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Ambrosetti–Prodi problems for the Robin (p, q)-Laplacian

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ABSTRACT

The classical Ambrosetti–Prodi problem considers perturbations of the linear Dirichlet Laplace operator by a nonlinear reaction whose derivative jumps over the principal eigenvalue of the operator. In this paper, we develop a related analysis for parametric problems driven by the nonlinear Robin (p, q)-Laplace operator (sum of a p-Laplacian and a q-Laplacian). Under hypotheses that cover both the $(p-1)$-linear and the $(p-1)$-superlinear case, we prove an optimal existence, multiplicity, and non-existence result, which is global in the parameter $\lambda > 0$.

In this paper, we are concerned with the solvability of the problem

$$Tu = f(z, u(z)) + \lambda\hat{\theta}(z) \text{ in } \Omega, \tag{*}$$

with a Robin boundary condition, where Ω is a smooth bounded domain, $T := T_1 + T_2$ is an unbalanced operator, f is a nonlinearity whose growth at $\pm\infty$ is different with respect to the principal eigenvalue of T_1, λ is a positive parameter, and $\hat{\theta}$ is a given function. Under such conditions, problem (*) is of Ambrosetti–Prodi type, in honor of the celebrated work [1].

In the present paper, we develop an original approach, whose features are the following:
(i) the problem is driven by a nonstandard differential operator, whose associated energy is a double-phase variational functional;
(ii) we consider the combined effects of a Robin boundary condition, an Ambrosetti–Prodi nonlinearity, and a parametric perturbation term;

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(iii) the main result establishes a global existence and multiplicity property, namely we show that there is a critical parameter $\lambda^* > 0$ such that

(a) the problem has at least two solutions for all $\lambda \in (0, \lambda^*)$;

(b) the problem has at least one solution if $\lambda = \lambda^*$;

(c) there is no solution for all $\lambda > \lambda^*$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following problem

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z, u(z)) + \lambda\hat{\theta}(z) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega, 1 < q < p, \lambda > 0. \end{cases} \tag{P_λ}$$

For $r \in (1, \infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2}Du) \text{ for all } u \in W^{1,r}(\Omega).$$

In problem (P_λ), the differential operator is the sum of two such Laplacians with different exponents (a (p,q)-Laplacian with $1 < q < p$) plus a potential term $\xi(z)|u|^{p-2}u$. The differential operator in (P_λ) is not homogeneous. In the reaction (right hand side of (P_λ)), we have a state independent parametric term $\lambda\hat{\theta}(\cdot)$ with $\hat{\theta} \in L^\infty(\Omega)$, $\hat{\theta} \geq 0$ and $\lambda > 0$ is the parameter and there is a perturbation term $f(z,x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). We consider two different cases. In the first $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to +\infty$ and in the second $f(z,\cdot)$ is $(p-1)$-linear as $x \to +\infty$. In the boundary condition $\frac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative of $u(\cdot)$ corresponding to the (p,q)-Laplacian. This directional derivative is interpreted using the nonlinear Green's identity (see [2], p.210) and when $u \in C^1(\bar{\Omega})$, then

$$\frac{\partial u}{\partial n_{pq}} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N} + |Du|^{q-2}(Du, n)_{\mathbb{R}^N} = \left[|Du|^{p-2} + |Du|^{q-2}\right]\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and when $\beta \equiv 0$, we recover the Neumann problem. Our aim is to prove an existence and multiplicity theorem which is global with respect to the parameter $\lambda > 0$.

The double-phase problem (P_λ) is motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born–Infeld equation [3] that appears in electromagnetism:

$$-\operatorname{div}\left(\frac{Du}{(1 - 2|Du|^2)^{1/2}}\right) = h(u) \text{ in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{2 \cdot 2^2}x^2 + \frac{5!!}{3! \cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1} + \cdots \text{ for } |x| < 1.$$

Taking $x = 2|Du|^2$ and adopting the first order approximation, we obtain problem (P_λ) for $p = 4$ and $q = 2$. Furthermore, the nth order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n}u.$$

We also refer to the following fourth-order relativistic operator

$$u \mapsto \operatorname{div}\left(\frac{|Du|^2}{(1 - |Du|^4)^{3/4}}\,Du\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1-x^4)^{-3/4} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots .$$

This shows that the fourth-order relativistic operator can be approximated by the following autonomous double phase operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

Problem (P_λ) belongs to a class of problems known as "Ambrosetti-Prodi-type problems". Their investigation was initiated with the work of Ambrosetti–Prodi [1]. Since then, Ambrosetti–Prodi problems for the Dirichlet p-Laplacian were studied by Arcoya–Ruiz [4], Koizumi–Schmitt [5] (the $(p-1)$-linear case), Arias–Cuesta [6], Miotto [7] (the $(p-1)$-superlinear case) and Aizicovici–Papageorgiou–Staicu [8] (both the linear and superlinear cases). For the Dirichlet (p,q)-Laplacian, there is only the work of Miotto–Miotto [9]. Beyond the Dirichlet problem, very little work has been done and there are only the papers of de Paiva–Montenegro [10] and Presoto–de Paiva [11], both dealing with Neumann problems. In [10] the equation is driven by the p-Laplacian, while in [11] the equation is semilinear driven by the Laplacian and the reaction is gradient dependent. Finally, for other relevant topics involving double phase problems and elliptic equations, we refer to the papers [12–19] and the references therein.

2. Mathematical background and hypotheses

In the analysis of problem (P_λ) we will use the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\bar\Omega)$ and the boundary Lebesgue spaces $L^s(\partial\Omega)$, $1 \le s < \infty$.

By $\|\cdot\|$ we will denote the norm of $W^{1,p}(\Omega)$ given by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The space $C^1(\bar\Omega)$ will come up as a result of the regularity theory. This space is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\bar\Omega) : u(z) \ge 0 \text{ for all } z \in \bar\Omega\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \bar\Omega\right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Having this measure, we can define in the usual way the boundary Lebesgue spaces $L^s(\partial\Omega)$, $1 \le s < \infty$. The theory of Sobolev spaces says that there exists a continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\bar\Omega).$$

This map is known as the "trace map" and through it we extend the notion of boundary values to all Sobolev functions.

We have that

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega),\ \left(\frac{1}{p} + \frac{1}{p'} = 1\right),\ \ker\gamma_0 = W_0^{1,p}(\Omega).$$

The trace operator is compact into $L^s(\partial\Omega)$ for all $s \in [1, \frac{(N-1)p}{N-p})$ if $p < N$ and into $L^s(\partial\Omega)$ for all $s \in [1,\infty)$ if $N \le p$. In the sequel, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

By $L^\infty(\Omega)_+$ we denote the positive (order) cone of $L^\infty(\Omega)$, that is,

$$L^\infty(\Omega) = \{h \in L^\infty(\Omega) : h(z) \geq 0 \text{ for all } a.a\ z \in \Omega\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} L^\infty(\Omega)_+ = \left\{h \in L^\infty(\Omega)_+ : \operatorname*{ess\,inf}_{\Omega} h > 0\right\}.$$

In $C^1(\bar{\Omega})$ we will also consider another open cone given by

$$D_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

If $h_1, h_2 : \Omega \to \mathbb{R}$ are measurable functions, then we write $h_1 \prec h_2$ if and only $h_1(z) \leq h_2(z)$ for a.a. $z \in \Omega$, $h_1 \neq h_2$ (that is, $h_1(z) < h_2(z)$ on a subset of Ω of positive measure). Also, if $h_1 \leq h_2$, then we set

$$[h_1, h_2] = \{h \in W^{1,p}(\Omega) : h_1(z) \leq h(z) \leq h_2(z) \text{ for a.a. } z \in \Omega\},$$

$$[h_1) = \{h \in W^{1,p}(\Omega) : h_1(z) \leq h(z) \text{ for a.a. } z \in \Omega\},$$

and

$$\operatorname{int}_{C^1(\bar{\Omega})}[h_1, h_2] = \text{the interior in } C^1(\bar{\Omega}) \text{ of } [h_1, h_2] \cap C^1(\bar{\Omega}).$$

For every $x \in \mathbb{R}$, Let $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Then if $u : \Omega \to \mathbb{R}$ is a measurable function we define the measurable functions $u^\pm : \Omega \to \mathbb{R}$, by $u^+(z) = u(z)^+$ and $u^-(z) = u(z)^-$ for all $z \in \Omega$. We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W^{1,p}(\Omega)$, then $u^\pm \in W^{1,p}(\Omega)$.

