, there exists a positive constant $C_{a,b}$ such that

$$\left(\int_\Omega |x|^{-bq} |u|^q \, \mathrm{d}x \right)^{p/q} \leq C_{a,b} \int_\Omega |x|^{-ap} |\nabla u|^p \, \mathrm{d}x, \tag{1.1}$$

where

$$-\infty < a < \frac{N-p}{p}, \quad a \leq b \leq a+1, \quad q = \frac{Np}{N - p(1 + a - b)},$$

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and $\Omega \subseteq \mathbb{R}^N$ is an arbitrary open domain. Note that the Caffarelli–Kohn–Nirenberg inequality (1.1) reduces to the classical Sobolev inequality (if $a = b = 0$) and to the Hardy inequality (if $a = 0$ and $b = 1$). Inequality (1.1) proves to be an important tool in studying degenerate elliptic problems. It is also related to the understanding of some important phenomena, such as best constants, existence or nonexistence of extremal functions, symmetry properties of minimizers, compactness of minimizing sequences and concentration phenomena. We refer to [2–11] for relevant applications of the Caffarelli–Kohn–Nirenberg inequality.

In the years that followed this inequality was extensively studied (see, e.g. [2–4,11] and the references therein). An important consequence of the Caffarelli–Kohn–Nirenberg inequality is that it enabled the study of some degenerate elliptic equations which involve differential operators of the type

$$\operatorname{div}(a(x)|\nabla u(x)|^p),$$

where $a(x)$ is a nonnegative function satisfying $\inf_x a(x) = 0$. Thus, the resulting operator is not uniformly elliptic and consequently some of the techniques that can be applied in solving equations involving uniformly elliptic operators fail in this new context. Degenerate differential operators involving a nonnegative weight that is allowed to have zeros at some points or even to be unbounded are used in the study of many physical phenomena related to equilibrium of anisotropic continuous media.

The goal of this article is to obtain inequalities of type (1.1) in the case when the constant p is replaced by a function $p(x)$ of class C^1 and to use them in studying some degenerate elliptic equations involving variable exponent growth conditions. Our attempt will be considered in the context of bounded smooth domains $\Omega \subset \mathbb{R}^N$ with $N \geq 2$. Particularly, we supplement the result of [12, Theorem 3.3], regarding the positivity of the first eigenvalue of the $p(x)$-Laplace operator.

2. Variable exponent Lebesgue and Sobolev spaces

We recall some definitions and basic properties of the Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$ and $p(x) : \overline{\Omega} \to (1, \infty)$ is a continuous function. For further information and proofs we refer to [13,14]. On the other hand, regarding applications of variable exponent Lebesgue and Sobolev spaces to PDEs we refer to [15] while for some physical motivations of such problems we remember the contributions of Rajagopal and Ruzicka [16], Ruzicka [17] and Zhikov [18].

For any continuous function $h : \overline{\Omega} \to (1, \infty)$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

Given $p(x) \in C(\overline{\Omega}, (1, \infty))$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}.$$

$L^{p(x)}(\Omega)$ endowed with the *Luxemburg norm*, that is

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} \mathrm{d}x \le 1\right\},$$

is a reflexive Banach space.

If p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \tag{2.1}$$

holds true.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true:

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{2.2}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{2.3}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0 \tag{2.4}$$

$$|u_n|_{p(x)} \to \infty \Leftrightarrow \rho_{p(x)}(u_n) \to \infty. \tag{2.5}$$

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \|\nabla u\|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^\star(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \ge N$.

3. A Caffarelli–Kohn–Nirenberg-type inequality in bounded domains involving variable exponent growth conditions

Assume $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is an open, bounded and smooth set.

For each $x \in \Omega$, $x = (x_1, \ldots, x_N)$ and $i \in \{1, \ldots, N\}$ we denote

$$m_i = \inf_{x\in\Omega} x_i \quad M_i = \sup_{x\in\Omega} x_i.$$

For each $i \in \{1, \ldots, N\}$ let $a_i : [m_i, M_i] \to \mathbb{R}$ be functions of class C^1. Particularly, the functions a_i are allowed to vanish.

