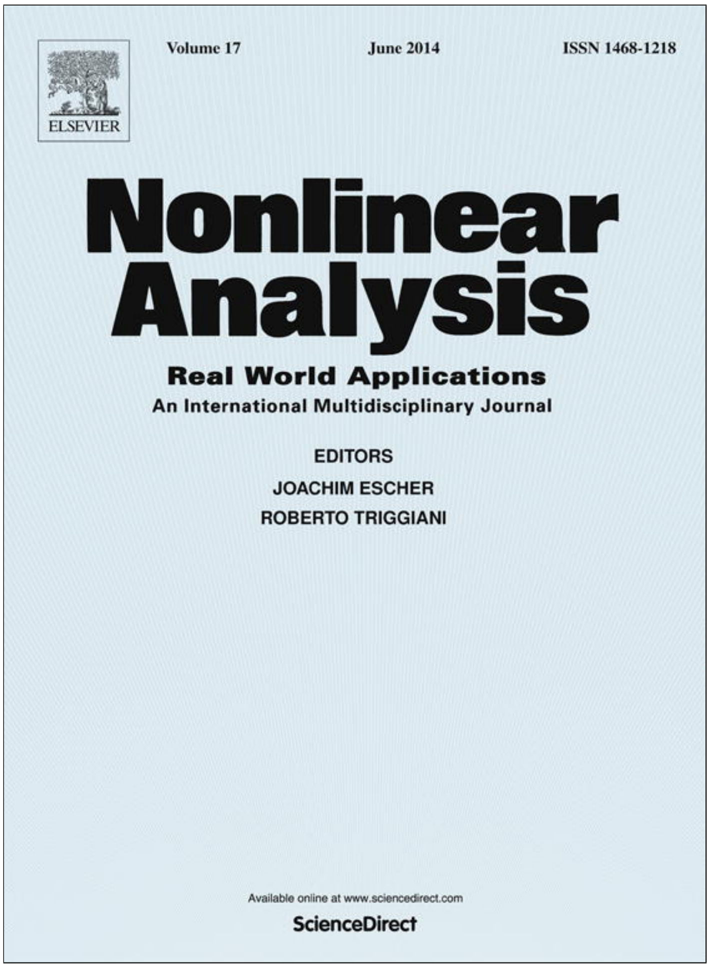
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Morse theory and local linking for a nonlinear degenerate problem arising in the theory of electrorheological fluids

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ABSTRACT

Many electrorheological fluids are suspensions of solid particles that are exposed to a strong electric field. This causes a dramatic increase of their effective viscosity. In this paper we are concerned with a mathematical problem that is related with this non-Newtonian behavior. More precisely, we study the nonlinear stationary equation $-\mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f(x, u)$ in Ω, under Dirichlet boundary conditions, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $p > 1$ is a continuous function, and $f(x, u)$ has a sublinear growth near the origin. Under various natural assumptions, by using the Morse theory in combination with local linking arguments, we obtain the existence of nontrivial weak solutions.

1. Introduction and main results

The qualitative analysis of nonlinear partial differential equations involving differential operators with variable exponent is motivated by wide applications to various fields. Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery on electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their effective viscosity depends on the electric field in the fluid. The dramatic increase of the effective viscosity (or shear stress) is due to the existence of special particle structures that appear in the presence of an electric field hindering the flow. Winslow noticed that in such fluids (for instance lithium polymethacrylate) the effective viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the effective viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. For a general account of the underlying physics we refer to Halsey [1]. For overviews of microscopic models in relationship with applications to electrorheology we refer the reader to Parthasarathy and Klingenberg [2] and Růžička [3]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. We also point out an interesting recent mathematical model developed by Rajagopal and Růžička [4]. The model takes into account the delicate interaction between the electromagnetic fields and the moving fluids. Particularly, in the context of continuum mechanics, these fluids are seen as non-Newtonian fluids. Other relevant applications of nonlinear equations involving differential operators with variable exponent include obstacle problems [5,6], porous medium equation [7–10], and Kirchhoff problems [11,12].

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In the recent paper [13] it is introduced a method to study the existence and multiplicity of solutions for a class of nonlinear stationary partial differential equations described by a non-standard differential operator with variable exponent. We continue this analysis in the present paper, where we describe how powerful tools in modern nonlinear analysis (Morse theory and linking theory) can be used to establish some qualitative properties of solutions to these equations.

1.1. Statement of the problem

The goal of this paper is to investigate the existence of nontrivial solutions to the following $p(x)$-Laplacian problem:

$$\begin{cases} -\triangle_{p(x)} u + |u|^{p(x)-2}u := -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $p \in C(\overline{\Omega})$ and $1 < p_- := \min_{x\in\bar{\Omega}} p(x) \le p(x) \le p_+ := \max_{x\in\bar{\Omega}} p(x) < \infty$ and $F(x,t) = \int_0^t f(x,s)ds$, $\mathcal{F}(x,t) = f(x,t)t - p_+F(x,t)$. For simplicity, we denote by C and C_k various positive constants whose exact value is irrelevant.