The presence of the state independent parametric term $\lambda\hat{\theta}(\cdot)$, will lead to some functionals with a mild nonsmoothness at the origin. We can overcome this difficulty using the subdifferential theory of Clarke [20].

So, let X be a Banach space. A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, we can find $U \subseteq X$ an open set containing x such that

$$|\varphi(v) - \varphi(y)| \leq k_U \|v - y\|$$

for all $v, y \in U$ and for some $k_U > 0$.

By $\mathrm{Lip}_{loc}(X)$ we denote the space of all locally Lipschitz functions on X. Given $\varphi \in \mathrm{Lip}_{loc}(X)$, we can define the generalized directional derivative of φ at x in the direction h by

$$\varphi^0(x; h) = \limsup_{x' \to x, t \to 0^+} \frac{\varphi(x' + th) - \varphi(x')}{t}.$$

Then $\varphi^0(x; \cdot)$ is finite, sublinear and $|\varphi^0(x; h)| \leq c\|h\|$ for all $h \in X$ and some $c > 0$. So, we can define the generalized subdifferential (or Clarke subdifferential) of $\varphi(\cdot)$ at $x \in X$ by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

We know that $\partial\varphi(x) \subseteq X^*$ is nonempty, convex and w^*-compact. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. This notion is proved to be very fruitful and has a rich calculus and many applications. We refer to the book of Clarke [20] for details.

Let $\mathrm{Lip}_{loc}(X)$ and set

$$m_\varphi(x) = \inf\{\|x^*\|_* : x^* \in \partial\varphi(x)\} \text{ for all } x \in X.$$

We say that $\varphi(\cdot)$ satisfies the "nonsmooth C-condition", if the following property holds:

"every sequence $\{x_n\}_{n\in\mathbb{N}}$ such that
$\{\varphi(x_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded,
$(1+\|x_n\|)m_\varphi(x_n) \to 0$,
admits a strongly convergent subsequence."

If $\varphi \in C^1(X)$, then since $\partial\varphi(x) = \{\varphi'(x)\}$ for all $x \in X$, we see that the above notion coincides with the classical C-condition of the smooth critical point theory (see [21], p.366). We also refer to Tang and Cheng [22] who proposed a new approach to restore the compactness of Palais–Smale sequences and to Tang and Chen [23] who introduced an original method to recover the compactness of minimizing sequences. A related approach has been developed by Chen and Tang [24] in the framework of Cerami sequences.

The nonsmooth C-condition, is a compactness condition on the functional $\varphi(\cdot)$ which compensates for the fact that the ambient space X need not be locally compact (in most applications X is infinite dimensional). It leads to a deformation lemma, from which one can have an extension of the classical (smooth) critical point theory to locally Lipschitz functions. For details, we refer to the book of Gasinski–Papageorgiou [25].

Given $\varphi \in \mathrm{Lip}_{loc}(X)$, we define

$$K_\varphi = \{u \in X : 0 \in \partial\varphi(u)\},$$

(the critical set of φ). If $\varphi \in C^1(X)$, then

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

We introduce the conditions on the potential $\xi(\cdot)$, the boundary coefficient $\beta(\cdot)$ and on the function $\hat{\theta}(\cdot)$ from the parametric term in the reaction.

H_0 : $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ $(0<\alpha<1)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$, $\xi \not\equiv 0$ or $\beta \not\equiv 0$ and $\hat{\theta} \in \operatorname{int} L^\infty(\Omega)_+$.

Remark. If $\beta \equiv 0$, then we recover the Neumann problem and recall that $\hat{\theta} \in \operatorname{int} L^\infty(\Omega)_+$ means that $\hat{\theta} \in L^\infty(\Omega)$ and $0 < \operatorname{ess\,inf}_\Omega \hat{\theta}$.

In what follows, we denote by $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ the C^1-functional defined by

$$\gamma_p(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p \mathrm{d}z + \int_{\partial\Omega} \beta(z)|u|^p \mathrm{d}\sigma$$

for all $u \in W^{1,p}(\Omega)$.

From Papageorgiou–Qin–Rădulescu [26], we know that

$$c_0\|u\|^p \leq \gamma_p(u) \text{ for some } c_0 > 0, \text{all } u \in W^{1,p}(\Omega). \tag{1}$$

We consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue", if problem (2) has a nontrivial solution $\hat{u} \in W^{1,p}(\Omega)$ known as an "eigenfunction" corresponding to $\hat{\lambda}$. From Papageorgiou–Rădulescu [27] and Fragnelli–Mugnai–Papageorgiou [28], we know that there exists a smallest eigenvalue $\hat{\lambda}_1(p)$ such that

$$\hat{\lambda}_1(p) = \inf\left\{\frac{\gamma_p(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0\right\}. \tag{3}$$

From (1) it is clear that $\hat{\lambda}_1(p) > 0$ and this eigenvalue is isolated and simple and the infimum is realized on the corresponding one dimensional eigenspace, the elements of which have fixed sign. Moreover, the nonlinear regularity theory (see [29,30]) implies that the eigenfunctions $\hat{u} \in C^1(\bar{\Omega})$. Moreover, by the nonlinear maximum principle the eigenfunctions for $\hat{\lambda}_1(p) > 0$ belong in $\pm \operatorname{int} C_+$.

These properties lead to the following proposition (see Gasinski–O'Regan–Papageorgiou [31], Lemma 2.1).

Proposition 1. *If $\theta \in L^\infty(\Omega)$ and $\theta \prec \hat{\lambda}_1(p)$, then there exists $\hat{c} > 0$ such that for all $u \in W^{1,p}(\Omega)$,*

$$\hat{c}\|u\|^p \leq \gamma_p(u) - \int_\Omega \theta(z)|u|^p \mathrm{d}z.$$

We mention that if $\hat{\lambda} > 0$ is an eigenvalue and $\hat{\lambda} \neq \hat{\lambda}_1(p)$, then the corresponding eigenfunctions are nodal (sign changing).

We will also consider a weighted version of (2). So, let $m \in L^\infty(\Omega)_+ \backslash \{0\}$ and consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \tilde{\lambda} m(z)|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Again we have a smallest eigenvalue $\tilde{\lambda}_1(p,m) > 0$ which is isolated, simple and

$$\tilde{\lambda}_1(p,m) = \inf\left\{ \frac{\gamma_p(u)}{\int_\Omega m(z)|u|^p \mathrm{d}z} : u \in W^{1,p}(\Omega), u \neq 0 \right\}.$$

The infimum is realized on the corresponding one dimensional eigenspace the elements of which are in $\pm \operatorname{int} C_+$. Hence we have the following monotonicity property for the map $m \mapsto \tilde{\lambda}_1(p,m)$.

Proposition 2. *If $m, \hat{m} \in L^\infty(\Omega)_+ \backslash \{0\}$, $m(z) \leq \hat{m}(z)$ for a.a. $z \in \Omega$, $m \neq \hat{m}$, then*

$$\tilde{\lambda}_1(p,\hat{m}) < \tilde{\lambda}_1(p,m).$$

For $r \in (1,\infty)$, let $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \text{ for all } u, h \in W^{1,r}(\Omega).$$

This map has the following properties (see, for example, Gasinski–Papageorgiou [32, p. 279]).

Proposition 3. *The operator $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ is bounded (maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$\text{“}u_n \xrightarrow{w} u \text{ in } W^{1,r}(\Omega) \text{ and } \limsup_{n\to\infty}\langle A_r(u_n), u_n - u\rangle \leq 0 \text{ imply that } u_n \to u \text{ in } W^{1,r}(\Omega)\text{”}.$$

Now we introduce the hypotheses on the perturbation $f(z,x)$. Recall that $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ if $p \geq N$.

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leq a(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p \leq r < p^*$;

(ii) there exists a function $\eta \in L^\infty(\Omega)$ such that $\hat{\lambda}_1(p) \prec \eta$ and

$$\eta(z) \leq \liminf_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exists a function $\theta \in L^{\infty}(\Omega)$ such that $\theta \prec \hat{\lambda}_1(p)$ and

$$\limsup_{x \to -\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iv) $f(z,x)x \geq 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ and there exist a function $\eta_0 \in L^{\infty}(\Omega)_+ \backslash \{0\}$ and $\tau \in (1,q)$ such that

$$\eta_0(z) \leq \liminf_{x \to 0} \frac{f(z,x)}{|x|^{\tau-2}x} \text{ uniformly for a.a. } z \in \Omega;$$

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \mapsto f(z,x) + \hat{\xi}_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

Remark. Evidently, hypothesis H_1(ii) incorporates both the $(p-1)$-superlinear and the $(p-1)$-linear case.

