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Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian

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Abstract

In this paper, we investigate the multiplicity of solutions for a p-Kirchhoff system driven by a nonlocal integro-differential operator with zero Dirichlet boundary data. As a special case, we consider the following fractional p-Kirchhoff system

$$\begin{cases} \left(\sum_{i=1}^{k}[u_i]_{s,p}^{p}\right)^{\theta-1}(-\Delta)_p^s u_j(x) = \lambda_j|u_j|^{q-2}u_j + \sum_{i\neq j}\beta_{ij}|u_i|^m|u_j|^{m-2}u_j & \text{in } \Omega, \\ u_j = 0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where $[u_j]_{s,p} = \left(\iint_{\mathbb{R}^{2N}}\frac{|u_j(x)-u_j(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{1/p}$, $j = 1, 2, \ldots, k$, $k \geqslant 2$, $\theta \geqslant 1$, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $N > ps$ with $s \in (0, 1)$, $(-\Delta)_p^s$ is the fractional p-Laplacian, $\lambda_j > 0$ and $\beta_{ij} = \beta_{ji}$ for $i \neq j$, $j = 1, 2, \cdots, k$. When $1 < q < \theta p < 2m < p_s^*$ and $\beta_{ij} > 0$ for all $1 \leqslant i < j \leqslant k$, two distinct solutions are obtained by using the Nehari manifold method. When $1 < \theta p < 2m \leqslant q < p_s^*$ and $\beta_{ij} \in \mathbb{R}$ for all $1 \leqslant i < j \leqslant k$ or $1 < \theta p < q < 2m < p_s^*$ and $\beta_{ij} > 0$ for all $1 \leqslant i < j \leqslant k$, the existence of infinitely many solutions is obtained by applying the symmetric mountain pass theorem. To our best knowledge, our results for the above system are new in the study of Kirchhoff problems.

Keywords: fractional p-Kirchhoff system, multiple solutions, Nehari manifold, mountain pass theorem
Mathematics Subject Classification numbers: 35R11, 35A15, 47G20

1. Introduction and main results

In this paper we study nonlinear fractional p-Kirchhoff type systems of mixed couplings. More precisely, we consider the following k equations:

$$\begin{cases} \left(\sum_{i=1}^{k}[u_i]_{s,p}^{p}\right)^{\theta-1} \mathcal{L}_K u_j(x) = \lambda_j |u_j|^{q-2}u_j + \sum_{i\neq j}\beta_{ij}|u_i|^m|u_j|^{m-2}u_j & \text{in } \Omega, \\ u_j = 0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.1}$$

where $[u_j]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} |u_j(x)-u_j(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y\right)^{1/p}$, $j=1,2,\ldots,k, k\geqslant 2$, $\theta\geqslant 1$, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipshcitz boundary $\partial\Omega$, $N>ps$ with $s\in(0,1)$, $\lambda_j>0$ and $\beta_{ij}=\beta_{ji}$ for $i\neq j$, $j=1,2,\cdots,k$, and $\mathcal{L}_K$ is a nonlocal integro-differential operator defined as

$$\mathcal{L}_K\varphi(x) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))K(x-y)\,\mathrm{d}y \quad \text{for } x\in\mathbb{R}^N,$$

along any $\varphi\in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ of radius ε centered at x, and the singular kernel $K:\mathbb{R}^N\setminus\{0\}\to\mathbb{R}^+$ is a measurable function with the following property

$$\begin{cases} \gamma K\in L^1(\mathbb{R}^N), \quad \text{where } \gamma(x)=\min\{|x|^p,1\}; \\ \text{there exists } K_0>0 \text{ such that } K(x)\geqslant K_0\,|x|^{-(N+ps)} \text{ for any } x\in\mathbb{R}^N\setminus\{0\}. \end{cases} \tag{1.2}$$

Especially, when $K(x)=|x|^{-(N+ps)}$ for $x\in\mathbb{R}^N\setminus\{0\}$, $\mathcal{L}_K$ reduces to the following fractional p-Laplace operator

$$(-\Delta)_p^s\varphi(x) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,\mathrm{d}y,$$

see [12, 31] and the references therein for further details on the fractional p-Laplacian. In this case, if $s\to 1^-$, $p=2$ and $\theta=1$, then problem (1.1) becomes

$$\begin{cases} -\Delta u_j = \lambda_j|u_j|^{q-2}u_j + \sum_{i\neq j}\beta_{ij}|u_i|^m|u_j|^{m-2}u_j & \text{in } \Omega, \\ u_j = 0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.3}$$

where $j=1,2,\ldots,k, k\geqslant 2$, see [12, proposition 4.4]. For $k=2$, system (1.2), also known as the Gross–Pitaevskii system, has applications in many physical models such as in nonlinear optics (for example, the beam in Kerr-like photorefractive media) and in Bose–Einstein condensates for multi-species condensates, see [9, 13] and the references therein. In the latter case, λ_j and β_{ij} are physically the intraspecies and interspecies scattering lengths respectively. The sign of β_{ij} determines whether the interactions of states are repulsive or attractive, i.e. the interaction is attractive if $\beta_{ij} > 0$, and repulsive if $\beta_{ij} < 0$; the two states are in strong competition when β_{ij} is negative and very large. These phenomena have been documented in experiments as well as in numerical simulations. For more details about their applications and some recent results, we refer to [9, 11, 17, 32] and the references therein.

In [14], *Fiscella* and *Valdinoci* proposed a stationary Kirchhoff variational model, in bounded regular domains of $\mathbb{R}^N$, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. In fact, problem (1.1) is a fractional version of a model, the so-called Kirchhoff problems, introduced by *Kirchhoff*. To be more precise, Kirchhoff established a model given by the following:

$$\rho\frac{\partial^2 u}{\partial t^2} - M(\|\nabla u\|^2_{L^2([0,L])})\frac{\partial^2 u}{\partial x^2} = f(x,u), \tag{1.4}$$

where $M(\xi) = p_0/h + (E/(2L))\xi$ with $\xi \geqslant 0$, ρ, p_0, h, E, L are constants, which extends the classical D'Alambert wave equation by considering the effects of the changes in the length of the strings during the vibrations. When M is this type, problem (1.4) is said to be non-degenerate if $M(0) > 0$, while it is called degenerate if $M(0) = 0$. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model. Obviously, a novel feature of our results is to cover the degenerate case in the fractional p-Laplacian setting. Here we would like to mention some recent results, for example, see [22, 24, 29] for the non-degenerate case and [10, 21, 23, 30] for the degenerate case.

Recently, great attention has been focused on the study of problems involving fractional Laplacian operators, more generally, nonlocal operators. This type of operator arises in a quite natural way in many different applications, such as finance, physics, fluid dynamics, population dynamics, image processing, minimal surfaces and game theory. In particular, nonlocal integro-differential operators for problem (1.1) arise naturally in the study of stochastic processes with jumps, and more precisely in Lévy processes. In this case, the fractional Laplacian operator can be viewed as the infinitesimal generator of radially symmetric stable processes in Lévy processes, see [1, 5, 16, 25] and the references therein for more details. Indeed, the literature on non-local fractional Laplacian operators and their application to differential equations is quite large—we refer the interested reader to [2, 4, 6], [18–20, 26–28] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the reader to [12] for a short introduction.

In [7], *Chen* and *Deng* investigated the existence of two nontrivial solutions to the fractional p-Laplacian system involving concave–convex nonlinearities via the Nehari method. However, they just considered the sublinear case $1 < q < p$. Also, we refer the readers to [8, 15] for more applications by using the Nehari method. By applying the symmetric mountain pass lemma, *Chen* in [9] obtained the existence of infinitely many nonnegative solutions for a class of the quasilinear Schrödinger system in $\mathbb{R}^N$ in the Laplacian setting. In that paper, the authors considered the superlinear case $p < q$. Inspired by the above papers, we will study the multiplicity of solutions for problem (1.1) from both the sublinear and superlinear cases, in turn, in the Kirchhoff setting. The main difficulties when dealing with this problem come from the degeneracy of the Kirchhoff function and the nonlocal nature of the fractional Laplacian.

To our best knowledge, there is no result on fractional p-Laplacian systems in the Kirchhoff context.