Let $\vec{a} : \Omega \to \mathbb{R}^N$ be defined by

$$\vec{a}(x) = (a_1(x_1), \ldots, a_N(x_N)).$$

We assume that there exists $a_0 > 0$ a constant such that

$$\operatorname{div} \vec{a}(x) \geq a_0 > 0 \quad \forall x \in \overline{\Omega}. \tag{3.1}$$

Next, we consider $p : \overline{\Omega} \to (1, N)$ is a function of class C^1 satisfying

$$\vec{a}(x) \cdot \nabla p(x) = 0 \quad \forall x \in \Omega. \tag{3.2}$$

We prove the following result:

Theorem 1 *Assume that $\vec{a}(x)$ and $p(x)$ are defined as above and satisfy conditions* (3.1) *and* (3.2). *Then there exists a positive constant C such that*

$$\int_\Omega |u(x)|^{p(x)} \, \mathrm{d}x \leq C \int_\Omega |\vec{a}(x)|^{p(x)} |\nabla u(x)|^{p(x)} \, \mathrm{d}x \quad \forall u \in C_c^1(\Omega). \tag{3.3}$$

Proof The proof of Theorem 1 is inspired by the ideas in [19, Théorème 20.7].

Simple computations based on relation (3.2) show that for each $u \in C_c^1(\Omega)$ the following equality holds true:

$$\begin{aligned}
\operatorname{div}(|u(x)|^{p(x)} \vec{a}(x)) &= \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(|u(x)|^{p(x)} a_i(x_i) \right) \\
&= |u(x)|^{p(x)} \operatorname{div} \vec{a}(x) \\
&\quad + \sum_{i=1}^N a_i(x_i) \left[p(x) |u(x)|^{p(x)-2} u(x) \frac{\partial u}{\partial x_i} + |u(x)|^{p(x)} \log(|u(x)|) \frac{\partial p}{\partial x_i} \right] \\
&= |u(x)|^{p(x)} \operatorname{div} \vec{a}(x) + p(x) |u(x)|^{p(x)-2} u(x) \nabla u(x) \cdot \vec{a}(x) \\
&\quad + |u(x)|^{p(x)} \log(|u(x)|) \nabla p(x) \cdot \vec{a}(x) \\
&= |u(x)|^{p(x)} \operatorname{div} \vec{a}(x) + p(x) |u(x)|^{p(x)-2} u(x) \nabla u(x) \cdot \vec{a}(x).
\end{aligned}$$

On the other hand, the flux-divergence theorem implies that for each $u \in C_c^1(\Omega)$ we have

$$\int_\Omega \operatorname{div}(|u(x)|^{p(x)} \vec{a}(x)) \, \mathrm{d}x = \int_{\partial\Omega} |u(x)|^{p(x)} \vec{a}(x) \cdot \vec{n} \, \mathrm{d}\sigma(x) = 0.$$

Using the above pieces of information we infer that for each $u \in C_c^1(\Omega)$ it holds true

$$\int_\Omega |u(x)|^{p(x)} \operatorname{div} \vec{a}(x) \, \mathrm{d}x \leq p^+ \int_\Omega |u(x)|^{p(x)-1} |\nabla u(x)| |\vec{a}(x)| \, \mathrm{d}x.$$

Next, we recall that for each $\epsilon > 0$, each $x \in \Omega$ and each $A, B \geq 0$ the following Young type inequality holds true (see, e.g. [20, the footnote on p. 56])

$$AB \leq \epsilon A^{\frac{p(x)}{p(x)-1}} + \frac{1}{\epsilon^{p(x)-1}} B^{p(x)}.$$

We fix $\epsilon > 0$ such that

$$p^+\epsilon < a_0,$$

where a_0 is given by relation (3.1).

The above facts and relation (3.1) yield

$$a_0 \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x \le p^+\left[\epsilon \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x + \int_\Omega \left(\frac{1}{\epsilon}\right)^{p(x)-1} |\vec{a}(x)|^{p(x)} |\nabla u(x)|^{p(x)}\,\mathrm{d}x\right],$$

for any $u \in C_c^1(\Omega)$, or

$$(a_0 - \epsilon p^+) \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x \le \left[\left(\frac{1}{\epsilon}\right)^{p^- -1} + \left(\frac{1}{\epsilon}\right)^{p^+ -1}\right] p^+ \int_\Omega |\vec{a}(x)|^{p(x)} |\nabla u(x)|^{p(x)}\,\mathrm{d}x,$$

for any $u \in C_c^1(\Omega)$. The conclusion of Theorem 1 is now clear. ■

Remark 1 The result of Theorem 1 implies the fact that under the hypotheses (3.1) and (3.2) there exists a positive constant D such that

$$\int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x \le D \int_\Omega |\nabla u(x)|^{p(x)}\,\mathrm{d}x \quad \forall u \in C_c^1(\Omega).$$