We introduce the following set:

$$\mathscr{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a nontrivial solution}\}$$

(set of admissible parameters).

In the next section we will show that $\mathscr{L} \neq \emptyset$. To do this, we need some auxiliary material.

On account of hypotheses H_1(ii) and H_1(iii) given $\varepsilon > 0$ we can find $c_1 = c_1(\varepsilon) > 0$ such that

$$f(z,x) \geq [\theta(z) + \varepsilon]|x|^{p-2}x - c_1 \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0. \tag{4}$$

Then we consider the following auxiliary Robin problem:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)|u(z)|^{p-2}u(z) = [\theta(z)+\varepsilon]|u(z)|^{p-2}u(z) - c_1 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega, u \leq 0. \end{cases} \tag{5}$$

Proposition 4. *If hypotheses H_0 hold, then for all $\varepsilon > 0$ small, problem (5) has a unique negative solution $\bar{u} \in -\mathrm{int}\, C_+$.*

Proof. We consider the functional $\hat{\psi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|Du\|_q^q - \frac{1}{p}\int_\Omega [\theta(z)+\varepsilon](u^-)^p \mathrm{d}z - \int_\Omega c_1 u^- \mathrm{d}z$$

for all $u \in W^{1,p}(\Omega)$.

This functional is locally Lipschitz (differentiability fails at $u = 0$). We have

$$c_2\|u^+\|^p \leq \frac{1}{p}\gamma_p(u^+) \text{ with } c_2 = \frac{c_0}{p} \quad (\text{see } (1)), \tag{6}$$

and

$$\begin{aligned} &\frac{1}{p}\left[\gamma_p(u^-) - \int_\Omega \theta(z)(u^-)^p \mathrm{d}z - \varepsilon\|u^-\|_p^p\right] \\ \geq &\frac{1}{p}\left[\hat{c} - \frac{\varepsilon}{\hat{\lambda}_1(p)}\right]\|u^-\|^p, \quad (\text{see Proposition 1}). \end{aligned}$$

Choosing $\varepsilon \in (0, \hat{c}\hat{\lambda}_1(p))$, we obtain

$$c_3\|u^-\|^p \leq \frac{1}{p}\left[\gamma_p(u^-) - \int_\Omega \theta(z)(u^-)^p \mathrm{d}z - \varepsilon\|u^-\|_p^p\right] \text{ for some } c_3 > 0. \tag{7}$$

Then using (6) and (7), we have

$$\begin{aligned}&\hat{\psi}(u) \geq c_4\|u\|^p - c_5\|u\| \quad \text{for some } c_4, c_5 > 0,\\ \Rightarrow\ &\hat{\psi}(\cdot) \text{ is coercive (recall } p > 1).\end{aligned}$$

Also using the Sobolev embedding theorem, we see that $\hat{\psi}(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}&\hat{\psi}(\bar{u}) = \inf\left\{\hat{\psi}(u) : u \in W^{1,p}(\Omega)\right\},\\ \Rightarrow\ &0 \in \partial\hat{\psi}(\hat{u}) \quad \text{(see Clarke [20], p.38).}\end{aligned} \tag{8}$$

We have

$$\partial\psi(\bar{u}) = \gamma_p'(\bar{u}) + A_q(\bar{u}) - [\theta(z)+\varepsilon]|\bar{u}|^{p-2}\bar{u} + g(\bar{u}), \tag{9}$$

where

$$g(\bar{u})(z) = \begin{cases} -c_1, & \text{if } \bar{u}(z) < 0,\\ \{-c_1 v : 0 \leq v \leq 1\}, & \text{if } \bar{u}(z) = 0,\\ 0, & \text{if } \bar{u}(z) < 0,\end{cases} \tag{10}$$

(see Clarke [20], p.39). On (9) we act with $\bar{u}^+ \in W^{1,p}(\Omega)$ and using (8) and (10), we have

$$\begin{aligned}&\gamma_p(\bar{u}^+) + \|D\bar{u}^+\|_q^q = 0,\\ \Rightarrow\ &\bar{u} \leq 0, \bar{u} \neq 0 \quad \text{(see (1)).}\end{aligned}$$

From Proposition 2.10 of Papageorgiou–Rădulescu [29], we have that $\bar{u} \in L^\infty(\Omega)$ and then the nonlinear regularity theory of Lieberman [30] implies that $\bar{u} \in C_+\backslash\{0\}$. We have

$$\begin{aligned}&\Delta_p(-\bar{u}) + \Delta_q(-\bar{u}) \leq \xi(z)(-\bar{u})^{p-1} \text{ (recall } \bar{u} \leq 0),\\ \Rightarrow\ &\bar{u} \in -\operatorname{int} C_+\end{aligned}$$

(see Pucci–Serrin [33], pp.111,120).

Next we prove the uniqueness of this negative solution. To this end we consider the integral functional $j : L^1(\Omega) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \dfrac{1}{p}\|D(u^-)^{1/q}\|_p^p + \dfrac{1}{q}\|D(u^-)^{1/q}\|_q^q + \dfrac{1}{p}\displaystyle\int_\Omega \xi(z)(u^-)^{p/q}\mathrm{d}z, & \text{if } u \leq 0, (u^-)^{1/q} \in W^{1,p}(\Omega),\\ +\infty, & \text{otherwise;}\end{cases}$$

(note that $\bar{u}^- = -\bar{u}$).

From Díaz–Saá [34], we know that $j(\cdot)$ is convex. Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$). Suppose that $\bar{v} \in W^{1,p}(\Omega)$ is another negative solution of (5). Again we have $\bar{v} \in -\operatorname{int} C_+$. Then using Proposition 4.1.22, p.274, of [21], we infer that

$$\frac{|\bar{u}|}{|\bar{v}|} \in L^\infty(\Omega), \frac{|\bar{v}|}{|\bar{u}|} \in L^\infty(\Omega).$$

Therefore, if $h = |\bar{u}|^q - |\bar{v}|^q$, then for $|t| < 1$ small we have

$$|\bar{u}|^q + th \in \operatorname{dom} j, \ \ |\bar{v}|^q + th \in \operatorname{dom} j.$$

Exploiting the convexity of $j(\cdot)$, we see that the directional derivative of $j(\cdot)$ at $|\bar{u}|^q$ and at $|\bar{v}|^q$ in the direction h exists and using Green's identity, we have

$$\begin{aligned}j'(|\bar{u}|^q)(h) =& \frac{1}{q}\int_\Omega \frac{\Delta_p\bar{u} + \Delta_q\bar{u} - \xi(z)|\bar{u}|^{p-2}(-\bar{u})}{|\bar{u}|^{q-2}\bar{u}} h\mathrm{d}z - \frac{1}{q}\int_{\partial\Omega}\beta(z)|\bar{u}|^{p-q}h\mathrm{d}\sigma\\ =& \frac{1}{q}\int_\Omega\left(-[\theta(z)+\varepsilon]|\bar{u}|^{p-q} + \frac{\hat{c}_1}{|\bar{u}|^{q-1}}\right)h\mathrm{d}z\\ &- \frac{1}{q}\int_{\partial\Omega}\beta(z)|\bar{u}|^{p-q}h\mathrm{d}\sigma,\end{aligned}$$

and

$$
\begin{aligned}
j'(|\bar v|^q)(h) =& \frac{1}{q}\int_\Omega \frac{\Delta_p \bar v + \Delta_q \bar v - \xi(z)|\bar v|^{p-2}(-\bar v)}{|\bar v|^{q-2}\bar v} h \mathrm{d}z - \frac{1}{q}\int_{\partial\Omega} \beta(z)|\bar v|^{p-q} h \mathrm{d}\sigma \\
=& \frac{1}{q}\int_\Omega \left(-[\theta(z)+\varepsilon]|\bar v|^{p-q} + \frac{c_1}{|\bar v|^{q-1}}\right) h \mathrm{d}z \\
& - \frac{1}{q}\int_{\partial\Omega} \beta(z)|\bar v|^{p-q} h \mathrm{d}\sigma.
\end{aligned}
$$