To give our main result precisely, we introduce first some notation. In the following, we denote $\mathcal{Q} = \mathbb{R}^{2N} \setminus \mathcal{O}$, where

$$\mathcal{O} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N},$$

and $\mathcal{C}(\Omega) = \mathbb{R}^N \setminus \Omega$. Let $W_j = W_0$ be the fractional Sobolev space with the norm

$$\|u\|_{W_j} = \left(\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x-y) \mathrm{d}x\mathrm{d}y \right)^{1/p}.$$

By the fractional Sobolev inequality, we have for $1 \leqslant \mu \leqslant p_s^*$ that

$$|u|_\mu := \|u\|_{L^\mu(\mathbb{R}^N)} \leqslant C_* \|u\|_{W_j}, \quad \forall\, u \in W_j \tag{1.5}$$

with some $C_* > 0$. For the product space $W = W_1 \times \cdots \times W_k$, we introduce the norm

$$\|\mathbf{u}\|_W = (\|u_1\|_{W_1}^p + \cdots + \|u_k\|_{W_k}^p)^{1/p}, \quad \forall\, \mathbf{u} = (u_1, \cdots, u_k) \in W.$$

Then $(W, \|\cdot\|_W)$ is a reflexive Banach space, see [29, lemma 2.4].

From now on we write $\|u\|_{L^q(\Omega)} := |u|_q$ $(1 \leqslant q \leqslant \infty)$. Let $I(\mathbf{u}) : W \to \mathbb{R}$ be the corresponding functional of system (1.1), which is defined by

$$I(\mathbf{u}) = \frac{1}{\theta p} \|\mathbf{u}\|_W^{\theta p} - \frac{1}{q} \sum_{j=1}^k \lambda_j |u_j|_q^q - \frac{1}{m} \sum_{j=1}^k \sum_{i<j} \beta_{ij} |u_i u_j|_m^m, \quad \forall\, \mathbf{u} \in W.$$

Then $I \in C^1(W, \mathbb{R})$ and for any $\mathbf{v} = (v_1, v_2, \cdots, v_k) \in W$,

$$\begin{aligned}\langle I'(\mathbf{u}), \mathbf{v} \rangle = \|\mathbf{u}\|_W^{(\theta-1)p} \sum_{j=1}^k \iint_{\mathcal{Q}} |u_j(x) - u_j(y)|^p (u_j(x) - u_j(y))(v_j(x) - v_j(y)) K(x-y) \mathrm{d}x\mathrm{d}y \\ - \sum_{j=1}^k \int_\Omega \lambda_j |u_j|^{q-2} u_j v_j \mathrm{d}x - \sum_{j=1}^k \sum_{i \neq j} \beta_{ij} \int_\Omega |u_i|^m |u_j|^{m-2} u_j v_j \mathrm{d}x.\end{aligned}$$

Now, we give the definition of (weak) solutions for problem (1.1).

Definition 1.1. We say that $\mathbf{u} = (u_1, u_2, \cdots, u_k) \in W \setminus \{0\}$ is a weak solution of problem (1.1), if

$$\begin{aligned}&\|\mathbf{u}\|_W^{(\theta-1)p} \sum_{j=1}^k \iint_{\mathcal{Q}} |u_j(x) - u_j(y)|^p (u_j(x) - u_j(y))(v_j(x) - v_j(y)) K(x-y) \mathrm{d}x\mathrm{d}y \\ &= \sum_{j=1}^k \int_\Omega \lambda_j |u_j|^{q-2} u_j v_j \mathrm{d}x + \sum_{j=1}^k \sum_{i \neq j} \beta_{ij} \int_\Omega |u_i|^m |u_j|^{m-2} u_j v_j \mathrm{d}x,\end{aligned}$$

for any $\mathbf{v} \in W$, with $\mathbf{v} = (v_1, v_2, \cdots, v_k)$.

Obviously, a weak solution of system (1.1) is equivalent to a critical point of the functional I.

For the sublinear case, we get the following main result:

Theorem 1.1. *Let $1 \leqslant \theta < N/(N - ps)$. If $1 < q < \theta p < 2m < p_s^*$, $\lambda_j > 0$ for all $j = 1, \cdots, k \geqslant 2$, and $\beta_{ij} = \beta_{ji} > 0$ for all $1 \leqslant i < j \leqslant n$, then there exists $\Lambda_1 = \Lambda_1(\theta, p, q, m, \beta_{ij}, C_*) > 0$ such that problem (1.1) has two distinct solutions in W, provided that $\left(\sum_{j=1}^{k} \lambda_j^{\theta p/(\theta p - q)}\right)^{(\theta p - q)/(\theta p)} \leqslant \Lambda_1$. In addition, if $1 < q < p$, then those two solutions have at most $k - 2$ zero components.*

Remark 1.1. If $k = 2$, then the two distinct solutions obtained by theorem 1.1 are not semi-trivial solutions.

For the superlinear case, we give the following assumptions:
(H_1) $\lambda_j > 0$ for all $j = 1, \cdots, k$;
(H_2) $\theta p < 2m \leqslant q < p_s^*$ and $\beta_{ij} = \beta_{ji}$ for all $1 \leqslant i < j \leqslant k$;
(H_3) $\theta p < q < 2m < p_s^*$ and $\beta_{ij} = \beta_{ji} > 0$ for all $1 \leqslant i < j \leqslant k$.

Theorem 1.2. *Let $1 \leqslant \theta < N/(N - ps)$. If (H_1)–(H_2) or (H_1), (H_3) hold, then problem (1.1) has infinitely many solutions in W.*

This paper is organized as follows. In section 2, by using the Nehari method, we establish the existence of two distinct solutions for problem (1.1). In section 3, the existence of infinitely many solutions for problem (1.1) is obtained by applying the symmetric mountain pass theorem.

2. Sublinear case

We consider the Nehari manifold

$$\mathcal{N} = \{\mathbf{u} \in W \setminus \{0\} : \langle I'(\mathbf{u}), \mathbf{u}\rangle = 0\}.$$

Then $\mathbf{u} \in \mathcal{N}$ if and only if

$$\|\mathbf{u}\|_W^{\theta p} - \sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q \, dx - 2\sum_{j=1}^{k}\sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^m \, dx = 0, \tag{2.1}$$

see [3] for some enlightening discussions on this aspect. Next we will prove the existence of solutions of system (1.1) by studying the existence of minimizers of functional I on $\mathcal{N}$. The Nehari manifold is related to the behavior of fibering maps of the form $H_{\mathbf{u}} : t \to I(t\mathbf{u})$ for $t > 0$ defined by

$$H_{\mathbf{u}}(t) = I(t\mathbf{u}) = \frac{t^{\theta p}}{\theta p}\|\mathbf{u}\|^{\theta p} - \frac{t^q}{q}\sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q \, dx - \frac{t^{2m}}{m}\sum_{j=1}^{k}\sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^m \, dx. \tag{2.2}$$

Lemma 2.1. *Let $\mathbf{u} \in W \setminus \{0\}$, then $t\mathbf{u} \in \mathcal{N}$ if and only if $H'_{\mathbf{u}}(t) = 0$.*

Proof. The result easily follows from the fact that $H'_{\mathbf{u}}(t) = \langle I'(t\mathbf{u}), \mathbf{u}\rangle$. □

By lemma 2.1, we know that the elements in $\mathcal{N}$ correspond to stationary points of the maps $H_{\mathbf{u}}$. Hence it is natural to split $\mathcal{N}$ into three parts corresponding to local minimal, local maxima and points of inflexion of fibering maps. A simple calculation yields

$$H'_{\mathbf{u}}(t) = t^{\theta p - 1}\|\mathbf{u}\|_W^{\theta p} - t^{q-1}\sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q \, dx - 2t^{2m-1}\sum_{j=1}^{k}\sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^m \, dx, \tag{2.3}$$

and

$$H''_{\mathbf{u}}(t) = (\theta p-1)t^{\theta p-2}\|\mathbf{u}\|_W^{\theta p} - (q-1)t^{q-2}\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q\,\mathrm{d}x$$
$$-2(2m-1)t^{2m-2}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^m\,\mathrm{d}x. \tag{2.4}$$

By lemma 2.1, $\mathbf{u}\in\mathcal{N}$ if and only if $H'_{\mathbf{u}}(1)=0$. Thus for $\mathbf{u}\in\mathcal{N}$ we obtain by (2.3) and (2.4)

$$\begin{aligned}
H''_{\mathbf{u}}(1) &= (\theta p-1)\|\mathbf{u}\|_W^{\theta p} - (q-1)\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q\,\mathrm{d}x - 2(2m-1)\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^m\,\mathrm{d}x\\
&= (\theta p-q)\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q\,\mathrm{d}x + 2(\theta p-2m)\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^m\,\mathrm{d}x\\
&= (\theta p-q)\|\mathbf{u}\|_W^{\theta p} - 2(2m-q)\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^m\,\mathrm{d}x\\
&= (\theta p-2m)\|\mathbf{u}\|_W^{\theta p} + (2m-q)\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q\,\mathrm{d}x.
\end{aligned} \tag{2.5}$$

Hence we split $\mathcal{N}$ into the following three parts:

$$\mathcal{N}^+ = \{\mathbf{u}\in\mathcal{N} : H''_{\mathbf{u}}(1)>0\};$$
$$\mathcal{N}^0 = \{\mathbf{u}\in\mathcal{N} : H''_{\mathbf{u}}(1)=0\};$$
$$\mathcal{N}^- = \{\mathbf{u}\in\mathcal{N} : H''_{\mathbf{u}}(1)<0\}.$$

The following lemma shows that local minimizers on Nehari manifold $\mathcal{N}$ are usually critical points of I.