We assume that the reaction term $f(x,u)$ satisfies the following hypotheses:

(H1) $f \in C(\overline{\Omega}\times\mathbb{R})$ with $f(x,0) = 0$ and there exists $C_1 > 0$ such that

$$|f(x,t)| \le C_1(1+|t|^{q(x)-1}), \quad \forall (x,t) \in \Omega\times\mathbb{R},$$

where $q(x) \in C(\overline{\Omega})$, $1 < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $p^* = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, $p^*(x) = +\infty$ if $p(x) \ge N$;

(H2) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{p_+}} = +\infty$ uniformly for $x \in \bar{\Omega}$;

(H3) there exists $\theta \ge 1$ such that $\theta\mathcal{F}(x,t) \ge \mathcal{F}(x,st)$ for $(x,t) \in \Omega\times\mathbb{R}$ and $s \in [0,1]$;

(H4) there exists $\nu > 0$ such that

$$\frac{f(x,t)}{|t|^{p_+-2}t} \text{ is increasing in } t \ge \nu \text{ and decreasing in } t \le -\nu;$$

(H5) there are small constants r and R with $0 < r < R$ such that

$$C_2|t|^{\alpha(x)} \le p(x)F(x,t) \le C_3|t|^{p(x)}, \quad \text{for } t \in \mathbb{R} \text{ with } r \le |t| \le R, \text{ a.e. } x \in \Omega, \tag{1.2}$$

where C_2 and C_3 are constants with $0 < C_2 < C_3 < 1$, $\alpha(x) \in C(\bar{\Omega})$ and $1 < \alpha(x) < p(x)$. Moreover, there exists $C_4 > 0$ such that

$$F(x,t) \ge -C_4|t|^{p_+} \quad \text{for all } (x,t) \in \Omega\times\mathbb{R}. \tag{1.3}$$

The assumption (H2) implies that the problem (1.1) is superlinear at infinity. A lot of works concerning the superlinear elliptic boundary value problem have been done by using the usual Ambrosetti–Rabinowitz condition, which is originally due to Ambrosetti and Rabinowitz for the case $p = 2$ in [14], that is,

(AR): there exist $\mu > p_+$ and $M > 0$ such that

$$0 < p_+F(x,t) \le f(x,t)t \quad \text{for all } x \in \Omega \text{ and } |t| \ge M. \tag{1.4}$$

From (1.4) it follows that for some $a, b > 0$

$$F(x,t) \ge a|t|^{\mu} - b \quad \text{for } (x,t) \in \Omega\times\mathbb{R}. \tag{1.5}$$

Obviously, (1.5) implies the much weaker condition (H2).

Let us consider the following function (for simplicity we drop the x-dependence):

$$f(x,t) = |t|^{p_+-2}t\left(p_+\log(1+|t|) + \frac{|t|}{1+|t|}\right), \tag{1.6}$$

then $F(x,t) = |t|^{p_+}\log(1+|t|)$. Then f does not satisfy the (AR) condition for any $\mu > p_+$, but it satisfies our conditions (H2) and (H3). Furthermore, we can show that the function fulfills all hypotheses (H1)–(H5).

In [15], Tan and Fang studied (1.1) under the assumptions (H1), (H2), (H3), (1.3) and

(F): There exists $\delta > 0$ such that $F(x,t) \le 0$ for all $x \in \Omega$ and $|t| \le \delta$.

Using Morse theory, they obtained the existence of one nontrivial weak solution in $W_0^{1,p(x)}(\Omega)$. However, the function $f(x,t)$ in (1.6) does not satisfy the condition (F). Liu and Su [16] used Morse theory and local linking to study the existence of multiple nontrivial solutions for p-Laplacian equations under the following assumption:

(L): There is some $r > 0$ small and $\lambda_1 < \hat{\lambda} < \bar{\lambda}$ such that

$$\lambda_1|t|^p \leq pF(x,t) \leq \hat{\lambda}|t|^p, \quad \text{for a.e. } x \in \Omega \text{ and } |t| \leq r,$$

where λ_1 is the first eigenvalues of $-\triangle_p$ on $W_0^{1,p}(\Omega)$. More attention has been focused on the existence of nontrivial solutions for p-Laplacian equations via Morse theory, see for example [16–21] and references therein. In particular, Guo and Liu [22] studied the existence and multiplicity of nontrivial solutions for a class of p-sublinear p-Laplacian equations. Inspired by their works, we introduce the assumption (1.2). Here the assumption (1.2) means that $f(x,t)$ is sublinear near zero, that is, the function behaves like $O(|u|^{\alpha(x)-2}u)$ near zero for some $\alpha(x) \in C(\bar{\Omega})$ and $1 < \alpha(x) < p(x)$. To the best of our knowledge, our results are new even for the case $p(x) \equiv p$.

1.2. Main results

Our first two results in this paper are the following theorems.

Theorem 1. *Assume that* (H1), (H2), (H3) *and* (H5) *hold. Then the problem* (1.1) *has at least one nontrivial weak solution in* $W_0^{1,p(x)}(\Omega)$.

Theorem 2. *Assume that* (H1), (H2), (H4) *and* (H5) *hold. Then the problem* (1.1) *has at least one nontrivial weak solution in* $W_0^{1,p(x)}(\Omega)$.

Before the statement of Theorem 3, we recall some known results about the eigenvalues of $-\triangle_{p(x)}$ on $W_0^{1,p(x)}(\Omega)$. We say that λ is an eigenvalue of $-\triangle_{p(x)}$ with Dirichlet boundary conditions means that the following boundary value problem:

$$\begin{cases} -\triangle_{p(x)} u = \lambda|u|^{p(x)-2}u & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.7}$$

has nonzero solution. Fan et al. [23] obtained the principal eigenvalue $\lambda_* > 0$ by introducing the following condition:
(P): there exists a vector $l \in \mathbb{R}^N \setminus \{0\}$ such that for any $x \in \Omega$, $c(t) = p(x+tl)$ is monotone in $t \in I_x = \{t : x + tl \in \Omega\}$.