The convexity of $j(\cdot)$ implies that the monotonicity of $j'(\cdot)$. Hence

$$
\begin{aligned}
0 \le & \int_\Omega -[\theta(z)+\varepsilon](|\bar u|^{p-q} - |\bar v|^{p-q})(|\bar u|^q - |\bar v|^q)\mathrm{d}z \\
& + \int_\Omega c_1 \left(\frac{1}{|\bar u|^{p-q}} - \frac{1}{|\bar v|^{p-q}}\right)(|\bar u|^q - |\bar v|^q)\mathrm{d}z \\
& - \int_{\partial\Omega} \beta(z)(|\bar u|^{p-q} - |\bar v|^{p-q})(|\bar u|^q - |\bar v|^q)\mathrm{d}\sigma \\
\le & 0 \quad \text{(recall that } \beta \ge 0\text{)}, \\
\Rightarrow & \bar u = \bar v.
\end{aligned}
$$

This proves the uniqueness of the negative solution $\bar u \in -\operatorname{int} C_+$ of problem (5). ☐

3. Existence and multiplicity of solutions

First we show that $\mathscr{L} \neq \emptyset$. To do this we will use the solution $\bar u \in -\operatorname{int} C_+$ from Proposition 4. Using $\bar u \in \operatorname{int} C_+$ we introduce the following Carathéodory function

$$
\hat f(z,x) = \begin{cases} f(z,\bar u(z)), & \text{if } x \le \bar u(z), \\ f(z,-x^-), & \text{if } \bar u(z) < x. \end{cases} \tag{11}
$$

We set $\widehat F(z,x) = \int_0^x \hat f(z,s)\mathrm{d}s$. Also we introduce the function $T_\lambda(z,x)$ defined by

$$
T_\lambda(z,x) = \begin{cases} \lambda\hat\theta(z)\bar u(z), & \text{if } x \le \bar u(z), \\ \lambda\theta(z)x^-, & \text{if } \bar u(z) < x. \end{cases} \tag{12}
$$

Evidently for all $x \in \mathbb{R}$, $z \mapsto T_\lambda(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto T_\lambda(z,x)$ is Lipschitz continuous (hence $T_\lambda(\cdot,\cdot)$ is jointly measurable). We consider the locally Lipschitz functional $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$
\psi_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \widehat F(z,u)\mathrm{d}z - \int_\Omega T_\lambda(z,u)\mathrm{d}z
$$

for all $u \in W^{1,p}(\Omega)$.

Proposition 5. *If hypotheses H_0 and H_1 hold, then $\mathscr{L} \neq \emptyset$.*

Proof. For every $u \in W^{1,p}(\Omega)$ we have

$$
\begin{aligned}
\psi_\lambda(u) \ge & \frac{1}{p}\gamma_p(u) - \frac{1}{p}\int_\Omega [\theta(z)+\varepsilon](u^-)^p \mathrm{d}z - c_6[\lambda\|u\| + 1] \\
& \text{for some } c_6 > 0 \text{ (see (4), (11) and (12))} \\
= & \frac{1}{p}\left[\gamma_p(u) - \int_\Omega \theta(z)|u|^p \mathrm{d}z - \varepsilon\|u\|_p^p\right] - c_6[\lambda\|u\| + 1] \\
\ge & \frac{1}{p}\left[\hat c - \frac{\varepsilon}{\hat\lambda_1(p)}\right]\|u\|^p - \lambda c_6\|u\| - c_6 \\
& \text{(see (3) and Proposition 1).}
\end{aligned}
$$

Choosing $\varepsilon \in (0, \hat{c}\hat{\lambda}_1(p))$, we obtain

$$\begin{aligned}&\psi_\lambda(u) \geq c_7\|u\|^p - \lambda c_6\|u\| - c_6 \text{ for some } c_7 > 0,\\ \Rightarrow\ &\psi_\lambda(\cdot) \text{ is coercive for all } \lambda > 0.\end{aligned}$$

Using the Sobolev embedding theorem we see that $\psi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_0) = \inf\left\{\psi_\lambda(u) : u \in W^{1,p}(\Omega)\right\}. \tag{13}$$

Let $\hat{m} = \max_{\bar{\Omega}} \bar{u} < 0$ (recall that $\bar{u} \in -\mathrm{int}\, C_+$) On account of hypothesis H_1(iv), given $\varepsilon > 0$ we can find $\delta \in (0, -\hat{m})$ such that

$$F(z,x) \geq \frac{1}{\tau}[\eta_0(z) - \varepsilon]|x|^\tau \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta.$$

So, if $\mu \in (0, \delta)$, then using (11) and (12) we have

$$\begin{aligned}\psi_\lambda(\mu) \leq &\frac{|\mu|^p}{p}\left[\int_\Omega \xi(z)\mathrm{d}z + \int_{\partial\Omega}\beta(z)\mathrm{d}\sigma\right] - \frac{|\mu|^\tau}{\tau}\left[\int_\Omega \eta_0(z)\mathrm{d}z - \varepsilon|\Omega|_N\right]\\ &+ \lambda|\mu|\int_\Omega \hat{\theta}(z)\mathrm{d}z\end{aligned}$$

with $|\cdot|_N$ denoting the Lebesgue measure on $\mathbb{R}^N$.

Since $\int_\Omega \eta_0(z)\mathrm{d}z > 0$ (see hypothesis H_1(iv)), choosing $\varepsilon > 0$ small and using hypothesis H_0, we have that

$$\psi_\lambda(\mu) \leq c_8\left[|\mu|^p + \lambda|\mu|\right] - c_9|\mu|^\tau, \text{ for some } c_8, c_9 > 0. \tag{14}$$

Since $\tau < q < p$, choosing $|\mu| < 1$ small, we have

$$c_9|\mu|^\tau - c_8|\mu|^p \geq d_0 > 0. \tag{15}$$

Then we choose $\hat{\lambda} > 0$ small so that

$$\lambda c_8|\mu| < d_0 \text{ for all } \lambda \in (0, \hat{\lambda}). \tag{16}$$

Using (15), (16) in (14) we infer that

$$\begin{aligned}&\psi_\lambda(\mu) < 0 \text{ for } |\mu| \text{ and } \lambda > 0 \text{ small},\\ \Rightarrow\ &\psi_\lambda(u_0) < 0 = \psi_\lambda(0) \text{ (see (13))}\\ \Rightarrow\ &u_0 \neq 0.\end{aligned}$$

Recall that $\psi_\lambda(\cdot)$ is locally Lipschitz. Hence

$$0 \in \partial\psi_\lambda(u_0),$$

$$\Rightarrow \gamma_p'(u_0) + A_q(u_0) = N_{\hat{f}}(u_0) + l \text{ in } W^{1,p}(\Omega)^* \tag{17}$$

with $N_{\hat{f}}(u)(\cdot) = \hat{f}(\cdot, u(\cdot))$ for all $u \in W^{1,p}(\Omega)$ (the Nemytskii map corresponding to the function $\hat{f}(z,x)$) and $l \in L^{r'}(\Omega)$, $\frac{1}{r} + \frac{1}{r'} = 1$, $l(z) \in \partial_x T_\lambda(z, u_0(z))$ for a.a. $z \in \Omega$ (see Clarke [20], p.80). We know that

$$\partial_x T_\lambda(z,x) = \begin{cases} 0, & \text{if } x < \bar{u}(z),\\ -\lambda\hat{\theta}(z), & \text{if } \bar{u}(z) < x < 0,\\ \{-\lambda\hat{\theta}(z)v : 0 \leq v \leq 1\}, & \text{if } x = \bar{u}(z) \text{ or } x = 0,\\ 0, & \text{if } 0 < x.\end{cases} \tag{18}$$

On (17) we act with $u_0^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}&\gamma_p(u_0^+) + \|Du_0^+\|_q^q \le 0 \ \ \text{(see (11), (18))},\\ \Rightarrow\ & u_0 \le 0, u_0 \ne 0.\end{aligned}$$

Also on (17) we act with $(\bar{u} - u_0)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}&\langle \gamma_p'(u_0), (\bar{u} - u_0)^+\rangle + \langle A_q(u_0), (\bar{u} - u_0)^+\rangle\\ &= \int_\Omega f(z, \bar{u})(\bar{u} - u_0)^+ \mathrm{d}z\\ &\ge \int_\Omega \Big([\theta(z) + \varepsilon]|\bar{u}|^{p-2}\bar{u} - c_1\Big)(\bar{u} - u_0)^+ \mathrm{d}z \ \text{(see (1))}\\ &= \langle \gamma_p'(\bar{u}), (\bar{u} - u_0)^+\rangle + \langle A_q(\bar{u}), (\bar{u} - u_0)^+\rangle \ \text{(see Proposition 4)},\\ &\Rightarrow \bar{u} \le u_0.\end{aligned}$$