Thus, we deduce that in the hypothesis of Theorem 1 we have

$$\inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x}{\displaystyle\int_\Omega |u|^{p(x)}\,\mathrm{d}x} > 0. \tag{3.4}$$

The above relation asserts that in this case the first eigenvalue of the $p(x)$-Laplace operator (i.e. $\Delta_{p(x)} u := \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$) is positive. That fact is not obvious as Fan et al. pointed out in [12]. Actually, the infimum of the set of eigenvalues corresponding to the $p(x)$-Laplace operator can be 0 [12, Theorem 3.1]. On the other hand, a necessary and sufficient condition such that (3.4) holds true has not been obtained yet except the case when $N = 1$ (in that case, the infimum is positive if and only if $p(x)$ is a monotone function, see [12, Theorem 3.2]). However, the authors of [12] pointed out that in the case $N > 1$ a sufficient condition to have (3.4) is a vector $l \in \mathbb{R}^N \setminus \{0\}$ to exist such that, for any $x \in \Omega$, the function $f(t) = p(x + tl)$ is monotone, for $t \in I_x := \{s; x + sl \in \Omega\}$ [12, Theorem 3.3]. Assuming p is of class C^1 the monotony of function f reads as follows: either

$$\nabla p(x + tl) \cdot l \ge 0, \quad \text{for all } t \in I_x,\ x \in \Omega,$$

or

$$\nabla p(x + tl) \cdot l \le 0, \quad \text{for all } t \in I_x,\ x \in \Omega.$$

The above conditions seem to be related to condition (3.2) in this article. On the other hand, in the case when $N = 1$ relation (3.2) implies that $p(x)$ should be a constant function. The two results do not contradict each other but they seem to supplement each other.

Example 1 We point out an example of functions $\vec{a}(x)$ and $p(x)$ satisfying conditions (3.1) and (3.2) in the case when $\vec{a}(x)$ can vanish in some points of Ω. Let $N \geq 3$ and $\Omega = B_{\frac{1}{\sqrt{N}}}(0)$, the ball centred in the origin of radius $\frac{1}{\sqrt{N}}$. We define $\vec{a}(x) : \Omega \to \mathbb{R}^N$ by

$$\vec{a}(x) = (-x_1, x_2, x_3, \ldots, x_{N-1}, x_N),$$

(more exactly, function $\vec{a}(x)$ is associated with a vector $x \in \Omega$ the vector obtained from x by changing in the first position x_1 by $-x_1$ and keeping unchanged x_i for $i \in \{2, \ldots, N\}$). Clearly, $\vec{a}(x)$ is of class C^1, $\vec{a}(0) = 0$ and we have

$$\operatorname{div}(\vec{a}(x)) = N - 2 \geq 1 \quad \forall x \in \Omega.$$

Thus, condition (3.1) is satisfied.

Next, we define $p : \overline{\Omega} \to (1, N)$ by

$$p(x) = x_1(x_2 + x_3 + \cdots + x_{N-1} + x_N) + 2 \quad \forall x \in \overline{\Omega}.$$

It is easy to check that p is of class C^1 and some elementary computations show that

$$\nabla p(x) \cdot \vec{a}(x) = (x_2 + \cdots + x_N)(-x_1) + x_1 x_2 + \cdots + x_1 x_N = 0 \quad \forall x \in \Omega.$$

It means that condition (3.2) is satisfied, too.

Example 2 We point out a second example, for $N = 2$. Taking $\Omega = B_{\frac{1}{3^{1/3}}}(0)$, $\vec{a}(x) = (-x_1, 2x_2)$ and $p(x) = x_1^2 x_2 + \frac{3}{2}$ it is easy to check that relations (3.1) and (3.2) are fulfilled.

Remark 2 If N, a and p are as in Example 1 or Example 2 then the result of Theorem 1 reads as follows: there exists a positive constant $C > 0$ such that

$$\int_\Omega |u(x)|^{p(x)}\, \mathrm{d}x \leq C \int_\Omega |x|^{p(x)} |\nabla u(x)|^{p(x)}\, \mathrm{d}x \quad \forall u \in C_c^1(\Omega). \tag{3.5}$$

4. Applications in solving PDEs involving variable exponent growth conditions

In this section we assume that N, Ω, $\vec{a}(x)$ and $p(x)$ are as in Example 1 or Example 2. We denote by $\mathcal{D}_0^{1,p(x)}(\Omega)$ the closure of $C_c^1(\Omega)$ under the norm

$$\|u\| = \big| |x| |\nabla u(x)| \big|_{p(x)}.$$

Undoubtedly, $(\mathcal{D}_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a reflexive Banach space.