Theorem 3. *Assume that conditions* (P), (H1), (H5) *are fulfilled and*
(H6): $\limsup_{|t|\to\infty} \frac{p(x)F(x,t)}{|t|^{p(x)}} < \frac{p_-}{p_+}\lambda_* + 1$ *uniformly on* $x \in \bar{\Omega}$;
(H7): $F(x,t) \geq 0$ *for all* $x \in \Omega$ *and* $|t| \leq r$.
Then the problem (1.1) *has at least two nontrivial weak solutions in* $W_0^{1,p(x)}(\Omega)$.

The following example illustrates the hypotheses of Theorem 3.

Example 1. Let $\Omega = B(x_0, 1) := \{x \in \mathbb{R}^N : |x - x_0| < 1\}$, where $x_0 = (x_{01}, x_{02}, \ldots, x_{0N})$, $x_{0k} > 1$, $k = 1, 2, \ldots, N$, $p(x) = |x|$ for all $x \in \Omega$. Then $p(x)$ satisfies the condition (P). If we take $f(x,t) = \beta|t|^{p_- -2}t$, $0 < \beta < \frac{p_-}{p_+}\lambda_* + 1$, then it is easy to verify that $f(x,t)$ satisfies all assumptions of Theorem 3.

The paper is organized as follows. In Section 2, we present some necessary preliminary knowledge. In Section 3, we verify the (C) condition and the (PS) condition. In Section 4, we compute the critical groups at zero and at infinity. In Section 5, we give the proof of Theorems 1–3.

2. Preliminary results

Most materials can be modeled with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant. With the emergence of nonlinear problems in applied sciences, Lebesgue and Sobolev spaces L^p and $W^{1,p}$ have demonstrated their limitations in applications. A class of nonlinear problems with variable exponent is a new research field and reflects a new kind of physical phenomena. For instance, for some materials with inhomogeneities, e.g., the electrorheological fluids (sometimes referred to as "smart fluids"), this is not adequate, but rather the exponent p should be able to vary.

To the best of our knowledge, variable exponent Lebesgue spaces appeared in the literature for the first time already in a 1931 article by W. Orlicz [24]. In the 1950s this study was carried on by Nakano [25], who made the first systematic study of spaces with variable exponent (called *modular spaces*). Nakano mentioned explicitly variable exponent Lebesgue spaces as an example of more general spaces he considered, see Nakano [25, p. 284]. Later, the Polish mathematicians investigated the modular function spaces (see e.g. Musielak [26]). Since Kováčik and Rákosník discussed $L^{p(x)}$ space and $W^{k,p(x)}$ space in [27], many results have been obtained concerning this kind of variable exponent spaces, see for instance [28–32] for basic properties of variable exponent spaces and [5–10] for the applications of variable exponent spaces on elliptic equations. For the application backgrounds of the $p(x)$-Laplacian equations we refer to [33–37]. For an overview of variable exponent spaces with various applications to differential equations we refer to [38].

In this section we first recall some necessary facts about variable exponent spaces. For the details we refer to [27,28,32] and the references therein.

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1, +\infty)$ and

$$\|u\|_{p(x)} = \inf\left\{t > 0 : \int_\Omega \left|\frac{u}{t}\right|^{p(x)} dx \le 1\right\}. \tag{2.1}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\int_\Omega |u(x)|^{p(x)}dx < \infty$. Then the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$ with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \tag{2.2}$$

For $u \in L^{p(x)}(\Omega)$, we define

$$|||u||| = \inf\left\{t > 0 : \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{t^{p(x)}} dx \le 1\right\}. \tag{2.3}$$

We deduce that

$$\frac{1}{2}\|u\|_{1,p(x)} \le |||u||| \le 2\|u\|_{1,p(x)}. \tag{2.4}$$

With these norms, the function spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces. We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, then $W_0^{1,p(x)}(\Omega)$ is also a separable and reflexive Banach space.

Proposition 1 (See [32]). *For all $u \in L^{p(x)}(\Omega)$, we define $\rho(u) = \int_\Omega |u(x)|^{p(x)}dx$. Then the following properties are fulfilled:*

(1) *If $\|u\|_{p(x)} \ge 1$, then $\|u\|_{p(x)}^{p-} \le \rho(u) \le \|u\|_{p(x)}^{p+}$.*
(2) *If $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p+} \le \rho(u) \le \|u\|_{p(x)}^{p-}$.*

A related result is the following.

Proposition 2 (See [39]). *Let $I(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx$. If $u \in W^{1,p(x)}(\Omega)$, then the following properties hold:*

(1) *If $u \ne 0$, then $|||u||| = l \Leftrightarrow I(\frac{u}{l}) = 1$.*
(2) *If $|||u||| \ge 1$, then $|||u|||^{p-} \le I(u) \le |||u|||^{p+}$.*
(3) *If $|||u||| \le 1$, then $|||u|||^{p-} \le I(u) \le |||u|||^{p+}$.*
(4) *If $|||u||| = 1$, then $I(u) = 1$.*

For $p(x) \in \mathcal{P}(\Omega)$, let $p' : \Omega \to \mathbb{R}$ be such that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad \text{a.e. } x \in \Omega.$$

Proposition 3 (See [27]).

(i) *For any $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|u\|_{p(x)}\|v\|_{p'(x)} \le 2\|u\|_{p(x)}\|v\|_{p'(x)}. \tag{2.5}$$

(ii) *If $p_1, p_2 \in C(\bar\Omega)$, $1 < p_1(x) \le p_2(x)$ for any $x \in \bar\Omega$, then the embedding $L^{p_2(x)}(\Omega) \to L^{p_1(x)}(\Omega)$ is continuous.*

Proposition 4 (See [27]). *Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and assume that*

$$|F(x, t)| \le a(x) + b|t|^{\frac{p_1(x)}{p_2(x)}}, \quad \forall (x, t) \in \Omega \times \mathbb{R}$$

where $a(x) \in L^\infty(\Omega)$, b is a positive constant, $p_1(x), p_2(x) \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$. We define the Nemytsky operator as follows:

$$(N_F(u))(x) = F(x, u(x)).$$

Then the mapping $N_F : L^{p_1(x)}(\Omega) \to L^{p_2(x)}(\Omega)$ is continuous and bounded.