We have proved that

$$u_0 \in [\bar{u}, 0], u_0 \ne 0. \tag{19}$$

The nonlinear regularity theory (see Lieberman [30]) implies that $u_0 \in (-C_+)\backslash\{0\}$. Moreover, from (19), (11), (18) and (17) we have

$$\begin{aligned}&\Delta_p(-u_0) + \Delta_q(-u_0) - \xi(z)(-u_0)^{p-1} = f(z, u_0) + l(z) \text{ in } \Omega,\\ \Rightarrow\ &\Delta_p(-u_0) + \Delta_q(-u_0) \le \|\xi\|_\infty(-u_0)^{p-1}\\ &\quad \text{(see hypothesis } H_1\text{(iv) and (18))},\\ \Rightarrow\ &u_0 \in -\operatorname{int} C_+ \ \text{(see Pucci–Serrin [33])}.\end{aligned}$$

Let $\rho = \|u_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(v). We have

$$\begin{aligned}&-\Delta_p\bar{u} - \Delta_q\bar{u} + [\xi(z) + \hat{\xi}_\rho]|\bar{u}|^{p-2}\bar{u}\\ &= [\theta(z) + \varepsilon]|\bar{u}|^{p-2}u - c_1 + \hat{\xi}_\rho|\bar{u}|^{p-2}\bar{u} \ \text{(see Proposition 4)}\\ &\le f(z, \bar{u}) + \hat{\xi}_\rho|\bar{u}|^{p-2}\bar{u} + \lambda\hat{\theta}(z) \ \text{(see (1) and hypotheses } H_0)\\ &\le f(z, u_0) + \hat{\xi}_\rho|u_0|^{p-2}u_0 + \lambda\hat{\theta}(z) \ \text{(see (19) and } H_1\text{(v))}\\ &= -\Delta_p u_0 - \Delta_q u_0 + [\xi(z) + \hat{\xi}_\rho]|u_0|^{p-2}u_0.\end{aligned} \tag{20}$$

Since $\lambda\hat{\theta}(z) \ge \lambda\hat{m} > 0$ ($\hat{m} = \operatorname{ess\,inf}\hat{\theta}$, see H_0), from (20) and Proposition 2.10 of Papageorgiou–Rădulescu–Repovš [35] we infer that

$$\bar{u}(z) < u_0(z) \text{ for all } z \in \Omega.$$

Hence from (18) we infer that $l(z) = \lambda\hat{\theta}(z)$ and so (17) implies that $u_0 \in -\operatorname{int} C_+$ is a solution of (P_λ), $\lambda \in (0, \hat{\lambda})$. Therefore $(0, \hat{\lambda}) \subseteq \mathscr{L} \ne \emptyset$. $\square$

Let S_λ denote the set of nontrivial solutions of (P_λ). From Proposition 2.10 of Papageorgiou–Rădulescu [29] we have that $S_\lambda \subseteq L^\infty(\Omega)$. Then we apply the nonlinear regularity theory of Lieberman [30] and conclude that

$$S_\lambda \subseteq C^1(\bar{\Omega})\backslash\{0\} \text{ for all } \lambda > 0. \tag{21}$$

Next we establish a structural property of the set $\mathscr{L}$, namely we show that $\mathscr{L}$ is connected.

Proposition 6. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathscr{L}$ and $\mu \in (0, \lambda)$, then $\mu \in \mathscr{L}$.*

Proof. Since $\lambda \in \mathscr{L}$ we can find $u_\lambda \in S_\lambda \subset C^1(\bar{\Omega})\backslash\{0\}$ (see (21)). Let $g_\mu : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$g_\mu(z,x) = \begin{cases} f(z,x) + \mu\hat{\theta}(z), & \text{if } x \le u_\lambda(z), \\ f(z,u_\lambda(z)) + \mu\hat{\theta}(z), & \text{if } u_\lambda(z) < x. \end{cases} \tag{22}$$

We set $G_\mu(z,x) = \int_0^x g_\mu(z,s)\mathrm{d}s$ and consider the C^1-functional $\varphi_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\mu(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|Du\|_q^q - \int_\Omega G_\mu(z,u)\mathrm{d}z \text{ for all } u \in W^{1,p}(\Omega).$$

Hypotheses H_1(i), H_1(iii) and (22) imply that given $\varepsilon > 0$, we can find $c_{10} = c_{10}(\varepsilon) > 0$ such that

$$G_\mu(z,x) \le \frac{1}{p}[\theta(z)+\varepsilon]|x|^p + c_{10}|x| \text{ for a.a. } z \in \Omega, \text{all } x \le 0. \tag{23}$$

Also hypothesis H_1(i) and (22) imply that

$$G_\mu(z,x) \le c_{11}x \text{ for a.a. } z \in \Omega, \text{all } x \ge 0, \text{some } c_{11} > 0. \tag{24}$$

Then for $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}
\varphi_\lambda(u) =& \frac{1}{p}\gamma_p(u^+) + \frac{1}{q}\|Du^+\|_q^q - c_{11}\|u^+\|_1 \\
&+ \frac{1}{p}\gamma_p(u^-) + \frac{1}{q}\|Du^-\|_q^q - \frac{1}{p}\int_\omega \theta(z)(u^-)^p\mathrm{d}z \\
&- \frac{\varepsilon}{p}\|u^-\|_p^p - c_{10}\|u^-\|_1 \text{ (see (23) and (24))} \\
\ge& c_0\|u^+\|^p + \frac{1}{p}(\hat{c}-\varepsilon)\|u^-\|^p - c_{12}\|u\| \\
&\text{for soame } c_{12} > 0 \text{ (see (1) and Proposition 1)} \\
\ge& c_{13}\|u\|^p - c_{12}\|u\| \\
&\text{for some } c_{13} > 0 \text{ (choosing } \varepsilon \in (0,\hat{c})), \\
\Rightarrow& \varphi_\lambda(\cdot) \text{ is coercive.}
\end{aligned}$$

Using the Sobolev embedding theorem, we see that $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\varphi_\mu(u_\mu) = \inf\left\{\varphi_\mu(u) : u \in W^{1,p}(\Omega)\right\}, \\
\Rightarrow\ &\varphi_\mu'(u_\mu) = 0, \\
\Rightarrow\ &\langle \gamma_p'(u_\mu), h\rangle + \langle A_q(u_\mu), h\rangle = \int_\Omega g_\mu(z,u_\mu)h\mathrm{d}z
\end{aligned} \tag{25}$$

for all $h \in W^{1,p}(\Omega)$.

We choose $h = (u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle \gamma_p'(u_\mu), (u_\mu - u_\lambda)^+\rangle + \langle A_q(u_\mu), (u_\mu - u_\lambda)^+\rangle \\
=& \int_\Omega \left[f(z,u_\lambda) + \mu\hat{\theta}\right](u_\mu - u_\lambda)^+\mathrm{d}z \text{ (see (22))} \\
\le& \int_\Omega \left[f(z,u_\lambda) + \lambda\hat{\theta}\right](u_\mu - u_\lambda)^+\mathrm{d}z \text{ (since } \mu < \lambda \text{ and } \hat{\theta} \ge 0) \\
=& \langle \gamma_p'(u_\lambda), (u_\mu - u_\lambda)^+\rangle + \langle A_q(u_\lambda), (u_\mu - u_\lambda)^+\rangle \text{ (since } u_\lambda \in S_\lambda), \\
\Rightarrow& u_\mu \le u_\lambda.
\end{aligned}$$

From (22) and (25) it follows that

$$u_\mu \in S_\mu \text{ and so } \mu \in \mathscr{L}.$$

The proof is now complete. □

A byproduct of the above proof is the following corollary.