Lemma 2.2. *Assume $\mathbf{u}_0$ is a local minimizer of I on $\mathcal{N}$ and $\mathbf{u}_0\notin\mathcal{N}^0$. Then $\mathbf{u}_0$ is a critical point of I.*

Proof. This lemma is proved by following the same discussion as in [3, theorem 2.3]. □

Set

$$J_{\mathbf{u}}(t) = t^{\theta p-2m}\|\mathbf{u}\|_W^{\theta p} - t^{q-2m}\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q\,\mathrm{d}x.$$

Then we obtain

Lemma 2.3. *Assume $\mathbf{u}\in W\setminus\{0\}$. Then the function $J_{\mathbf{u}}$ satisfies the following properties:*

(1) *$J_{\mathbf{u}}(t)$ has a unique critical point at $t=t_{\max}(\mathbf{u}) = \left(\frac{(2m-q)\sum_{j=1}^{k}\lambda_j\,|\,u_j|_q^q}{(2m-\theta p)\,\|\,u\,\|_W^{\theta p}}\right)^{1/(\theta p-q)} > 0$;*

(2) *$J_{\mathbf{u}}(t)$ is strictly increasing on $(0, t_{\max}(\mathbf{u}))$ and strictly decreasing on $(t_{\max}(\mathbf{u}), +\infty)$;*

(3) *$\lim_{t\to 0^+} J_{\mathbf{u}}(t) = -\infty$.*

Proof. It is easy to see that

$$J'_{\mathbf{u}}(t) = (\theta p - 2m)t^{\theta p-2m-1}\|\mathbf{u}\|_W^{\theta p} - (q-2m)t^{q-2m-1}\sum_{j=1}^{k}\lambda_j|u_j|_q^q.$$

Set $J'_{\mathbf{u}}(t) = 0$, then there exists a unique $t_{\max} \in (0, +\infty)$ such that $J'_{\mathbf{u}}(t_{\max}) = 0$ and $J''_u(t_{\max}) < 0$, with

$$t_{\max} = \left(\frac{(2m-q)\sum_{j=1}^{k}\lambda_j|u_j|_q^q}{(2m-\theta p)\|\mathbf{u}\|^{\theta p}}\right)^{1/(\theta p - q)}.$$

Moreover, we have $J'_u(t) > 0$ for $t \in (0, t_{\max})$ and $J'_u(t) < 0$ for $t \in (t_{\max}, +\infty)$. Assertion (3) follows from the fact that $q < \theta p$. □

Lemma 2.4. *Let $t > 0$. Then $t\mathbf{u} \in \mathcal{N}^+$(or $\mathcal{N}^-$) if and only if $J'_{\mathbf{u}}(t) > 0$ (or < 0).*

Proof. For $t > 0$, it is easy to see that $t\mathbf{u} \in \mathcal{N}$ if and only if

$$J_{\mathbf{u}}(t) = 2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_i u_j|^m\,\mathrm{d}x. \tag{2.6}$$

Moreover, if $t\mathbf{u} \in \mathcal{N}$, then

$$t^{2m-1}J'_{\mathbf{u}}(t) = H''_{\mathbf{u}}(t), \tag{2.7}$$

which implies that $t\mathbf{u} \in \mathcal{N}^+$ (or $\mathcal{N}^-$) if and only if $J'_{\mathbf{u}}(t) > 0$ (or $J'_{\mathbf{u}}(t) < 0$). □

Set

$$\Lambda_0 = \frac{2m-\theta p}{2m-q}\left(\frac{\theta p - q}{2m-q}\right)^{\frac{\theta p - q}{2m-\theta p}}\left(2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\right)^{-\frac{\theta p - q}{2m-\theta p}} C_*^{-\frac{\theta p(2m-q)}{2m-\theta p}}$$

and

$$\Theta_{\Lambda_0} = \left\{(\lambda_1, \lambda_2, \cdots, \lambda_k) \in (\mathbb{R}^+)^k : 0 < \left(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}} < \Lambda_0\right\}.$$

Lemma 2.5. *Assume $\mathbf{u} \in W \setminus \{0\}$. Then for all $(\lambda_1, \cdots, \lambda_k) \in \Lambda_0$ there exist $t^+, t^- > 0$ such that $t^+ < t_{\max} < t^-$, $t^\pm\mathbf{u} \in \mathcal{N}^\pm$ and*

$$I(t^+\mathbf{u}) = \inf_{0\leqslant t\leqslant t_{\max}} I(t\mathbf{u}), \quad I(t^-\mathbf{u}) = \sup_{t\geqslant 0} I(t\mathbf{u}).$$

Proof. Fix $\mathbf{u} \in W$ with $\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_i u_j|^m\,\mathrm{d}x > 0$. By lemma 2.3, one has $\lim_{t\to 0^+}J_{\mathbf{u}}(t) = -\infty$, $\lim_{t\to+\infty}J_{\mathbf{u}}(t) = 0$, and there exists a unique $t_{\max} > 0$ such that $J_{\mathbf{u}}(t)$ achieves its maximum at $t_{\max}$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, +\infty)$. Obviously, $t\mathbf{u} \in \mathcal{N}^+$ (or $\mathcal{N}^-$) if and only if $J'_{\mathbf{u}}(t) > 0$ or (<0). Moreover, by the Hölder inequality, we have

$$
\begin{aligned}
J_{\mathbf{u}}(t_{\max}) &= \frac{\theta p - q}{2m - \theta p}\left(\frac{2m - q}{2m - \theta p}\right)^{\frac{q-2m}{\theta p - q}} \frac{\left(\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q\,\mathrm{d}x\right)^{(\theta p-2m)/(\theta p-q)}}{(\|\mathbf{u}\|_W^{\theta p})^{(q-2m)/(\theta p-q)}} \\
&\geqslant \frac{\theta p - q}{2m - \theta p}\left(\frac{2m - q}{2m - \theta p}\right)^{\frac{q-2m}{\theta p - q}} \frac{\left(\sum_{j=1}^{k}\lambda_j C_*^q\|u_j\|_{W_0}^q\right)^{(\theta p-2m)/(\theta p-q)}}{(\|\mathbf{u}\|_W^{\theta p})^{(q-2m)/(\theta p-q)}} \\
&\geqslant \frac{\theta p - q}{2m - \theta p}\left(\frac{2m - q}{2m - \theta p}\right)^{\frac{q-2m}{\theta p - q}} \frac{\left(C_*^q(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p-q}})^{\frac{\theta p-q}{\theta p}}(\sum_{j=1}^{k}\|u_j\|_{W_j}^{\theta p})^{\frac{q}{\theta p}}\right)^{(\theta p-2m)/(\theta p-q)}}{(\|\mathbf{u}\|_W^{\theta p})^{(q-2m)/(\theta p-q)}} \\
&\geqslant \frac{\theta p - q}{2m - \theta p}\left(\frac{2m - q}{2m - \theta p}\right)^{\frac{q-2m}{\theta p - q}} \frac{\left(C_*^q(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p-q}})^{\frac{\theta p-q}{\theta p}}\right)^{(\theta p-2m)/(\theta p-q)}\|\mathbf{u}\|_{W_0}^{\frac{q(\theta p-2m)}{\theta p-q}}}{(\|\mathbf{u}\|_W^{\theta p})^{(q-2m)/(\theta p-q)}} \\
&= \frac{\theta p - q}{2m - \theta p}\left(\frac{2m - q}{2m - \theta p}\right)^{\frac{q-2m}{\theta p - q}} \frac{\|\mathbf{u}\|_W^{2m}}{\left(C_*^q(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p-q}})^{\frac{\theta p-q}{\theta p}}\right)^{(2m-\theta p)/(\theta p-q)}}.
\end{aligned}
$$