Proposition 5 (See [40]).

(i) *Let $p \in C(\bar\Omega)$ and $1 \le q(x) < p^*(x)$ for each $x \in \bar\Omega$. Then the embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$ is compact.*
(ii) *There is a constant $C > 0$ such that*

$$\|u\|_{p(x)} \le C\|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

From Proposition 5 (ii) we know that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. In the following, we will use $\|\nabla u\|_{p(x)}$ to replace $\|u\|_{1,p(x)}$.

3. Verification of the compactness conditions

From the assumption (H1) we deduce that the energy functional $\Phi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \int_\Omega F(x,u)dx$$

is well defined and of class C^1. The derivative of Φ at u is given by

$$\langle \Phi'(u), v\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v + |u|^{p(x)-2}u \cdot v)dx - \int_\Omega f(x,u)v dx \tag{3.1}$$

for $v \in W_0^{1,p(x)}(\Omega)$.

Set

$$I(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx, \qquad J(u) = \int_\Omega F(x,u)dx.$$

Then $\Phi(u) = I(u) - J(u)$.

Lemma 1. (i) *(see [39]) We have $I \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and*

$$\langle I'(u), \varphi\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + |u|^{p(x)-2}u\varphi)dx, \quad \forall u, \varphi \in W_0^{1,p(x)}(\Omega).$$

Moreover, the mapping $I' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is a bounded homeomorphism and is of type (S^+), namely

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to\infty}\langle I'(u_n) - I'(u), u_n - u\rangle \le 0 \text{ imply } u_n \to u.$$

(ii) *(see [40]) Under the assumption* (H1), *we have $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and*

$$\langle J'(u), \varphi\rangle = \int_\Omega f(x,u)\varphi dx \quad \forall u, \varphi \in W_0^{1,p(x)}(\Omega).$$

Moreover, the mapping $I' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is weakly–strongly continuous, namely,

$$u_n \rightharpoonup u \text{ implies } u_n \to u,$$

where $\rightharpoonup$ and $\to$ denote the weak and the strong convergence in $W_0^{1,p(x)}(\Omega)$, respectively.

For simplicity, from now on we denote $X = W_0^{1,p(x)}(\Omega)$ and $X^* = (W_0^{1,p(x)}(\Omega))^*$.

Definition 1. The function $u \in X$ is called a weak solution of problem (1.1) if for any $\phi \in X$

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\phi + |u|^{p(x)-2}u \cdot \phi)dx - \int_\Omega f(x,u)\phi dx = 0. \tag{3.2}$$

From (3.1) and (3.2) we know that the critical points of Φ are weak solutions of (1.1). From (3.2) we know that 0 is a critical point of Φ. Now we are interested in seeking nontrivial critical points of Φ.

Definition 2. The functional Φ satisfies the (C) condition if for $c \in \mathbb{R}$, any sequence $\{u_n\} \subset X$ such that $\Phi(u) \to c$, $(1 + \|u_n\|)\|\Phi'(u_n)\|_{X^*} \to 0$ has a convergent subsequence. The functional Φ satisfies the (PS) condition if any sequence $\{u_n\} \subset X$ such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ has a convergent subsequence.

Theorem 4. *Suppose Φ satisfies* (H1), (H2) *and* (H3). *Then Φ satisfies the (C) condition.*

Proof. Similar to the proof of Lemma 3.1 in [41], and for the reader's convenience, we will give the key steps of the proof. We first verify the boundedness of (C) sequences. Assume Φ has an unbounded (C) sequence $\{u_n\}$. Up to a subsequence we may assume that

$$\Phi(u) \to c, \qquad \|\!|u_n|\!\| \to \infty, \qquad \langle\Phi'(u_n), u_n\rangle \to 0 \quad \text{as } n \to \infty.$$

Let $v_n = \|\!|u_n|\!\|^{-1}u_n$, then $\|\!|v_n|\!\| = 1$. By Proposition 5, up to a subsequence we have

$$v_n \rightharpoonup v \text{ in } X, \qquad v_n \to v \text{ in } L^{q(x)}(\Omega), \qquad v_n \to v \ \text{ a.e. } x \in \Omega. \tag{3.3}$$

If $v = 0$, as in [42], we choose a sequence $\{t_n\} \subset [0, 1]$ such that $\Phi(t_n u_n) = \max_{t\in[0,1]} \Phi(tu_n)$. For any positive integer m, we can choose $r = (2mp_+)^{\frac{1}{p_-}}$ such that $r\|u_n\|^{-1} \in (0, 1)$ as n large enough. Since $\|v_n\| = 1$, by Proposition 2, we see that

$$\int_\Omega (|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx = 1. \tag{3.4}$$

Since $v_n \to 0$ in $L^{q(x)}(\Omega)$ and (H1), from Proposition 4 we know that $F(\cdot, rv_n) \to 0$ in $L^1(\Omega)$. Thus

$$\lim_{n\to\infty} \int_\Omega F(x, rv_n)dx = 0. \tag{3.5}$$

Hence, for n large enough, relations (3.4) and (3.5) yield

$$\Phi(t_n u_n) \geq \Phi(r\|u_n\|^{-1}u_n) = \Phi(rv_n) \geq \frac{r^{p_-}}{p^+} - \int_\Omega F(x, rv_n)dx \geq m.$$