Corollary 7. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathscr{L}$, $u_\lambda \in S_\lambda$ and $\mu \in (0, \lambda)$, then $\mu \in \mathscr{L}$ and there exists $u_\mu \in S_\mu$ such that $u_\mu \leq u_\lambda$.*

We can improve this corollary as follows:

Proposition 8. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathscr{L}$, $u_\lambda \in S_\lambda$ and $\mu \in (0, \lambda)$, then $\mu \in \mathscr{L}$ and there exists $u_\mu \in S_\mu$ such that $u_\lambda - u_\mu \in D_+$.*

Proof. From Corollary 7 we already know that $\mu \in \mathscr{L}$ and there exists $u_\mu \in S_\mu$ such that

$$u_\mu \leq u_\lambda. \tag{26}$$

Let $\rho = \max\{\|u_\mu\|_\infty, \|u_\lambda\|_\infty\}$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(v). We have

$$\begin{aligned}
&-\Delta_p u_\mu - \Delta_q u_\mu + [\xi(z) + \hat{\xi}_\rho]|u_\mu|^{p-2}u_\mu \\
&= f(z, u_\mu) + \mu\hat{\theta} + \hat{\xi}_\rho|u_\mu|^{p-2}u_\mu \\
&\leq f(z, u_\lambda) + \lambda\hat{\theta} + \hat{\xi}_\rho|u_\lambda|^{p-2}u_\lambda \text{ (see (26), } H_1\text{(v) and since } \mu < \lambda) \\
&= -\Delta_p u_\lambda - \Delta_q u_\lambda + [\xi(z) + \hat{\xi}_\rho]|u_\lambda|^{p-2}u_\lambda \text{ (since } u_\lambda \in S_\lambda).
\end{aligned} \tag{27}$$

The hypothesis on $\hat{\theta}(\cdot)$ (see H_0) implies that

$$0 < (\lambda - \mu)\hat{m} \leq (\lambda - \mu)\hat{\theta} \quad (\hat{m} = \operatorname*{ess\,inf}_{\Omega} \hat{\theta} > 0).$$

So, (27) and Proposition 2.10 of [35], imply that

$$u_\lambda - u_\mu \in D_+.$$

The proof is now complete. $\square$

Let $\lambda^* = \sup \mathscr{L}$.

Proposition 9. *If hypotheses H_0 and H_1 hold, then $\lambda^* < \infty$.*

Proof. Arguing by contradiction, suppose we can find $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathscr{L}$ such that $\lambda_n \to +\infty$. Let $u_n \in S_{\lambda_n}$, $n \in \mathbb{N}$. We have

$$\gamma_p'(u_n) + A_q(u_n) = N_f(u_n) + \lambda_n\hat{\theta} \quad \text{in } W^{1,p}(\Omega)^*, n \in \mathbb{N} \tag{28}$$

(recall $N_f(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in W^{1,p}(\Omega)$). Acting with $-u_n^- \in W^{1,p}(\Omega)$ on (28), we obtain

$$\begin{aligned}
&\gamma_p(u_n^-) + \|Du_n^-\|_q^q \\
&= \int_\Omega f(z, u_n)(-u_n^-)\mathrm{d}z - \lambda_n \int_\Omega \hat{\theta} u_n^- \mathrm{d}z \\
&\leq \int_\Omega f(z, -u_n^-)(-u_n^-)\mathrm{d}z \text{ (recall } \hat{\theta} \geq 0) \\
&\leq \int_\Omega [\theta(z) + \varepsilon](u_n^-)^p \mathrm{d}z + c_1 \int_\Omega u_n^- \mathrm{d}z \quad \text{for all } n \in \mathbb{N} \text{ (see (4))}, \\
&\Rightarrow \gamma_p(u_n^-) - \int_\Omega \theta(z)(u_n^-)^p \mathrm{d}z - \varepsilon\|u_n^-\|^p \leq c_{14}\|u_n^-\| \\
&\qquad \text{for some } c_{14} > 0, \text{all } n \in \mathbb{N}, \\
&\Rightarrow [\hat{c} - \varepsilon]\|u_n^-\|^{p-1} \leq c_{14} \quad \text{for all } n \in \mathbb{N} \quad \text{(see Proposition 1)}.
\end{aligned}$$

Choosing $\varepsilon \in (0, \hat{c})$, we conclude that

$$\{u_n^-\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{29}$$

Suppose that $\|u_n^+\| \to \infty$ and let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{30}$$

From (28) we have

$$\gamma_p'(y_n) = \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} + \frac{\lambda_n}{\|u_n^+\|^{p-1}}\hat{\theta} + v_n^* \tag{31}$$

with $v_n^* \in W^{1,p}(\Omega)^*$, $v_n^* \to 0$ in $W^{1,p}(\Omega)^*$ (see (29)), and $N_f(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in W^{1,p}(\Omega)$ (the Nemytskii operator for $f(z,x)$).

First assume that $f(z,\cdot)$ is $(p-1)$-linear as $x \to +\infty$ (that is, $r = p$ in H_1(i)). In (31) we see that the left hand side is bounded. Note that

$$\left\{\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}}\right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded (see } H_1\text{(i) with } r = p).$$

Therefore,

$$\left\{\frac{\lambda_n}{\|u_n^+\|^{p-1}}\right\}_{n\in\mathbb{N}} \subseteq (0, +\infty) \text{ must be bounded.}$$

Acting on (31) with $y_n - y \in W^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (30), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0, \\ \Rightarrow\ & y_n \to y \text{ in } W^{1,p}(\Omega) \text{ and so } \|y\| = 1, y \geq 0. \end{aligned} \tag{32}$$

From (31) and Proposition 2.10 of Papageorgiou–Rădulescu [29], we have that $\{y_n\}_{n\in\mathbb{N}} \subseteq L^\infty(\Omega)$ is bounded. So, we have that

$$\left\{\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} + \frac{\lambda_n}{\|u_n^+\|^{p-1}}\hat{\theta}\right\}_{n\in\mathbb{N}} \subseteq L^\infty(\Omega) \text{ is bounded.}$$

Therefore, we may assume that

$$\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} + \frac{\lambda_n}{\|u_n^+\|^{p-1}}\hat{\theta} \xrightarrow{w^*} \hat{g}_* \text{ in } L^\infty(\Omega) \tag{33}$$

with $\hat{g}_*(z) = \hat{\eta}(z)y(z)^{p-1} + \hat{b}(z)$, $\hat{\eta}, \hat{b} \in L^\infty(\Omega)$, $\eta(z) \leq \hat{\eta}(z)$ for a.a. $z \in \Omega$ and $\hat{b} = \mu\hat{\theta}$, $\mu \geq 0$. If in (31) we pass to the limit as $n \to \infty$ and use (32) and (33), we have

$$\begin{aligned} &\gamma_p'(y) = \hat{\eta}y^{p-1} + \hat{b} \text{ in } W^{1,p}(\Omega)^*, \\ \Rightarrow &\begin{cases} -\Delta_p y + \xi(z)y^{p-1} = \hat{\eta}(z)y^{p-1} + \hat{b}, & \text{in } \Omega, \\ \frac{\partial y}{\partial n_p} + \beta(z)y^{p-1} = 0, & \text{on } \partial\Omega, \end{cases} \end{aligned} \tag{34}$$

with $\frac{\partial y}{\partial n_p} = |Dy|^{p-2}\frac{\partial y}{\partial n}$. From [29] we have that $y \in L^\infty(\Omega)$ and then the nonlinear regularity theory of Lieberman [30] says that $y \in C_+\backslash\{0\}$. Moreover, form (34) we have

$$\begin{aligned} &\Delta_p y \leq \|\xi\|_\infty y^{p-1} \text{ in } \Omega, \\ \Rightarrow\ & y \in \operatorname{int} C_+ \text{ (by the nonlinear maximum principle, see [2], p.738).} \end{aligned}$$

First suppose $\hat{b} \equiv 0$. From Proposition 2 and since $\eta \leq \hat{\eta}$ we have

$$\tilde{\lambda}_1(p, \hat{\eta}) < \tilde{\lambda}_1(p, \hat{\lambda}_1(p)) = 1.$$

Therefore, from (34), it follows that y is sign changing, a contradiction.

Next suppose $\hat{b} \neq 0$. For $\lambda \in (0,1)$, let $\hat{\eta}_\lambda = \lambda\hat{\eta}$ and consider the following auxiliary Robin problem

$$\begin{cases} -\Delta_p v + \xi(z)|v|^{p-2}v = \hat{\eta}_\lambda |v|^{p-2}v + \hat{b}(z), & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n_p} + \beta(z)|v|^{p-2}v = 0, & \text{on } \partial\Omega, \end{cases} \tag{35}$$

Evidently, $y \in \operatorname{int} C_+$ is an upper solution for (35). Also $v = 0$ is a lower solution of (35). Truncating the reaction of (35) at $\{0, y(z)\}$ and using the direct method of the calculus of variations, we produce $u_\lambda(\cdot)$ a solution of (35) such that $u_\lambda \in [0, y] \cap (C_+ \backslash \{0\})$. But for $\lambda \in (0,1)$ small the antimaximum principle (see Motreanu–Motreanu–Papageorgiou [36, p. 263]), implies that $u_\lambda \in$ -int C_+, a contradiction.