For $(\lambda_1, \lambda_2, \cdots, \lambda_k) \in \Theta_{\Lambda_0}$, we have by the Hölder inequality

$$
\begin{aligned}
0 < 2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^m\,\mathrm{d}x &\leqslant 2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^m\,\mathrm{d}x \\
&\leqslant 2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}C_*^{2m}\|\mathbf{u}\|_W^{2m} \\
&< J_{\mathbf{u}}(t_{\max}).
\end{aligned}
$$

Hence there exist $t^+, t^- > 0$ such that $t^+ < t_{\max} < t^-$,

$$
J_{\mathbf{u}}(t^+) = J_{\mathbf{u}}(t^-) = 2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^m\,\mathrm{d}x,
$$

and $J'_{\mathbf{u}}(t^+) > 0, J'_{\mathbf{u}}(t^-) < 0$. Obviously, $H'_{\mathbf{u}}(t^+) = H'_{\mathbf{u}}(t^-) = 0$. Moreover, by $H''_{\mathbf{u}}(t) = t^{2m-1}J'_{\mathbf{u}}(t)$, we have

$$
H''_{\mathbf{u}}(t^+) > 0, \quad H''_{\mathbf{u}}(t^-) < 0.
$$

Thus $H_{\mathbf{u}}(t)$ has a local minimum at t^+ and a local maximum at t^- such that $t^+\mathbf{u} \in \mathcal{N}^+$ and $t^-\mathbf{u} \in \mathcal{N}^-$. Since $I(t\mathbf{u}) = H_{\mathbf{u}}(t)$, we obtain

$$
I(t^-\mathbf{u}) \geqslant I(t\mathbf{u}) \geqslant I(t^+\mathbf{u}), \ \forall\, t \in [t^+, t^-]
$$

and

$$I(t^+\mathbf{u}) \leqslant I(t\mathbf{u}), \ \forall\, t \in [0, t_{\max}].$$

Hence

$$I(t^+\mathbf{u}) = \inf_{0\leqslant t\leqslant t_{\max}} I(t\mathbf{u}), \ \ I(t^-\mathbf{u}) = \sup_{t\geqslant 0} I(t\mathbf{u}).$$

The proof is thus complete. □

Lemma 2.6. *For any* $(\lambda_1, \lambda_2, \cdots, \lambda_k) \in \Theta_{\lambda_0}$, *we have* $\mathcal{N}^0 = \varnothing$.

Proof. Arguing by contradiction, we assume there exist positive constants $\lambda_1, \cdots, \lambda_k$ with

$$0 < \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}} < \Lambda_0,$$

such that $\mathcal{N}^0 \neq \varnothing$. Then for $\mathbf{u} \in \mathcal{N}^0$, we have by (2.5) that $H_u''(1) = 0$ and

$$\begin{aligned} 0 = \langle I'(\mathbf{u}), \mathbf{u}\rangle &= (\theta p - q)\|\mathbf{u}\|_W^{\theta p} - 2(2m-q)\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_i u_j|^m\,\mathrm{d}x \\ &= (\theta p - 2m)\|\mathbf{u}\|_W^{\theta p} + (2m-q)\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q\,\mathrm{d}x. \end{aligned}$$

Hence, by the Hölder inequality and fractional Sobolev inequality (1.5), we obtain

$$\begin{aligned} \|\mathbf{u}\|_W^{\theta p} &\leqslant \frac{2m-q}{2m-\theta p} C_*^q \sum_{j=1}^{k}\lambda_j \|u_j\|_{W_j}^q \\ &\leqslant \frac{2m-q}{2m-\theta p} C_*^q \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}} \left(\sum_{j=1}^{k}\|u_j\|_{W_j}^{\theta p}\right)^{\frac{q}{\theta p}} \\ &\leqslant \frac{2m-q}{2m-\theta p} C_*^q \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}} \|\mathbf{u}\|_W^q. \end{aligned}$$

It follows from $q < \theta p$ that

$$\|\mathbf{u}\|_W \leqslant \left[\frac{2m-q}{2m-\theta p} C_*^q \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\right]^{\frac{1}{\theta p - q}}. \tag{2.8}$$

Moreover, by the Hölder inequality and (1.5), we have

$$\begin{aligned}\|\mathbf{u}\|_W^{\theta p} &= \frac{2(2m-q)}{\theta p - q}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_i u_j|^m \,\mathrm{d}x \\ &\leqslant \frac{2(2m-q)}{\theta p - q}C_*^{2m}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\|\mathbf{u}\|_W^{2m}.\end{aligned}$$

Thus,

$$\|\mathbf{u}\|_W \geqslant \left(\frac{\theta p - q}{2(2m-q)C_*^{2m}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}}\right)^{\frac{1}{2m-\theta p}}. \tag{2.9}$$

Combining (2.8) with (2.9), we get

$$\left(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}} \geqslant \frac{2m-\theta p}{2m-q}\left(\frac{\theta p - q}{2m-q}\right)^{\frac{\theta p - q}{2m-\theta p}}\left[\frac{1}{2C_*^{2m}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}}\right]^{\frac{\theta p - q}{2m-\theta p}} C_*^{-q} = \Lambda_0,$$

which is a contradiction. □

Lemma 2.7. *The functional I is coercive and bounded from below on $\mathcal{N}$.*

Proof. For any $\mathbf{u}\in\mathcal{N}$, by (2.1), the Hölder inequality and fractional Sobolev inequality (1.5), we have

$$\begin{aligned}I(\mathbf{u}) &= \left(\frac{1}{\theta p} - \frac{1}{2m}\right)\|\mathbf{u}\|_W^{\theta p} - \left(\frac{1}{q} - \frac{1}{2m}\right)\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{\theta p} - \frac{1}{2m}\right)\|\mathbf{u}\|_W^{\theta p} - \left(\frac{1}{q} - \frac{1}{2m}\right)C_*^q\left(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\|\mathbf{u}\|_W^q,\end{aligned}$$

this together with $q<\theta p$ implies that I is coercive and bounded from below on $\mathcal{N}$. □

By lemmas 2.6 and 2.7, for any $(\lambda_1,\cdots,\lambda_k)\in\Theta_{\Lambda_0}$, we obtain $\mathcal{N}=\mathcal{N}^+\cup\mathcal{N}^-$ and I is coercive and bounded from below on $\mathcal{N}^+$ and $\mathcal{N}^-$. Define

$$c = \inf_{\mathbf{u}\in\mathcal{N}} I(\mathbf{u}),\quad c^+ = \inf_{\mathbf{u}\in\mathcal{N}^+} I(\mathbf{u}),\quad c^- = \inf_{\mathbf{u}\in\mathcal{N}^-} I(\mathbf{u}),$$

and set

$$\Theta_{\Lambda_1} = \left\{(\lambda_1,\cdots,\lambda_k)\in(\mathbb{R}^+)^k : \left(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}} < \Lambda_1\right\},$$

where $\Lambda_1 = \frac{q}{\theta p}\Lambda_0 < \Lambda_0$. Obviously, $\Theta_{\Lambda_1}\subset\Theta_{\Lambda_0}$. Then we have the following lemma.

Lemma 2.8. *If $(\lambda_1,\cdots,\lambda_k)\in\Theta_{\Lambda_1}$, then*

(1) $c\leqslant c^+<0$;
(2) *there exists* $\mathcal{C}_0=\mathcal{C}_0(p,q,\theta,N,\lambda_1,\cdots,\lambda_k,\beta_{ij})>0$ *such that* $c^->\mathcal{C}_0$.

Proof. For $\mathbf{u} \in \mathcal{N}^+$, we have $H''_{\mathbf{u}}(1) > 0$. Hence, by (2.5), we get

$$(\theta p - q)\|\mathbf{u}\|_W^{\theta p} > 2(2m - q) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^m \, dx.$$

This implies that

$$\begin{aligned} I(\mathbf{u}) &= \left(\frac{1}{\theta p} - \frac{1}{q}\right)\|\mathbf{u}\|_W^{\theta p} + 2\left(\frac{1}{q} - \frac{1}{2m}\right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^m \, dx \\ &\leqslant \left[\left(\frac{1}{\theta p} - \frac{1}{q}\right) + \left(\frac{1}{q} - \frac{1}{2m}\right)\frac{\theta p - q}{2m - q}\right]\|\mathbf{u}\|_W^{\theta p} \\ &= \frac{\theta p - q}{q}\left(\frac{\theta p - 2m}{2m\theta p}\right)\|\mathbf{u}\|_W^{\theta p} \\ &< 0, \end{aligned}$$

thanks to $q < \theta p < 2m$. Hence $c \leqslant c^+ < 0$.