We deduce that $\Phi(t_n u_n) \to +\infty$. But $\Phi(0) = 0$, $\Phi(u_n) \to c$, so $t_n \in (0, 1)$ and

$$\langle \Phi'(t_n u_n), t_n u_n\rangle = t_n \frac{d}{dt}\Big|_{t=t_n} \Phi(tu_n) = 0.$$

Now using (H3), we get

$$\begin{aligned} \frac{1}{\theta}\Phi(t_n u_n) &= \frac{1}{\theta}\left(\Phi(t_n u_n) - \frac{1}{p_+}\langle \Phi'(t_n u_n), t_n u_n\rangle\right) \\ &\leq \int_\Omega t^{p(x)}\left(\frac{1}{p(x)} - \frac{1}{p_+}\right)(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx + \frac{1}{p_+}\int_\Omega \frac{\mathcal{F}(x, t_n u_n)}{\theta}dx \\ &\leq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p_+}\right)(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx + \frac{1}{p_+}\int_\Omega \mathcal{F}(x, u_n)dx \\ &= \Phi(u_n) - \frac{1}{p_+}\langle \Phi'(u_n), u_n\rangle \to c. \end{aligned} \tag{3.6}$$

This contradicts the fact that $\Phi(t_n u_n) \to +\infty$.

If $v \neq 0$, for the set $\Omega' := \{x \in \Omega : v(x) \neq 0\}$ we have $|u_n(x)| \to +\infty$, $\forall x \in \Omega'$. Since the set Ω' has positive Lebesgue measure and $\|u_n\| > 1$ for n large, using (1.3) and (H2), we have

$$\begin{aligned} \frac{1}{p_-} &\geq \frac{1}{p_-}\int_\Omega \frac{1}{\|u_n\|^{p(x)}}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx \\ &\geq \int_\Omega \frac{1}{p(x)\|u_n\|^{p(x)}}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx \\ &\geq \frac{\Phi(u_n)}{\|u_n\|^{p_+}} + \int_{\Omega'} \frac{F(x, u_n)}{\|u_n\|^{p_+}}dx + \int_{\Omega\setminus\Omega'} \frac{F(x, u_n)}{\|u_n\|^{p_+}}dx \\ &\geq \frac{\Phi(u_n)}{\|u_n\|^{p_+}} + \int_{\Omega'} \frac{F(x, u_n)}{|u_n|^{p_+}}|v_n|^{p_+}dx - C_4\int_{\Omega\setminus\Omega'} |v_n|^{p_+}dx \to +\infty. \end{aligned}$$

This is impossible. Therefore we have proved that $\{u_n\}$ is bounded in X.

Next, with the same arguments as in the proof of Lemma 5.1 in [41], we prove that any (C) sequence has a convergent subsequence. Since $\{u_n\}$ is bounded in X, by Proposition 5, we may assume that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } X, \qquad u_n \to u \text{ in } L^{q(x)}(\Omega). \tag{3.7}$$

By the boundedness of $\{u_n\}$ in $L^{p(x)}(\Omega)$, we have

$$\sup_n \int_\Omega |u_n|^{p(x)}dx < \infty. \tag{3.8}$$

By (H1), (2.5), (3.7) and (3.8), as in the proof of Lemma 3.11 in Willem [43], we have

$$\int_\Omega (f(x, u_n) - f(x, u))(u_n - u)dx \to 0 \quad \text{as } n \to \infty.$$

Note that $\Phi'(u_n) \to 0$, hence

$$\langle I'(u_n) - I'(u), u_n - u\rangle = \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle + \int_\Omega (f(x, u_n) - f(x, u))(u_n - u)dx \to 0.$$

By Lemma 1, we obtain $u_n \to u$ in X. The proof is complete.

Theorem 5. *Suppose Φ satisfies* (H1), (H2) and (H4). *Then Φ satisfies the* (C) *condition.*

Proof. We only need to verify that (3.6) is fulfilled, the other part of the proof being completely similar to the proof of Theorem 4. From the proof of Theorem 1.2 in [15] we know that due to (H4) there exists a constant $C > 0$ such that

$$\mathcal{F}(x, s) \le \mathcal{F}(x, t) \text{ for } x \in \Omega \text{ and } 0 \le s \le t \text{ or } t \le s \le 0.$$

Therefore

$$\begin{aligned}
\Phi(t_n u_n) &= \Phi(t_n u_n) - \frac{1}{p_+}\langle \Phi'(t_n u_n), t_n u_n\rangle \\
&\le \int_\Omega t^{p(x)}\left(\frac{1}{p(x)} - \frac{1}{p_+}\right)(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx + \frac{1}{p_+}\int_\Omega \mathcal{F}(x, t_n u_n)dx \\
&\le \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p_+}\right)(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx + \frac{1}{p_+}\int_\Omega \mathcal{F}(x, u_n)dx + \frac{C}{p_+}|\Omega| \\
&= \Phi(u_n) - \frac{1}{p_+}\langle \Phi'(u_n), u_n\rangle + \frac{C}{p_+}|\Omega| \to c + \frac{C}{p_+}|\Omega|
\end{aligned}$$

as $n \to \infty$, where $|\Omega|$ denote the measure of the domain Ω.