Now we assume that $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to +\infty$, that is, $\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty$ uniformly for a.a. $z \in \Omega$ (see H_1(ii)). In this case, from (31) it follows that $y = 0$,

$$\left\{ \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \right\}_{n\in\mathbb{N}} \subseteq L^{r'}(\Omega) \text{ and } \left\{ \frac{\lambda_n}{\|u_n^+\|^{p-1}} \right\}_{n\in\mathbb{N}} \subseteq (0, +\infty) \text{ are bounded.}$$

Acting with $y_n \in W^{1,p}(\Omega)$ we obtain

$$\begin{aligned} &\gamma_p(y_n) + \frac{1}{\|u_n^+\|^{p-q}} \|Dy_n\|_q^q = \int_\Omega \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} y_n \mathrm{d}z + \frac{\lambda_n}{\|u_n^+\|^{p-1}} \int_\Omega \hat{\theta} y_n \mathrm{d}z, \\ &\Rightarrow \gamma_p(y_n) \leq \varepsilon_n, \text{ for all } n \in \mathbb{N}, \text{ with } \varepsilon_n \to 0^+, \\ &\Rightarrow y_n \to 0 \text{ in } W^{1,p}(\Omega) \text{ (see (1))}, \end{aligned}$$

a contradiction (recall $\|y_n\| = 1$, $n \in \mathbb{N}$).

Therefore in both cases, we have a contradiction and this means that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. This and (29) imply that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. Using this in (28) we have a contradiction to the fact that $\lambda_n \to \infty$. we conclude that $\lambda^* < \infty$. $\square$

Therefore we can say that

$$(0, \lambda^*) \subseteq \mathscr{L} \subseteq (0, \lambda^*]. \tag{36}$$

For $\lambda \in (0, \lambda^*)$ we have a multiplicity result.

Proposition 10. *If hypotheses H_0 and H_1 hold, and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has at least two solutions $u_\lambda, \hat{u}_\lambda \in C^1(\bar{\Omega}) \backslash \{0\}$, $u_\lambda \neq \hat{u}_\lambda$.*

Proof. Let $0 < \mu < \lambda < \nu < \lambda^*$, we know that $\mu, \lambda, \nu \in \mathscr{L}$ (see (36)). On account of Proposition 8, we can find $u_\nu \in S_\nu$, $u_\lambda \in S_\lambda$ and $u_\mu \in S_\mu$ such that

$$\begin{aligned} &u_\nu - u_\lambda \in D_+ \text{ and } u_\lambda - u_\mu \in D_+, \\ &\Rightarrow u_\lambda \in \operatorname{int}_{C^1(\bar{\Omega})}[u_\mu, u_\nu]. \end{aligned} \tag{37}$$

We introduce the Carathéodory functions $\hat{f}(z,x)$ and $\hat{f}_*(z,x)$ defined by

$$\hat{f}(z,x) = \begin{cases} f(z, u_\mu(z)), & \text{if } x \leq u_\mu(z), \\ f(z,x), & \text{if } u_\mu(z) < x, \end{cases} \tag{38}$$

$$\hat{f}_*(z,x) = \begin{cases} \hat{f}(z,x), & \text{if } x \le u_\nu(z), \\ \hat{f}(z,u_\nu(z)), & \text{if } u_\nu(z) < x. \end{cases} \tag{39}$$

We set $\hat{F}(z,x) = \int_0^x \hat{f}(z,s)\mathrm{d}s$ and $\hat{F}_*(z,x) = \int_0^x \hat{f}_*(z,s)\mathrm{d}s$.

Also we introduce the following Lipschitz integrands

$$\hat{T}_\lambda(z,x) = \begin{cases} \lambda\hat{\theta}(z)u_\mu(z), & \text{if } x \le u_\mu(z), \\ \lambda\hat{\theta}(z)x, & \text{if } u_\mu(z) < x, \end{cases} \tag{40}$$

$$\hat{T}^*_\lambda(z,x) = \begin{cases} \hat{T}_\lambda(z,x), & \text{if } x \le u_\nu(z), \\ \hat{T}_\lambda(z,u_\nu(z)), & \text{if } u_\nu(z) < x. \end{cases} \tag{41}$$

We consider the locally Lipschitz functionals $\hat{k}_\lambda, \hat{k}^*_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{k}_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \hat{F}(z,u)\mathrm{d}z - \int_\Omega \hat{T}_\lambda(z,u)\mathrm{d}z$$

$$\hat{k}^*_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \hat{F}_*(z,u)\mathrm{d}z - \int_\Omega \hat{T}^*_\lambda(z,u)\mathrm{d}z$$

for all $u \in W^{1,p}(\Omega)$. Using (38), (39), (40) and (41), we show that

$$K_{\hat{k}_\lambda} \subseteq [u_\mu) \cap C^1(\bar{\Omega}) \text{ and } K_{\hat{k}^*_\lambda} \subseteq [u_\mu, u_\nu] \cap C^1(\bar{\Omega}). \tag{42}$$

From (42) it is clear that we may assume that

$$K_{\hat{k}^*_\lambda} = \{u_\lambda\}. \tag{43}$$

Otherwise we already have a second nontrivial smooth solution (see (38) $\to$ (41) and the proof of Proposition 8) and so we are done.

From (39) and (41), it is clear that $\hat{k}^*_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{k}^*_\lambda(\tilde{u}_\lambda) = \inf\left\{\hat{k}^*_\lambda(u) : u \in W^{1,p}(\Omega)\right\}, \\ &\Rightarrow \tilde{u}_\lambda \in K_{\hat{k}^*_\lambda}, \\ &\Rightarrow \tilde{u}_\lambda = u_\lambda \quad \text{(see (43))}. \end{aligned}$$

Note that

$$\hat{k}_\lambda|_{[u_\mu,u_\nu]} = \hat{k}^*_\lambda|_{[u_\mu,u_\nu]} \quad \text{(see (38) } \to \text{ (41))}.$$

From (37) it follows that

$$\begin{aligned} &u_\lambda \text{ is a local } C^1(\bar{\Omega})\text{-minimizer of } \hat{k}_\lambda(\cdot), \\ \Rightarrow\ &u_\lambda \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \hat{k}_\lambda(\cdot) \end{aligned} \tag{44}$$

(see Bai–Gasinski–Winkert–Zeng [37]).

We assume that $K_{\hat{k}_\lambda}$ is finite. Otherwise, on account of (42) we already have an infinity of nontrivial smooth solutions of (P_λ) and so we are done. Using Theorem 5.7.6, p.449 of Papageorgiou–Rădulescu–Repovš [21], we can find $\rho \in (0,1)$ small such that

$$\hat{k}_\lambda(u_\lambda) < \inf\left\{\hat{k}_\lambda(u) : \|u - u_\lambda\| = \rho\right\} = \hat{m}_\lambda. \tag{45}$$

Let $\hat{u}_1(p) \in W^{1,p}(\Omega)$ be the positive, L^p-normalized (that is, $\|\hat{u}_1(p)\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1(p) > 0$, then $\hat{u}_1(p) \in \operatorname{int} C_+$ and on account of hypothesis H_1(ii) we have

$$\hat{k}_\lambda(t\hat{u}_1(p)) \to -\infty \ \text{ as } t \to +\infty. \tag{46}$$

Claim: $\hat{k}_\lambda(\cdot)$ satisfies the nonsmooth C-condition.

Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ such that

$$\hat{k}_\lambda(u_n) \le c_{15} \text{ for some } c_{15} > 0, \text{all } n \in \mathbb{N}, \tag{47}$$

$$(1 + \|u_n\|)m_{\hat{k}_{\lambda_n}}(u_n) \to 0 \text{ as } n \to \infty. \tag{48}$$

We can find $\hat{u}_n^* \in \partial \hat{k}_\lambda(u_n)$ such that $m_{\hat{k}_{\lambda_n}}(u_n) = \|\hat{u}_n^*\|_*$ for all $n \in \mathbb{N}$. We have

$$\hat{u}_n^* = \gamma_p'(u_n) + A_q(u_n) - N_{\hat{f}}(u_n) - h_n^* \tag{49}$$

with $h_n^*(z) \in \partial T_\lambda(z, u_n(z))$ for a.a. $z \in \Omega$, all $n \in \mathbb{N}$.