4.1. *A compact embedding result*

We prove the following result:

THEOREM 2 *Assume that* N, Ω, $\vec{a}(x)$ *and* $p(x)$ *are as in Example* 1 *or Example* 2 *and* $p^- > \frac{2N}{2N-1}$. *Then* $\mathcal{D}_0^{1,p(x)}(\Omega)$ *is compactly embedded in* $L^q(\Omega)$ *for each* $q \in \left(1, \frac{2Np^-}{2N+p^-}\right)$.

Proof Let $\{u_n\}$ be a bounded sequence in $\mathcal{D}_0^{1,p(x)}(\Omega)$. There exists $\epsilon_0 \in (0, 1)$ such that we have $\overline{B}_{\epsilon_0}(0) \subset \Omega$. Let $\epsilon \in (0, \epsilon_0)$ be arbitrary but fixed. By Theorem 1 it follows that $\{u_n\}$ is a bounded sequence in $L^{p(x)}(\Omega)$. Consequently, $\{u_n\} \subset W^{1,p(x)}(\Omega \setminus \overline{B}_\epsilon(0))$

is a bounded sequence. Since $W^{1,p(x)}(\Omega\setminus\overline{B}_\epsilon(0))\subset W^{1,p^-}(\Omega\setminus\overline{B}_\epsilon(0))$ we deduce that $\{u_n\}$ is a bounded sequence in $W^{1,p^-}(\Omega\setminus\overline{B}_\epsilon(0))$. The classical compact embedding theorem shows that there exists a convergent subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, in $L^q(\Omega\setminus\overline{B}_\epsilon(0))$. Thus, for any n and m large enough we have

$$\int_{\Omega\setminus\overline{B}_\epsilon(0)}|u_n-u_m|^q\,\mathrm{d}x<\epsilon. \tag{4.1}$$

On the other hand, the Hölder inequality for variable exponent spaces implies

$$\begin{aligned}\int_{B_\epsilon(0)}|u_n-u_m|^q\,\mathrm{d}x&=\int_{B_\epsilon(0)}|x|^{-\frac{q}{2}}|x|^{\frac{q}{2}}|u_n-u_m|^q\,\mathrm{d}x\\&\le D_1\left||x|^{-\frac{q}{2}}\chi_{B_\epsilon(0)}\right|_{\left(\frac{p(x)}{q}\right)'}\left||x|^{\frac{q}{2}}|u_n-u_m|^q\right|_{\frac{p(x)}{q}},\end{aligned}$$

where D_1 is a positive constant.

Furthermore, inequality (3.5) and relations (2.2) and (2.3) imply

$$\begin{aligned}&\left||x|^{\frac{q}{2}}|u_n-u_m|^q\right|_{\frac{p(x)}{q}}\\&\le\left(\int_\Omega|x|^{\frac{p(x)}{2}}|u_n-u_m|^{p(x)}\,\mathrm{d}x\right)^{\frac{q}{p^-}}+\left(\int_\Omega|x|^{\frac{p(x)}{2}}|u_n-u_m|^{p(x)}\,\mathrm{d}x\right)^{\frac{q}{p^+}}\\&\le\left[(\sup_{x\in\Omega}|x|+1)^{\frac{qp^+}{2p^-}}+(\sup_{x\in\Omega}|x|+1)^{\frac{q}{2}}\right]\left[\left(\rho_{p(x)}(u_n-u_m)\right)^{\frac{q}{p^-}}+\left(\rho_{p(x)}(u_n-u_m)\right)^{\frac{q}{p^+}}\right]\\&\le D_2\left[\left(\int_\Omega|x|^{p(x)}|\nabla(u_n-u_m)|^{p(x)}\,\mathrm{d}x\right)^{\frac{q}{p^-}}+\left(\int_\Omega|x|^{p(x)}|\nabla(u_n-u_m)|^{p(x)}\,\mathrm{d}x\right)^{\frac{q}{p^+}}\right],\end{aligned}$$

where D_2 is a positive constant.