(2) For $\mathbf{u} \in \mathcal{N}^-$, we have $H''_{\mathbf{u}}(1) < 0$, by (2.5) we obtain

$$(\theta p - q)\|\mathbf{u}\|_W^{\theta p} < 2(2m - q) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^m \, dx.$$

By the Hölder inequality and (1.5), we get

$$\sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^m \, dx \leqslant C_*^{2m} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \|\mathbf{u}\|_W^{2m}.$$

Hence,

$$\|\mathbf{u}\|_W > \left[\frac{1}{C_*^{2m} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij}} \frac{\theta p - q}{2(2m - q)}\right]^{\frac{1}{2m - \theta p}}. \tag{2.10}$$

Therefore, we deduce

$$\begin{aligned} I(\mathbf{u}) &\geqslant \|\mathbf{u}\|_W^q \left[\left(\frac{1}{\theta p} - \frac{1}{2m}\right)\|\mathbf{u}\|_W^{\theta p - q} - \left(\frac{1}{q} - \frac{1}{2m}\right) C_*^q \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\right] \\ &\geqslant \|\mathbf{u}\|_W^q \left\{\left(\frac{1}{\theta p} - \frac{1}{2m}\right)\left(\frac{\theta p - q}{2m - q}\right)^{(\theta p - q)/(2m - \theta p)} \left[\frac{1}{2C_*^{2m} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij}}\right]^{\frac{\theta p - q}{2m - \theta p}} \right. \\ &\quad \left. - \left(\frac{1}{q} - \frac{1}{2m}\right) C_*^q \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\right\} \\ &\geqslant \mathcal{C}_0 > 0, \end{aligned}$$

due to (2.10) and $(\lambda_1, \cdots, \lambda_k) \in \Theta_{\Lambda_1}$. □

Lemma 2.9. *If* $(\lambda_1, \cdots, \lambda_k) \in \Theta_{\Lambda_1}$, *then* I *has a minimizer* $\mathbf{u}_1 \in \mathcal{N}^+$ *and satisfies*

(i) $I(\mathbf{u}_1) = c = c^+ < 0$;
(ii) $\mathbf{u}_1$ *is a solution of problem* (1.1).

Proof. By lemma 2.7, we know I is bounded from below on $\mathcal{N}$. Thus, there exists a minimizing sequence $\{\mathbf{u}_n\}$, bounded in W, so that up to a subsequence, there exists $\mathbf{u}_1 \in W$ such that

$$\mathbf{u}_n \rightharpoonup \mathbf{u}_1 \text{ in } W, \quad \text{as } n \to \infty. \tag{2.11}$$

By [29, lemma 2.3], as $n \to \infty$,

$$\mathbf{u}_n \to \mathbf{u}_1 \text{ in } L^{\nu}(\Omega) \text{ and a.e. in } \Omega, \tag{2.12}$$

for all $\nu \in [1, p_s^*)$. For any $1 \leqslant j \leqslant k$, we have by the Hölder inequality and $\mathbf{u}_n \to \mathbf{u}_1$ in $L^q(\Omega)$ that

$$\begin{aligned}\int_\Omega \Big| |(\mathbf{u}_n)_j|^q - |(\mathbf{u}_n)_j|^q \Big| \mathrm{d}x &= q \int_\Omega (|(\mathbf{u}_1)_j| + \tau(|(\mathbf{u}_n)_j - (u_1)_j|))^{q-1} |(\mathbf{u}_n)_j - (u_1)_j| \mathrm{d}x \\ &\leqslant q |(\mathbf{u}_n)_j + (\mathbf{u}_1)_j|_q^{q-1} |(\mathbf{u}_n)_j - (\mathbf{u}_1)_j|_q \\ &\leqslant C |(\mathbf{u}_n)_j - (\mathbf{u}_1)_j|_q \to 0,\end{aligned}$$

as $n \to \infty$, where $\tau \in (0, 1)$ and $C > 0$ denotes various constants. Hence,

$$\lim_{n\to\infty} \int_\Omega \Big| |(\mathbf{u}_n)_j|^q - |(\mathbf{u}_1)_j|^q \Big| \mathrm{d}x = 0 \quad \forall j \in \{1, 2, \cdots, k\},$$

which yields

$$\sum_{j=1}^k \lambda_j \int_\Omega |(\mathbf{u}_n)_j|^q \, \mathrm{d}x \to \sum_{j=1}^k \lambda_j \int_\Omega |(\mathbf{u}_1)_j|^q \, \mathrm{d}x \quad \text{as } n \to \infty.$$

Moreover, by the Hölder inequality. we have for any $1 \leqslant i, j \leqslant k$

$$\begin{aligned}&\int_\Omega \Big| |(\mathbf{u}_n)_i|^m |(\mathbf{u}_n)_j|^m - |(\mathbf{u}_1)_i (\mathbf{u}_1)_j|^m \Big| \mathrm{d}x \\ &\leqslant C \int_\Omega (|(\mathbf{u}_n)_i|^{2m-1} + (\mathbf{u}_1)_i|^{2m-1} + |(\mathbf{u}_n)_j|^{2m-1} + |(\mathbf{u}_1)_j|^{2m-1})(|(\mathbf{u}_n)_i - (\mathbf{u}_1)_i| + |(\mathbf{u}_n)_j - (\mathbf{u}_1)_j|) \mathrm{d}x \\ &\leqslant C(|(\mathbf{u}_n)_i|_{2m}^{2m} + |(\mathbf{u}_n)_j|_{2m}^{2m} + |(\mathbf{u}_1)_i|_{2m}^{2m} + |(\mathbf{u}_1)_j|_{2m}^{2m})^{\frac{2m-1}{2m}} (|(\mathbf{u}_n)_i - (\mathbf{u}_1)_i|_{2m}^{2m} + |(\mathbf{u}_n)_j - (\mathbf{u}_1)_j|_{2m}^{2m})^{\frac{1}{2m}} \\ &\leqslant C(|(\mathbf{u}_n)_i - (\mathbf{u}_1)_i|_{2m}^{2m} + |(\mathbf{u}_n)_j - (\mathbf{u}_1)_j|_{2m}^{2m})^{\frac{1}{2m}} \to 0,\end{aligned}$$

as $n \to \infty$, thanks to $1 < 2m < p_s^*$ and (2.12). This implies that

$$\lim_{n\to\infty} \sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}_n)_i (\mathbf{u}_n)_j|^m \, \mathrm{d}x = \sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}_1)_i (\mathbf{u}_1)_j|^m \, \mathrm{d}x$$

By lemma 2.5, there exists $t_1 > 0$ such that $t_1 \mathbf{u}_1 \in \mathcal{N}^+$. Next we show that $\mathbf{u}_n \to \mathbf{u}_1$ in W. If not, then $\|\mathbf{u}_1\|_W^{\theta p} < \liminf_{n\to\infty} \|\mathbf{u}_n\|_W^{\theta p}$. Thus, for $\mathbf{u}_n \in \mathcal{N}^+$, we obtain

$$\begin{aligned}\lim_{n\to\infty} H'_{\mathbf{u}_n}(t_1) &= \lim_{n\to\infty}\left(t_1^{\theta p-1}\|\mathbf{u}_n\|_W^{\theta p} - t_1^{q-1}\sum_{j=1}^{k}\int_\Omega |(\mathbf{u}_n)_i|^q\,\mathrm{d}x - 2t_1^{2m-1}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |(\mathbf{u}_n)_i(\mathbf{u}_n)_j|^m\,\mathrm{d}x\right)\\ &> t_1^{\theta p-1}\|\mathbf{u}_1\|_W^{\theta p} - t_1^{q-1}\sum_{j=1}^{k}\int_\Omega |(\mathbf{u}_1)_i|^q\,\mathrm{d}x - 2t_1^{2m-1}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |(\mathbf{u}_1)_i(\mathbf{u}_1)_j|^m\,\mathrm{d}x\\ &= H'_{\mathbf{u}_1}(t_1) = 0.\end{aligned}$$