Theorem 6. *Suppose Φ satisfies* (P), (H1) and (H6). *Then Φ satisfies the* (PS) *condition.*

Proof. We first prove that Φ is coercive. It follows from (H1) and (H6) that for some $\varepsilon > 0$ small, there exists a constant $C_\varepsilon > 0$ such that

$$|F(x, u)| \le \frac{1}{p(x)}\left[\frac{p_-}{p_+}(\lambda_* - \varepsilon) + 1\right]|u|^{p(x)} + C_\varepsilon. \tag{3.9}$$

Hence by means of (3.9) and (2.4), for $u \in W_0^{1,p(x)}(\Omega)$ and $|||u||| \ge 2$ we have

$$\begin{aligned}
\Phi(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \int_\Omega F(x, u)dx \\
&\ge \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \int_\Omega \frac{1}{p(x)}\left[\frac{p_-}{p_+}(\lambda_* - \varepsilon) + 1\right]|u|^{p(x)}dx - C_\varepsilon|\Omega| \\
&= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{p_-}{p_+}(\lambda_* - \varepsilon)\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - C_\varepsilon|\Omega| \\
&\ge \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{\lambda_* - \varepsilon}{\lambda_*}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - C_\varepsilon|\Omega| \\
&\ge C(\varepsilon)|||u|||^{p_-} - C_\varepsilon|\Omega| \to \infty
\end{aligned}$$

as $|||u||| \to \infty$. Next, similar to the idea of the proof in [16, Lemma 3.1], the desired result immediately follows from Lemma 1.

4. Computation of critical groups

Let X be a real Banach space and $\Phi \in C^1(X, \mathbb{R})$, $K = \{u \in X : \Phi'(u) = 0\}$, then the qth critical group of Φ at an isolated critical point $u \in K$ with $\Phi(u) = c$ is defined by

$$C_q(\Phi, u) := H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad q \in \mathbb{N} := \{0, 1, 2, \ldots\},$$

where $\Phi^c = \{u \in X : \Phi(u) \le c\}$, U is any neighborhood of u, containing the unique critical point, H_* is the singular relative homology with coefficients in an Abelian group G.

We say that $u \in K$ is a homological nontrivial critical point of Φ if at least one of its critical groups is nontrivial.

Now we first present the following critical point theorem which will be used later.

Proposition 6 (*See [16, Theorem 2.1]*). *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ satisfy the* (PS) *condition and is bounded from below. If Φ has a critical point that is homological nontrivial and is not a minimizer of Φ, then Φ has at least three critical points.*

If Φ satisfies the condition (C) and the critical values of Φ are bounded from below by some $a < \inf \Phi(K)$, then the critical groups of Φ at infinity were introduced by Bartsch and Li [44] as

$$C_q(\Phi, \infty) := H_q(X, \Phi^a), \quad q \in \mathbb{N}. \tag{4.1}$$

If Φ satisfies the condition (C), then Φ satisfies the deformation condition. By the deformation lemma, the right-hand side of (4.1) does not depend on the choice of a.

Remark 1. Morse theory [21,45] tells us that if $K = \{0\}$ then $C_q(\Phi, \infty) = C_q(\Phi, 0)$ for all $q \in \mathbb{N}$. It follows that if $C_q(\Phi, \infty) \neq C_q(\Phi, 0)$ for some $q \in \mathbb{N}$, then Φ must have a nontrivial critical point. So, we must compute the critical groups at zero and at infinity.

For the proofs of our theorems, in what follows we may assume that Φ has only finitely many critical points. Since Φ satisfies the condition (C), then the critical groups $C_q(\Phi, \infty)$ at infinity make sense.

Theorem 7. *Suppose that Φ satisfies* (H1), (H2) and (H3). *Then $C_q(\Phi, \infty) = 0$ for all $q \in \mathbb{N}$.*

Proof. We use some ideas from the proof of Theorem 1.1 in [15]. Let $S = \{u \in X : \|\|u\|\| = 1\}$. For $u \in S$, by the Fatou lemma and (H2) we have

$$\lim_{t\to+\infty} \int_\Omega \frac{F(x, tu)}{|t|^{p_+}} dx \geq \int_\Omega \lim_{t\to+\infty} \frac{F(x, tu)}{|tu|^{p_+}} |u|^{p_+} dx = +\infty.$$

Therefore

$$\Phi(tu) \leq \frac{t^{p_+}}{p_-} - \int_\Omega F(x, tu)dx \leq t^{p_+} \left(\frac{1}{p_-} - \int_\Omega \frac{F(x, tu)}{|t|^{p_+}} dx \right) \to -\infty \quad \text{as } t \to +\infty.$$

Choose $a < \min\{\inf_{\|u\|\leq 1} \Phi(u), 0\}$, then for any $u \in S$, there exists $t_0 > 1$ such that $\Phi(t_0 u) \leq a$. By (H3), we have

$$\mathcal{F}(x, m) \geq 0 \quad \text{for } (x, m) \in \Omega \times \mathbb{R}. \tag{4.2}$$

Therefore, if

$$\Phi(tu) = \int_\Omega \frac{t^{p(x)}}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \int_\Omega F(x, tu)dx \leq a,$$

then

$$\int_\Omega \left(t^{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\right)dx \leq p_+ a + \int_\Omega p_+ F(x, tu)dx.$$

Using (4.2), we obtain

$$\begin{aligned} \frac{d}{dt}\Phi(tu) &= \frac{1}{t}\left[\int_\Omega t^{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \int_\Omega f(x, tu)tudx\right] \\ &\leq \frac{1}{t}\left[p_+ a - \int_\Omega \mathcal{F}(x, tu)dx\right] < 0. \end{aligned}$$

Then by the implicit function theorem, there exists a unique $T \in C(S, \mathbb{R})$ such that $\Phi(T(u)u) = a$. As in [18], use the function T to construct a strong deformation retract from $X \setminus \{0\}$ to Φ^a. Therefore, we deduce

$$C_q(\Phi, \infty) = H_q(X, \Phi^a) = H_q(X, X \setminus \{0\}) = 0, \quad \forall q \in \mathbb{N}.$$

The proof is completed.