Combining the above pieces of information we find that there exists a positive constant M such that

$$\int_{B_\epsilon(0)}|u_n-u_m|^q\,\mathrm{d}x\le M\left||x|^{-\frac{q}{2}}\chi_{B_\epsilon(0)}\right|_{\left(\frac{p(x)}{q}\right)'}.$$

But using again relations (2.2) and (2.3) it is easy to see that

$$\left||x|^{-\frac{q}{2}}\chi_{B_\epsilon(0)}\right|_{\left(\frac{p(x)}{q}\right)'}\le\left[\rho_{\left(\frac{p(x)}{q}\right)'}\left(|x|^{-\frac{q}{2}}\chi_{B_\epsilon(0)}\right)\right]^{\left(\left(\frac{p(x)}{q}\right)'\right)^+}+\left[\rho_{\left(\frac{p(x)}{q}\right)'}\left(|x|^{-\frac{q}{2}}\chi_{B_\epsilon(0)}\right)\right]^{\left(\left(\frac{p(x)}{q}\right)'\right)^-},$$

where $\left(\frac{p(x)}{q}\right)'=\frac{p(x)}{p(x)-q}$, and assuming $\epsilon\in(0,1)$

$$\begin{aligned}\int_{B_\epsilon(0)}|x|^{\frac{-qp(x)}{2(p(x)-q)}}\,\mathrm{d}x&\le\int_{B_\epsilon(0)}|x|^{\frac{-qp^-}{2(p^--q)}}\,\mathrm{d}x\\&=\int_0^\epsilon\omega_N r^{N-1}r^{\frac{-qp^-}{2(p^--q)}}\,\mathrm{d}r\\&=\omega_N\frac{1}{\alpha}\epsilon^\alpha,\end{aligned}$$

where $\alpha=N-\frac{qp^-}{2(p^--q)}>0$ and ω_N is the area of the unit ball in $\mathbb{R}^N$.

Consequently,

$$\int_{B_\epsilon(0)} |u_n - u_m|^q \, \mathrm{d}x \leq M_1(\epsilon^{\alpha_1} + \epsilon^{\alpha_2}),$$

with $\alpha_1, \alpha_2 > 0$ and $M_1 > 0$ a constant.

The above inequality and relation (4.1) show that for any n and m large enough we have

$$\int_\Omega |u_n - u_m|^q \, \mathrm{d}x \leq M_2(\epsilon + \epsilon^{\alpha_1} + \epsilon^{\alpha_2}),$$

where M_2 is a positive constant. We infer that $\{u_n\}$ is a Cauchy sequence in $L^q(\Omega)$ and consequently $\mathcal{D}_0^{1,p(x)}(\Omega)$ is compactly embedded in $L^q(\Omega)$. The proof of Theorem 2 is complete. ■

Remark 3 The proof of Theorem 2 still holds true if we replace the space $L^q(\Omega)$ by $L^{q(x)}(\Omega)$, where $q : \overline{\Omega} \to (1, \infty)$ is a continuous function satisfying $1 < q^- \leq q^+ < \frac{2Np^-}{2N+p^-}$.

4.2. *Existence of solutions for a singular PDE involving variable exponent growth conditions*

Assume $q(x)$ is a function satisfying the hypothesis given in Remark 3. We investigate the existence of solutions of the problem

$$\begin{cases} -\mathrm{div}(|x|^{p(x)}|\nabla u(x)|^{p(x)-2}\nabla u(x)) = \lambda |u(x)|^{q(x)-2}u(x) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{4.2}$$

where λ is a positive constant.

We say that $u \in \mathcal{D}_0^{1,p(x)}(\Omega)$ is a weak solution of problem (4.2) if

$$\int_\Omega |x|^{p(x)}|\nabla u|^{p(x)-2}\nabla u \nabla v \, \mathrm{d}x - \lambda \int_\Omega |u|^{q(x)-2}uv \, \mathrm{d}x = 0 \quad \forall v \in \mathcal{D}_0^{1,p(x)}(\Omega).$$

We show the following existence result on problem (4.2):

THEOREM 3 *For each* $\lambda > 0$ *problem* (4.2) *has a nontrivial weak solution.*