That is, $H'_{\mathbf{u}_n}(t_1) > 0$ for large enough n. Since $\mathbf{u}_n \in \mathcal{N}^+$, it is easy to see that $H'_{\mathbf{u}_n}(t) < 0$ for $t \in (0, 1)$ and $H'_{\mathbf{u}_n}(1) = 0$ for all n. Then we deduce that $t_1 > 1$. On the other hand $H_{\mathbf{u}_1}(t)$ is decreasing on $(0, t_1)$ and so

$$I(t_1\mathbf{u}_1) \leqslant I(\mathbf{u}_1) < \lim_{n\to\infty} I(\mathbf{u}_n) = \inf_{\mathbf{u}\in\mathcal{N}^+} I(\mathbf{u}),$$

which is a contradiction. Hence $\mathbf{u}_n \to \mathbf{u}_1$ in W. This implies

$$I(\mathbf{u}_n) \to I(\mathbf{u}_1) = \inf_{\mathbf{u}\in\mathcal{N}^+} I(\mathbf{u}) = c^+ \quad \text{as } n \to \infty.$$

Namely, $\mathbf{u}_1$ is a minimizer of I on $\mathcal{N}^+$. By lemma 2.2, $\mathbf{u}_1$ is a solution of problem (1.1). □

Lemma 2.10. *For $(\lambda_1, \cdots, \lambda_k) \in \Theta_{\lambda_1}$, the functional I has a minimizer $\mathbf{u}_2$ in $\mathcal{N}^-$ satisfying*

(1) $I(\mathbf{u}_2) = c = c^-$;

(2) $\mathbf{u}_2$ *is a solution of problem* (1.1).

Proof. Since I is bounded from below on $\mathcal{N}^-$, there exists a minimizing sequence $\{\mathbf{u}_n\} \subset \mathcal{N}^-$ such that

$$\lim_{n\to\infty} I(\mathbf{u}_n) = c^-.$$

By the same argument given in the proof of lemma 2.9, there exists $\mathbf{u}_2 \in W$ such that, up to a subsequence, $\mathbf{u}_n \to \mathbf{u}_2$ in W and $I(\mathbf{u}_2) = c^-$ and for all $(\lambda_1, \cdots, \lambda_k) \in \Theta_{\Lambda_1}$, we have that $\mathbf{u}_2$ is a solution of problem (1.1). □

Proof of theorem 1.1. By lemmas 2.9, 2.10 and 2.2, we obtain that for all $(\lambda_1, \cdots, \lambda_k) \in \Theta_{\Lambda_1}$, problem (1.1) has two solutions $\mathbf{u}_1 \in \mathcal{N}^+$ and $\mathbf{u}_2 \in \mathcal{N}^-$. Note that $\mathcal{N}^+ \cap \mathcal{N}^- = \varnothing$, so that those two solutions are distinct. In view of lemma 2.8, we have $I(\mathbf{u}_1) < 0$ and $I(\mathbf{u}_2) > 0$.

Next we show that $\mathbf{u}_1$ and $\mathbf{u}_2$ have at most $k-2$ zero components. Arguing by contradiction, we assume $((\mathbf{u}_2)_1, 0, \cdots, 0)$ is a solution of problem (1.1), then

$$\|(\mathbf{u}_2)_1\|_{W_0}^{(\theta-1)p}\,\|(\mathbf{u}_2)_1\|_{W_0}^{p} = \lambda_1\int_\Omega |(\mathbf{u}_2)_1|^q\,\mathrm{d}x.$$

Thus,

$$\begin{aligned}I(\mathbf{u}_2) &= \frac{1}{\theta p}\|(\mathbf{u}_2)_1\|_{W_0}^{\theta p} - \frac{\lambda_1}{q}\int_\Omega |(\mathbf{u}_2)_1|^q\,\mathrm{d}x\\ &= -\frac{\theta p - q}{\theta p q}\|(\mathbf{u}_2)_1\|_{W_0}^{\theta p} < 0,\end{aligned}$$

which is a contradiction. Hence $\mathbf{u}_2$ has at most $k-2$ zero components.

Finally, we show that $\mathbf{u}_1$ has at most $k-2$ zero components. Otherwise, without loss of generality, we assume that $(\mathbf{u}_1)_1 \neq 0, (\mathbf{u}_1)_j = 0, j = 2, 3, \cdots, k$. Then $\|(\mathbf{u}_1)_1\|_{W_0}^{\theta p} = \lambda_1 \int_\Omega |(\mathbf{u}_1)_1|^q \, \mathrm{d}x > 0$. Let $\mathbf{v} = (v_1, v_2, \cdots, v_k)$, where $v_1 = (\mathbf{u}_1)_1$, $v_2 = \varepsilon(\mathbf{u}_1)_1$, $v_j = 0$ for all $j = 3, \cdots, k$ and $\varepsilon \in (0, 1)$. Obviously, $\mathbf{v} \in W \setminus \{0\}$. Then by lemma 2.5 there exists $0 < t^+ < t_{\max} < t^-$ such that $t^+\mathbf{v} \in \mathcal{N}^+$ and

$$I(t^+\mathbf{v}) = \inf_{0 \leqslant t \leqslant t_{\max}} I(t\mathbf{v}).$$

Moreover,

$$\begin{aligned} t_{\max} &= \left(\frac{(2m-q)\sum_{j=1}^k \lambda_j \int_\Omega |v_j|^q \, \mathrm{d}x}{(2m-\theta p)\|\mathbf{v}\|_W^{\theta p}} \right)^{\frac{1}{\theta p - q}} \\ &\geqslant \left(\frac{(2m-q)\lambda_1 \int_\Omega |v_1|^q \, \mathrm{d}x}{(2m-\theta p)\|\mathbf{v}\|_W^{\theta p}} \right)^{\frac{1}{\theta p - q}} \\ &= \left(\frac{(2m-q)\|(\mathbf{u}_1)_1\|_{W_0}^{\theta p})}{(2m-\theta p)(1+\varepsilon^p)^\theta \|(\mathbf{u}_1)_1\|_{W_0}^{\theta p}} \right)^{\frac{1}{\theta p - q}} = \left(\frac{2m-q}{(2m-\theta p)(1+\varepsilon^p)^\theta} \right)^{\frac{1}{\theta p - q}}. \end{aligned} \tag{2.13}$$

Now we choose $\varepsilon > 0$ small enough such that

$$\varepsilon^p \leqslant \left[\left(\frac{2m-q}{2m-\theta p} \right)^{\frac{1}{\theta}} - 1 \right],$$

thanks to $(2m-q)/(2m-\theta p) > 1$. Then

$$t_{\max} \geqslant \left(\frac{2m-q}{(2m-\theta p)(1+\varepsilon^p)^\theta} \right)^{\frac{1}{\theta p - q}} \geqslant 1.$$

Note that

$$\begin{aligned} I(\mathbf{v}) &= \frac{1}{\theta p}\|\mathbf{v}\|_W^{\theta p} - \frac{1}{q}\sum_{j=1}^k \lambda_j \int_\Omega |u_j|^q \, \mathrm{d}x - \frac{1}{m}\sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^m \, \mathrm{d}x \\ &\leqslant \frac{1}{\theta p}(1+\varepsilon^p)^\theta \|(\mathbf{u}_1)_1\|_{W_0}^{\theta p} - \frac{1}{q}\left(1 + \frac{\lambda_2}{\lambda_1}\varepsilon^q\right)\|(\mathbf{u}_1)_1\|_{W_0}^{\theta p} \\ &= \left(\frac{1}{\theta p} - \frac{1}{q}\right)\|(\mathbf{u}_1)_1\|_{W_0}^{\theta p} + \frac{(1+\varepsilon^p)^\theta - 1}{\theta p}\|(\mathbf{u}_1)_1\|_{W_0}^{\theta p} - \frac{1}{q}\frac{\lambda_2}{\lambda_1}\varepsilon^q \|(\mathbf{u}_1)_1\|_{W_0}^{\theta p} \\ &= I(\mathbf{u}_1) + \frac{(1+\varepsilon^p)^\theta - 1}{\theta p}\|(\mathbf{u}_1)_1\|_{W_0}^{\theta p} - \frac{1}{q}\frac{\lambda_2}{\lambda_1}\varepsilon^q \|(\mathbf{u}_1)_1\|_{W_0}^{\theta p}. \end{aligned} \tag{2.14}$$