Theorem 8. *Suppose that Φ satisfies* (H1), (H2) and (H4). *Then $C_q(\Phi, \infty) = 0$ for all $q \in \mathbb{N}$.*

Proof. The proof is completely similar to the proof of Theorem 1.2 in [15] and thus it is omitted.

As in [39], since X is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^\infty \subset X$ and $\{f_n\}_{n=1}^\infty \subset X^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & n \neq m \\ 0, & n = m \end{cases}$$

$$X = \overline{\text{span}\{e_n : n = 1, 2, \ldots\}}, \qquad X^* = \overline{\text{span}\{f_n : n = 1, 2, \ldots\}}.$$

For $k = 1, 2, \ldots$, we denote $Y_k = \text{span}\{e_1, e_2, \ldots, e_k\}$, hence Y_k has a closed complementing subspace Z_k in X. Thus, $X = Y_k \oplus Z_k$ (see [46]).

Lemma 2 *(See [39, Lemma 3.3]).* *Assume that $\varphi : X \to \mathbb{R}$ is weakly–strongly continuous, $\varphi(0) = 0$, $\rho > 0$ is a given positive number. Set $\eta_k = \sup_{u\in Z_k, \|u\|\leq\rho} |\varphi(u)|$. Then $\eta_k \to 0$ as $k \to \infty$.*

In order to compute the critical groups at zero, we need the following result.

Proposition 7 (*See [16, Proposition 2.1]*). *Assume that Φ has a critical point $u = 0$ with $\Phi(0) = 0$. Suppose that Φ has a local linking at 0 with respect to $X = V \oplus W$, $k = \dim V < \infty$, that is, there exists $\rho > 0$ small such that*

$$\begin{cases}\Phi(u) \le 0, & u \in V, \|\|u\|\| \le \rho;\\ \Phi(u) > 0, & u \in W, 0 < \|\|u\|\| \le \rho.\end{cases}$$

Then $C_k(\Phi, 0) \neq 0$, hence 0 is a homological nontrivial critical point of Φ.

Theorem 9. *Suppose that Φ satisfies* (H1) and (H5). *Then there exists $k_0 \in \mathbb{N}$ such that $C_{k_0}(\Phi, 0) \neq 0$.*

Proof. Since $f(x, 0) = 0$, the zero function 0 is a critical point of Φ. So we only need to prove that Φ has a local linking at 0 with respect to $X = Y_k \oplus Z_k$. We take two steps:

Step 1: Take $u \in Y_k$. Since Y_k is finite dimensional, we have that for given $R > 0$, there exists $0 < \rho < 1$ small such that

$$u \in Y_k,\ \|\|u\|\| < \rho \Rightarrow |u(x)| < R, \quad \forall x \in \Omega.$$

For $0 < r < R$, let $\Omega_1 = \{x \in \Omega : |u(x)| < r\}$, $\Omega_2 = \{x \in \Omega : r \le |u(x)| \le R\}$, $\Omega_3 = \{x \in \Omega : |u(x)| > R\}$. Then $\Omega = \cup_{i=1}^{3} \Omega_i$ and Ω_i are pairwise disjoint. For the sake of simplicity, let $G(x, u) = F(x, u) - \frac{C_2}{p(x)}|u|^{\alpha(x)}$. Therefore, from (H5) it follows that

$$\begin{aligned}\Phi(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3}\right) F(x, u)dx\\ &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \int_\Omega \frac{C_2}{p(x)}|u|^{\alpha(x)}dx - \int_{\Omega_1} G(x, u)dx - \int_{\Omega_2} G(x, u)dx - \int_{\Omega_3} G(x, u)dx\\ &\le \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \frac{C_2}{p_+}\int_\Omega |u|^{\alpha(x)}dx - \int_{\Omega_1} G(x, u)dx.\end{aligned}$$

In terms of the assumptions on $\alpha(x)$ and Ω is a bounded domain in $\mathbb{R}^N$, the embedding $L^{p(x)}(\Omega) \to L^{\alpha(x)}(\Omega)$ is continuous. This implies that there exists a constant $C > 1$ such that

$$\|u\|_{\alpha(x)} \le C\|u\|_{p(x)} \le C\|u\|_{1,p(x)} \le 2C\|\|u\|\| \le 2C\rho.$$

If $\rho \le \frac{1}{2C}$, then $\|u\|_{\alpha(x)} \le 1$. Note that the norms on Y_k are equivalent to each other, $\|u\|_{\alpha(x)}$ is equivalent to $\|\|u\|\|$. As in [47, Theorem 3.1], due to $\alpha(x), p(x) \in C(\bar{\Omega})$ and $\alpha(x) < p(x)$, for each $x \in \Omega$, there exists an open subset $B_\delta(x)$ of $\bar{\Omega}$ such that

$$\alpha_x := \sup_{x \in B_\delta(x)} \alpha(x) < \inf_{x \in B_\delta(x)} p(x) := p_x.$$

Then $\{B_\delta(x)\}_{x \in \bar{\Omega}}$ is an open covering of $\bar{\Omega}$. Since $\bar{\Omega}$ is compact, there is a finite subcovering $\{B_\delta(x_i)\}_{i=1}^m$. We can use all the hyperplanes, for each of which there exists at least one hypersurface of some $\{B_\delta(x_i)\}_{i=1}^m$ lying on it, to divide $\{B_\delta(x_i)\}_{i=1}^m$ into finite open hypercube $\{\mathcal{Q}\}_{j=1}^n$ which mutually have no common points. It is obvious that $\overline{\Omega} = \cup_{j=1}^n \overline{\mathcal{Q}_j}$ and

$$\alpha_{j+} := \sup_{x \in \mathcal{Q}_j} \alpha(x) < \inf_{x \in \mathcal{Q}_j} p(x) := p_{j-}.$$