Proof of Theorem 3 In order to prove Theorem 3 we define, for each $\lambda > 0$, the energetic functional associated with problem (4.2), $J_\lambda : \mathcal{D}_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$J_\lambda(u) = \int_\Omega \frac{|x|^{p(x)}}{p(x)} |\nabla u|^{p(x)} \, \mathrm{d}x - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, \mathrm{d}x,$$

for each $u \in \mathcal{D}_0^{1,p(x)}(\Omega)$. Standard arguments (see, e.g. [21]) show that $J_\lambda \in C^1(\mathcal{D}_0^{1,p(x)}(\Omega), \mathbb{R})$ and its derivative is given by

$$\langle J'_\lambda(u), v \rangle = \int_\Omega |x|^{p(x)}|\nabla u|^{p(x)-2}\nabla u \nabla v \, \mathrm{d}x - \lambda \int_\Omega |u|^{q(x)-2}uv \, \mathrm{d}x,$$

for all $u, v \in \mathcal{D}_0^{1,p(x)}(\Omega)$. We infer that u is a solution of problem (4.2) if and only if it is a critical point of J_λ. Consequently, we concentrate our efforts on finding critical points for J_λ. In this context we prove the following assertions:

(a) The functional J_λ is weakly lower semi-continuous.
(b) The functional J_λ is bounded from below and coercive.
(c) There exists $\psi \in \mathcal{D}_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that $J_\lambda(\psi) < 0$.

The arguments to prove (a), (b) and (c) are detailed below.

(a) Similar arguments as in the proof of [21, Proposition 3.6 (ii)] can be used in order to obtain the fact that J_λ is weakly lower semi-continuous.

(b) It is obvious that for any $u \in \mathcal{D}_0^{1,p(x)}(\Omega)$ we have

$$J_\lambda(u) \geq \frac{1}{p^+}\int_\Omega |x|^{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\,\mathrm{d}x$$
$$\geq \frac{1}{p^+}\int_\Omega |x|^{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x - \frac{\lambda}{q^-}\left(|u|^{q^-}_{q(x)} + |u|^{q^+}_{q(x)}\right).$$

If $\|u\| > 1$ the above inequality and Theorem 2 imply that there exists a positive constant K such that

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{K\lambda}{q^-}(\|u\|^{q^-} + \|u\|^{q^+}).$$

Taking into account that $1 < q^- \leq q^+ < \frac{2Np^-}{2N+p^-} < p^-$ the above inequality shows that $\lim_{\|u\|\to\infty} J_\lambda(u) = \infty$, that is J_λ is coercive.

On the other hand, it is clear that for any $u \in \mathcal{D}_0^{1,p(x)}(\Omega)$ we have

$$J_\lambda(u) \geq \frac{1}{p^+}\min\{\|u\|^{p^-}, \|u\|^{p^+}\} - \frac{K\lambda}{q^-}(\|u\|^{q^-} + \|u\|^{q^+}),$$

and thus, we deduce that J_λ is bounded from below.

(c) We fix $\varphi \in C_c^1(\Omega)$, $\varphi \neq 0$. Then for each $t \in (0, 1)$ we have

$$J_\lambda(t\varphi) = \int_\Omega \frac{|x|^{p(x)}t^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)}\,\mathrm{d}x - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\varphi|^{q(x)}\,\mathrm{d}x$$
$$\leq t^{p^-}\int_\Omega \frac{|x|^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)}\,\mathrm{d}x - \lambda t^{q^+}\int_\Omega \frac{1}{q(x)}|\varphi|^{q(x)}\,\mathrm{d}x.$$

Thus, there exist L_1 and L_2 two positive constants such that for each $t \in (0,1)$ we have

$$J_\lambda(t\varphi) \leq L_1 t^{p^-} - L_2 t^{q^+}.$$

Taking into account that $q^+ < p^-$, by the above inequality we infer that for any $t \in (0, \min\{1, (\frac{L_2}{L_1})^{\frac{1}{p^- - q^+}}\})$ we have

$$J_\lambda(t\varphi) < 0.$$

Next, we deduce by (a) and (b) that J_λ is weakly lower semi-continuous, bounded from below and coercive. These facts in relation to [22, Theorem 1.2] show that there exists $u_\lambda \in \mathcal{D}_0^{1,p(x)}(\Omega)$ a global minimum point of J_λ. Moreover, since (c) holds true it follows that $u_\lambda \neq 0$. Standard arguments based on Theorem 2 show that u_λ is actually a critical point of J_λ and thus, a nontrivial weak solution of problem (4.2). The proof of Theorem 3 is complete. ■

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