If $\varepsilon < \left(\frac{\lambda_2 p}{\lambda_1 q 2^{\theta-1}}\right)^{1/(p-q)}$, then we get

$$\frac{(1+\varepsilon^p)^\theta-1}{\theta p}-\frac{1}{q}\frac{\lambda_2}{\lambda_1}\varepsilon^q=\frac{(1+\tau)^{\theta-1}}{p}\varepsilon^p-\frac{1}{q}\frac{\lambda_2}{\lambda_1}\varepsilon^q\leqslant\frac{2^{\theta-1}}{p}\varepsilon^p-\frac{1}{q}\frac{\lambda_2}{\lambda_1}\varepsilon^q<0, \tag{2.15}$$

thanks to $q<p$. Taking

$$0<\varepsilon<\min\left\{\left[\left(\frac{2m-q}{2m-\theta p}\right)^{1/\theta}-1\right]^{1/p},\left(\frac{\lambda_2 p}{\lambda_1 q 2^{\theta-1}}\right)^{1/(p-q)}\right\},$$

we deduce from (2.13)–(2.15) that

$$c^+\leqslant I(t^+\mathbf{v})\leqslant I(\mathbf{v})<I(\mathbf{u}_1)=c^+$$

which is a contradiction. Thus $\mathbf{u}_1$ has at most $k-2$ zero components. □

3. Superlinear case

In this section, we consider the superlinear case of problem (1.1). We will use the following symmetric mountain pass theorem to get our second result.

Theorem 3.1. (See [10, theorem 2.2]) *Let X be a real infinite dimensional Banach space and $K\in C^1(X)$ a functional satisfying the (PS) condition as well as the following three properties:*

(i) $K(0)=0$ *and there exist two constant $\rho,\alpha>0$ such that $K|_{\partial B_\rho}\geqslant\alpha$;*
(ii) *K is even;*
(iii) *for all finite dimensional subspaces $\overline{X}\subset X$ there exists $R=R(\overline{X})>0$ such that*

$$K(u)\leqslant 0 \ \text{ for all } u\in X\setminus B_R(\overline{X}),$$

where $B_R(\overline{X})=\{u\in\overline{X}:\|u\|<R\}$. Then K possesses an unbounded sequence of critical values characterized by a minimax argument.

Lemma 3.1. *Assume that (H_1)–(H_2) or (H_1), (H_3) hold. Then I satisfies the (PS) condition.*

Proof. Since $I'(\mathbf{u}_n)\to 0$ in W^* as $n\to\infty$ and $I'(\mathbf{u}_n)$ is bounded, there exists $C>0$ such that $|\langle I'(\mathbf{u}_n),\mathbf{u}_n\rangle|\leqslant C\|\mathbf{u}_n\|_W$ and $|I(\mathbf{u}_n)|\leqslant C$. Thus,

$$\begin{aligned}C+C\|\mathbf{u}_n\|_W&\geqslant I(u_n)-\frac{1}{\mu}\langle I'(\mathbf{u}_n),\mathbf{u}_n\rangle\\&=\left(\frac{1}{\theta p}-\frac{1}{\mu}\right)\|\mathbf{u}_n\|_W^{\theta p}-\left(\frac{1}{q}-\frac{1}{\mu}\right)\sum_{j=1}^k\lambda_j|(\mathbf{u}_n)_j|_q^q\\&-\left(\frac{1}{2m}-\frac{1}{\mu}\right)\sum_{j=1}^k\sum_{i\neq j}\beta_{ij}|(\mathbf{u}_n)_i(\mathbf{u}_n)_j|_m^m,\end{aligned}\tag{3.1}$$

where C denotes various positive constants and $\mu>0$. For (H_1) and (H_2), we take $\mu=2m$. Then from $2m\leqslant q$ and (3.1), we obtain

$$C+C\|\mathbf{u}_n\|_W\geqslant\left(\frac{1}{\theta p}-\frac{1}{2m}\right)\|\mathbf{u}_n\|_W^{\theta p},$$

It follows from $1 < \theta p < 2m$ that $\{\mathbf{u}_n\}$ is bounded in W. Similarly, for (H_1) and (H_3), we choose $\mu = q$ in (3.1) and obtain the boundedness of $\{\mathbf{u}_n\}$ in W.

Since W is a reflexive Banach space, up to a subsequence, still denoted by $\{\mathbf{u}_n\}$, such that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in W. Let $\varphi \in W_0$ be fixed and denote by $B(\varphi, \cdot)$ the linear functional on W_0 defined by

$$B(\varphi, v) = \int\int_{\mathbb{R}^{2N}} |\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))(v(x) - v(y))K(x - y)\mathrm{d}x\mathrm{d}y$$

for all $v \in W_0$. Clearly, by the Hölder inequality, $B(\varphi, \cdot)$ is also continuous, being

$$|B(\varphi, v)| \leqslant \|\varphi\|_{W_0}^{p-1}\|v\|_{W_0} \quad \text{for all } v \in W_0.$$

Hence, the weak convergence of $\{\mathbf{u}_n\}$ in W_0 gives that

$$\lim_{n\to\infty} B((\mathbf{u})_j, (\mathbf{u}_n)_j - (\mathbf{u})_j) = 0 \ \text{ for all } 1 \leqslant j \leqslant k. \tag{3.2}$$

Clearly, $\langle I'(\mathbf{u}_n), \mathbf{u}_n - \mathbf{u}\rangle \to 0$. Thus, we obtain

$$\begin{aligned} &\langle I'(\mathbf{u}_n), \mathbf{u}_n - \mathbf{u}\rangle \\ = &\|\mathbf{u}_n\|_W^{(\theta-1)p}\sum_{j=1}^{k} B((\mathbf{u}_n)_j, (\mathbf{u}_n)_j - \mathbf{u}_j) - \sum_{j=1}^{k}\lambda_j\int_\Omega |(\mathbf{u}_n)_j|^{q-2}(\mathbf{u}_n)_j((\mathbf{u}_n)_j - (\mathbf{u})_j)\mathrm{d}x \\ &- \sum_{j=1}^{k}\sum_{i\neq j}\beta_{ij}\int_\Omega |(\mathbf{u}_n)_i|^m|(\mathbf{u}_n)_j|^{m-2}(\mathbf{u}_n)_j((\mathbf{u}_n)_j - \mathbf{u}_j)\mathrm{d}x \\ &\longrightarrow 0, \end{aligned} \tag{3.3}$$

as $n \to \infty$. Similar to the proof of lemma 2.7, we get

$$\sum_{j=1}^{k}\lambda_j\int_\Omega |(\mathbf{u}_n)_j|^q\,\mathrm{d}x \to \sum_{j=1}^{k}\lambda_j\int_\Omega |\mathbf{u}_j|^q\,\mathrm{d}x \ \text{ as } n \to \infty,$$

and

$$\lim_{n\to\infty}\sum_{j=1}^{k}\sum_{i\neq j}\beta_{ij}\int_\Omega |(\mathbf{u}_n)_i(\mathbf{u}_n)_j|^m\,\mathrm{d}x = \sum_{j=1}^{k}\sum_{i\neq j}\beta_{ij}\int_\Omega |\mathbf{u}_i\mathbf{u}_j|^m\,\mathrm{d}x.$$

Combining these facts with (3.3), we have as $n \to \infty$

$$\|\mathbf{u}_n\|_W^{(\theta-1)p}\sum_{j=1}^{k} B((\mathbf{u}_n)_j, (\mathbf{u}_n)_j - \mathbf{u}_j) \longrightarrow 0. \tag{3.4}$$

By (3.2) and (3.4), we have as $n \to \infty$

$$\|\mathbf{u}_n\|_W^{(\theta-1)p}\left[\sum_{j=1}^{k} B((\mathbf{u}_n)_j, (\mathbf{u}_n)_j - \mathbf{u}_j) - \sum_{j=1}^{k} B(\mathbf{u}_j, (\mathbf{u}_n)_j - \mathbf{u}_j)\right] \longrightarrow 0. \tag{3.5}$$

Let us now recall the well-known Simon inequalities:

$$|\xi-\eta|^p \leqslant \begin{cases} \widetilde{C}_p(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta) & \text{for } p\geqslant 2 \\ \overline{C}_p[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta)]^{p/2}(|\xi|^p+|\eta|^p)^{(2-p)/2} & \text{for } 1<p<2, \end{cases} \tag{3.6}$$

for all $\xi,\eta\in\mathbb{R}^N$, where $\widetilde{C}_p$ and $\overline{C}_p$ are positive constants depending only on p.