Notice that $\int_{\Omega_1} G(x, u)dx \to 0$ as $r \to 0$ and $\|\|u\|\|_{\mathcal{Q}_j} \le \|\|u\|\|$. Therefore, there is a constant $C > 0$ such that for $0 < \rho < 0$ small and r sufficiently small

$$\begin{aligned}\Phi(u) &\le \sum_{j=1}^n \left[\int_{\mathcal{Q}_j} (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \frac{C_2}{p_+}\int_{\mathcal{Q}_j} |u|^{\alpha(x)}dx\right] - \int_{\Omega_1} G(x, u)dx\\ &\le \sum_{j=1}^n \left(\|\|u\|\|_{\mathcal{Q}_j}^{p_{j-}} - C\|\|u\|\|_{\mathcal{Q}_j}^{\alpha_{j+}}\right) - \int_{\Omega_1} G(x, u)dx \le 0.\end{aligned}$$

Step 2: By (H1) and Young's inequality, there exists $C > 0$ such that

$$|F(x, u)| \le \frac{C_3}{p_+}|u|^{p(x)} + C|u|^{s(x)}, \quad \text{for all } x \in \Omega \text{ and } |u| \ge R, \tag{4.3}$$

where $s(x) \in C(\bar{\Omega})$ and $p(x) < s(x) < p^*(x)$. For the sake of simplicity, let $H(x,u) = F(x,u) - \frac{C_3}{p(x)}|u|^{p(x)}$. Therefore, if $u \in Z_k$ and $\|u\| \le 1$, from (H5) and (4.3) we deduce

$$\begin{aligned}\Phi(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3}\right) F(x,u)dx \\ &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \int_\Omega \frac{C_3}{p(x)}|u|^{p(x)}dx - \int_{\Omega_1} H(x,u)dx - \int_{\Omega_2} H(x,u)dx - \int_{\Omega_3} H(x,u)dx \\ &\ge \frac{1-C_3}{p_+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx - C\int_\Omega |u|^{s(x)}dx - \int_{\Omega_1} H(x,u)dx.\end{aligned}$$

We next consider the functional $\varphi : X \to \mathbb{R}$, $\varphi(u) = \int_\Omega |u|^{s(x)}dx$. In view of Proposition 5, the embedding $X \to L^{s(x)}(\Omega)$ is compact. Thus it is easy to show that φ is weakly–strongly continuous (see [32, Theorem 1.4]). Hence by Lemma 2, we have

$$\eta_k = \sup_{u\in Z_k, \|u\|\le 1} |\varphi(u)| \to 0 \quad \text{as } k \to \infty. \tag{4.4}$$

Note that $\int_{\Omega_1} H(x,u)dx \to 0$ as $r \to 0$. Therefore, using the same argument as in *Step* 1, we obtain

$$\begin{aligned}\Phi(u) &\ge \sum_{i=1}^{l}\left[\frac{1-C_3}{p_+}\int_{\mathbb{Q}_i}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - C\int_{\mathbb{Q}_i}|u|^{s(x)}dx\right] - \int_{\Omega_1} H(x,u)dx \\ &\ge \sum_{i=1}^{l}\left[\frac{1-C_3}{p_+}\|u\|_{\mathbb{Q}_i}^{p_{i+}} - C\eta_k\|u\|_{\mathbb{Q}_i}^{s_{i-}}\right] - \int_{\Omega_1} H(x,u)dx.\end{aligned}$$

From (4.4) we know that there exists $k_0 \in \mathbb{N}$ such that $\eta_{k_0} \le \frac{1-C_3}{2Cp_+}$. Then we get

$$\Phi(u) \ge \sum_{i=1}^{l}\frac{1-C_3}{2Cp_+}(\|u\|_{\mathbb{Q}_i}^{p_{i+}} - \|u\|_{\mathbb{Q}_i}^{s_{i-}}) - \int_{\Omega_1} H(x,u)dx > 0$$

as $0 < \rho < 1$ small and r sufficiently small. Thus, there exists $k_0 \in \mathbb{N}$ such that $\Phi(u) > 0$ as $u \in Z_{k_0}$ and $0 < \|u\| \le \rho$.

Combining *Step* 1 with *Step* 2, we complete the proof of Theorem 9 due to Proposition 7.

Remark 2. From the proof of Theorem 9, we deduce that the conclusion of Theorem 9 still holds under the assumptions (H1), (H5) and (H7).

5. Proof of main results

In this section we deduce the proofs of our main existence results.

Proof of Theorem 1. By Theorems 7 and 9, we have $C_{k_0}(\Phi,\infty) \ne C_{k_0}(\Phi,0)$ for some $k_0 \in \mathbb{N}$. Then Theorem 1 follows immediately from Theorem 4 and Remark 1. The proof is complete.

Proof of Theorem 2. By Theorems 8 and 9, we have $C_{k_0}(\Phi,\infty) \ne C_{k_0}(\Phi,0)$ for some $k_0 \in \mathbb{N}$. Then Theorem 2 follows immediately from Theorem 4 and Remark 1. This completes the proof.

Proof of Theorem 3. By Theorem 6, Φ satisfies the (PS) condition and is bounded from below. By the assumption (H7) and Remark 2, the trivial solution $u = 0$ is homological nontrivial and is not a minimizer. The conclusion follows from Proposition 6.

Remark 3. The method used in this paper can also be applied to study the following more general problem:

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

where $b(x) \in C(\bar{\Omega})$ and $1 < b_- \le b_+ < \infty$. Through taking the same arguments of this paper, results similar to Theorems 1–3 can be established under the corresponding assumptions.

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