We first assume that $\inf_n\|\mathbf{u}_n\|_{W_0}>0$. Then by (3.5), it follows that as $n\to\infty$

$$\sum_{j=1}^k[B((\mathbf{u}_n)_j,(\mathbf{u}_n)_j-\mathbf{u}_j)-B(\mathbf{u}_j,(\mathbf{u}_n)_j-\mathbf{u}_j)]\longrightarrow 0. \tag{3.7}$$

It follows from (3.6) that as $n\to\infty$

$$[B((\mathbf{u}_n)_j,(\mathbf{u}_n)_j-\mathbf{u}_j)-B(\mathbf{u}_j,(\mathbf{u}_n)_j-\mathbf{u}_j)]\longrightarrow 0, \tag{3.8}$$

for all $1\leqslant j\leqslant k$. It follows from (3.8) that as $n\to\infty$

$$\begin{aligned}&\sum_{j=1}^k\int\int_Q|(\mathbf{u}_n)_j(x)-(\mathbf{u}_n)_j(y)-\mathbf{u}_j(x)+\mathbf{u}_j(y))|^pK(x-y)\mathrm{d}x\mathrm{d}y\\ &\leqslant\widetilde{C}_p\sum_{j=1}^k[B((\mathbf{u}_n)_j,(\mathbf{u}_n)_i-\mathbf{u}_j)-B(\mathbf{u}_j,(\mathbf{u}_n)_i-\mathbf{u}_j)]\\ &\longrightarrow 0,\end{aligned} \tag{3.9}$$

as $p>2$ and

$$\begin{aligned}&\sum_{j=1}^k\int\int_Q|(\mathbf{u}_n)_j(x)-(\mathbf{u}_n)_j(y)-\mathbf{u}_j(x)+\mathbf{u}_j(y))|^pK(x-y)\mathrm{d}x\mathrm{d}y\\ &\leqslant\overline{C}_p\sum_{j=1}^k\{[B((\mathbf{u}_n)_j,(\mathbf{u}_n)_j-\mathbf{u}_j)-B(\mathbf{u}_j,(\mathbf{u}_n)_j-\mathbf{u}_j)]\}^{\frac{p}{2}}\\ &\qquad\times\left\{\int\int_Q(|(\mathbf{u}_n)_j(x)-(\mathbf{u}_n)_j(y)|+|\mathbf{u}_j(x)-\mathbf{u}_j(y)|)^pK(x-y)\mathrm{d}x\mathrm{d}y\right\}^{\frac{2-p}{2}}\\ &\leqslant C\sum_{j=1}^k\{[B((\mathbf{u}_n)_j,(\mathbf{u}_n)_j-\mathbf{u}_j)-B(\mathbf{u}_j,(\mathbf{u}_n)_j-\mathbf{u}_j)]\}^{\frac{p}{2}}\\ &\longrightarrow 0,\end{aligned} \tag{3.10}$$

as $1<p<2$. Combining (3.9) with (3.10), we get that $\mathbf{u}_n\to\mathbf{u}$ strongly in W as $n\to\infty$.

It remains to consider the case $\inf_n\|\mathbf{u}_n\|_W=0$. For this case, either $\mathbf{0}$ is an accumulation point of the sequence $\{\mathbf{u}_n\}$ and so there exists a subsequence of $\{\mathbf{u}_n\}$ strongly converging to $\mathbf{u}=\mathbf{0}$, or $\mathbf{0}$ is an isolated point of the sequence $\{\mathbf{u}_n\}$ and so there exists a subsequence, still denoted by $\{\mathbf{u}_n\}$, such that $\inf_n\|\mathbf{u}_n\|_W>0$. In the first case we are done, while in the latter case we can proceed as above. □

Proof of theorem 1.2. It is easy to verify that the functional I is even in W and $I(0) = 0$. By lemma 3.1, the functional I satisfies the (PS) condition. It follows from (1.5) that

$$\begin{aligned} I(\mathbf{u}) &= \frac{1}{\theta p}\|\mathbf{u}\|_W^{\theta p} - \frac{1}{q}\sum_{j=1}^k \lambda_j |u_j|_q^q - \frac{1}{m}\sum_{j=1}^k \sum_{i<j} \beta_{ij}|u_i u_j|_m^m \\ &\geqslant \frac{1}{\theta p}\|\mathbf{u}\|_W^{\theta p} - \frac{1}{q}C_*^q\left(\sum_{j=1}^k \lambda_j \|u_j\|_{W_0}^q\right)\|u_j\|_{W_0}^q - \frac{1}{m}C_*^{2m}\left(\sum_{j=1}^k\sum_{i<j}|\beta_{ij}|\right)||u_i|_{2m}^m |u_j|_{2m}^m \\ &\geqslant \frac{1}{\theta p}\|\mathbf{u}\|_W^{\theta p} - \frac{1}{q}C_*^q\left(\sum_{j=1}^k \lambda_j\right)\|\mathbf{u}\|_W^q - \frac{1}{m}C_*^{2m}\left(\sum_{j=1}^k\sum_{i<j}|\beta_{ij}|\right)\|\mathbf{u}\|_W^{2m}. \end{aligned}$$

Let

$$h(t) = \frac{1}{\theta p}t^{\theta p} - \frac{1}{q}C_*^q\left(\sum_{j=1}^k \lambda_j\right)t^q - \frac{1}{m}C_*^{2m}\left(\sum_{j=1}^k\sum_{i<j}|\beta_{ij}|\right)t^{2m} \quad \text{for all } t \geqslant 0.$$

Clearly, there exists $\rho > 0$ such that $h(\rho) = \max_{t\geqslant 0} h(t) > 0$. Thus, we take $\mathbf{u} \in W$ with $\|\mathbf{u}\|_W = \rho$. Then

$$I(\mathbf{u}) \geqslant h(\rho) := \alpha > 0 \quad \text{for all } \mathbf{u} \in W \text{ with } \|\mathbf{u}\|_W = \rho. \tag{3.11}$$

Moreover, for any finite dimensional subspace $E \subset W$ and the equivalency of all norms in E, there exists $C_E > 0$ such that $|\mathbf{u}|_\nu \geqslant C_E\|\mathbf{u}\|_W$. Thus, there exists $C > 0$ such that

$$I(\mathbf{u}) \leqslant \frac{1}{\theta p}\|\mathbf{u}\|_W^{\theta p} - C(\|\mathbf{u}\|_W^q + \|\mathbf{u}\|_W^{2m}).$$

Since $\theta p < q, 2m < p_s^*$, we obtain that there exists $R_0 > \rho$ such that for any $R > R_0$, $I(\mathbf{u}) < 0$ as $\|\mathbf{u}\|_W \geqslant R$. Then by theorem 3.1 problem (1.1) admits infinitely many solutions $\mathbf{u}_n \in W$ with $I(\mathbf{u}_n) \to \infty$ as $n \to \infty$. Hence we complete the proof of theorem 1.2. □

Finally, we give the following example as a direct application of the main results.

Example 3.1. Let $0 < s < 1 < p < \infty, N > ps$ and Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. We consider problem

$$\begin{cases} (\|u\|_{W_0}^p + \|v\|_{W_0}^p)^{\theta-1}(-\Delta)_p^s u = \lambda_1 |u|^{q-2}u + \beta|v|^m|u|^{m-2}v & \text{in } \Omega, \\ (\|u\|_{W_0}^p + \|v\|_{W_0}^p)^{\theta-1}(-\Delta)_p^s v = \lambda_2 |v|^{q-2}v + \beta|u|^m|v|^{m-2}v & \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.12}$$

where $\theta \geqslant 1$, λ_1, $\lambda_2 > 0$ and $\beta > 0$.

If $1 < q < \theta p < 2m < p_s^*$ and $1 \leqslant \theta < N/(N - ps)$, then from theorem 1.1 it follows that there exists $\Lambda_1 > 0$ such that problem (3.12) has two distinct solutions in $W_0 \times W_0$ whenever $(\lambda_1^{\theta p/(\theta p - q)} + \lambda_2^{\theta p/(\theta p - q)})^{(\theta p - q)/(\theta p)} \leqslant \Lambda_1$. In addition, if $1 < q < p$, then those two solutions are not semi-trivial solutions.

If $1 \leqslant \theta p < q$, $2m < p_s^*$ and $1 \leqslant \theta < N/(N - ps)$, then from theorem 1.2 it follows that problem (3.12) has infinitely many solutions in $W_0 \times W_0$.

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