From (48) and (49) we have

$$\left| \langle \gamma_p'(u_n), h\rangle + \langle A_q(u_n), h\rangle - \int_\Omega \hat{f}(z, u_n) h \mathrm{d}z - \int_\Omega h_n^* h \mathrm{d}z \right| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{50}$$

for all $h \in W^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$. In (50) we use the test function $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
\gamma_p(u_n^-) + \|Du_n^-\|_q^q &\le \int_\Omega \hat{f}(z, -u_n^-)(-u_n^-)\mathrm{d}z + c_{16} \\
&\qquad \text{for some } c_{16} > 0, \text{all } n \in \mathbb{N} \text{ (see (40))} \\
&\le \int_\Omega [\theta(z) + \varepsilon](u_n^-)^p \mathrm{d}z + c_{17}[1 + \|u_n^-\|] \\
&\qquad \text{for some } c_{17} > 0, \text{all } n \in \mathbb{N} \text{ (see (4))}, \\
\Rightarrow [\hat{c} - \varepsilon]\|u_n^-\|^p &\le c_{17}[1 + \|u_n^-\|] \text{ for all } n \in \mathbb{N} \quad \text{(see Proposition 1)}.
\end{aligned}$$

Choosing $\varepsilon \in (0, \hat{c})$ we infer that

$$\{u_n^-\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{51}$$

Suppose that $\|u_n^+\| \to \infty$ and let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{52}$$

From (50) we have

$$\begin{aligned}
&\langle \gamma_p'(y_n), h\rangle + \frac{1}{\|u_n^+\|^{p-q}} \langle A_q(y_n), h\rangle \\
&\le \varepsilon_n' \|h\| + \int_\Omega \frac{\hat{f}(z, u_n^+)}{\|u_n^+\|^{p-1}} h \mathrm{d}z + \int_\Omega \frac{h_n^*}{\|u_n^+\|^{p-1}} h \mathrm{d}z
\end{aligned} \tag{53}$$

for all $h \in W^{1,p}(\Omega)$, with $\varepsilon_n' \to 0^+$ (see (51)).

First we assume that $f(z, \cdot)$ is $(p-1)$-linear (that is, $r = p$ in H_1(i)). Let $h = y_n - y \in W^{1,p}(\Omega)$ in (53) and pass to the limit as $n \to \infty$. Using (52), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0, \\
&\Rightarrow y_n \to y \text{ in } W^{1,p}(\Omega) \text{ and so } \|y\| = 1, y \ge 0.
\end{aligned} \tag{54}$$

On account of hypothesis H_1(i) (with $r = p$), we have that

$$\left\{ \frac{\hat{f}(\cdot, u_n(\cdot))}{\|u_n^+\|^{p-1}} \right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega), \ \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$$

and so using also hypothesis H_1(ii), we can say that

$$\frac{\hat{f}(\cdot, u_n(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat{\eta}(\cdot)y^{p-1} \text{ in } L^{p'}(\Omega) \tag{55}$$

with $\hat{\eta} \in L^\infty(\Omega)$, $\eta(z) \leq \hat{\eta}(z)$ for a.a. $z \in \Omega$.

Passing to the limit as $n \to \infty$ in (53) and using (54) and (55), we obtain

$$\langle \gamma_p'(y), h\rangle = \int_\Omega \hat{\eta}(z)y^{p-1}h\,dz \text{ for all } h \in W^{1,p}(\Omega),$$

$$\Rightarrow \begin{cases} -\Delta_p y + \xi(z)y^{p-1} = \hat{\eta}(z)y^{p-1}, & \text{in } \Omega, \\ \frac{\partial y}{\partial n_p} + \beta(z)y^{p-1} = 0, & \text{on } \partial\Omega. \end{cases} \tag{56}$$

Using Proposition 2, we have

$$\tilde{\lambda}_1(p, \hat{\eta}) < \tilde{\lambda}_1(p, \hat{\lambda}_1(p)) = 1.$$

So, from (56) it follows that $y(\cdot)$ must be nodal, a contradiction.

Now we assume that $f(z,\cdot)$ is $(p-1)$-superlinear. Then from (47) and (51), we have

$$\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\mathrm{d}z \leq \frac{1}{p}\gamma_p(y_n) + \varepsilon_n'' \quad \text{with } \varepsilon_n'' \to 0^+.$$

Since the right hand side is bounded and

$$\frac{F(\cdot, u_n^+(\cdot))}{\|u_n^+\|^p} = \frac{F(\cdot, u_n^+(\cdot))}{(u_n^+(\cdot))^p}(y_n(\cdot))^p, \ n \in \mathbb{N},$$

we see that $y = 0$. In (53) we choose $h = y_n \in W^{1,p}(\Omega)$ and as before (see the proof of Proposition 9), we have

$$\gamma_p(y_n) \to 0,$$
$$\Rightarrow y_n \to 0 \text{ in } W^{1,p}(\Omega) \quad (\text{see (11)}),$$

a contradiction to the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

So, in both cases we have proved that

$$\{u_n^+\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded},$$
$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded} \quad (\text{see (51)}).$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{57}$$

From (48) and (49), we have

$$\left| \langle \gamma_p'(u_n), h\rangle + \langle A_q(u_n), h\rangle - \int_\Omega \hat{f}(z,u_n)h\mathrm{d}z - \int_\Omega h_n^* h\mathrm{d}z \right| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|}$$

for all $h \in W^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$.

Choosing $h = u_n - u \in W^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (57), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle A_p(u_n), u_n - u\rangle = 0, \\
&\Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 3)}, \\
&\Rightarrow \hat{k}_\lambda(\cdot) \text{ satisfies the nonsmooth } C\text{-condition.}
\end{aligned}$$

This proves the Claim.

Then (45), (46) and the Claim permit the use of the mountain pass theorem. So, we can find $\hat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\hat{u}_\lambda \in K_{\hat{k}_\lambda} \text{ and } \hat{m}_\lambda \le \hat{k}_\lambda(\hat{u}_\lambda), \\
\Rightarrow\ &\hat{u}_\lambda \in \operatorname{int} C_+, \hat{u}_\lambda \ge u_\mu \text{ and } \hat{u}_\lambda \ne u_\lambda \quad \text{(see (42), (45))}.
\end{aligned}$$

Moreover, as in the proof of Proposition 8, using the comparison principle of [35] (Proposition 2.10), we have that

$$\begin{aligned}
&u_\mu(z) < \hat{u}_\lambda(z), \\
\Rightarrow\ &\hat{u}_\lambda \in S_\lambda.
\end{aligned}$$

The proof is now complete. □

Next we show the admissibility of the critical parameter value λ^*.

Proposition 11. *If hypotheses H_0 and H_1 hold, then $\lambda^* \in \mathscr{L}$.*

Proof. Consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0, \lambda^*)$ such that $\lambda_n \uparrow \lambda^*$ and let $u_n \in S_{\lambda_n} \subseteq C^1(\bar{\Omega})\backslash\{0\}$ for all $n \in \mathbb{N}$. We claim that

$$\bar{u} \le u_n \text{ for all } n \in \mathbb{N}. \tag{58}$$

For fixed $n \in \mathbb{N}$, consider the Carathéodory function

$$e(z,x) = \begin{cases} (\theta(z)+\varepsilon)|x^-|^{p-2}(-x^-) - c_1, & \text{if } x \le -u_n^-(z), \\ (\theta(z)+\varepsilon)|u_n^-(z)|^{p-2}(-u_n^-(z)) - c_1, & \text{if } -u_n^-(z) < x. \end{cases} \tag{59}$$

We set $E(z,x) = \int_0^x e(z,s)\mathrm{d}s$ and consider the C^1-functional $\psi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|Du\|_q^q - \int_\Omega E(z,u)\mathrm{d}z \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (59) and using Proposition 1 and (1), we show that $\psi(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\psi(\tilde{u}) = \inf\left\{\psi(u) : u \in W^{1,p}(\Omega)\right\}, \\
\Rightarrow\ &\psi'(\tilde{u}) = 0, \\
\Rightarrow\ &\langle \gamma_p'(\tilde{u}), h\rangle + \langle A_q(\tilde{u}), h\rangle = \int_\Omega e(z,\tilde{u})h\mathrm{d}z
\end{aligned} \tag{60}$$

for all $h \in W^{1,p}(\Omega)$.

In (60) we choose $u = (\tilde{u} - u_n)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle \gamma_p'(\tilde{u}), (\tilde{u}-u_n)^+\rangle + \langle A_q(\tilde{u}), (\tilde{u}-u_n)^+\rangle \\
=&\int_\Omega \left([\theta(z)+\varepsilon]|u_n^-|^{p-2}(-u_n^-) - c_1\right)(\tilde{u}-u_n)^+ \mathrm{d}z \\
&\quad \text{(note that } (\tilde{u}-u_n)^+ \Rightarrow \tilde{u} \geq -u_n^-, \text{ and see (59))} \\
\leq &\int_\Omega \left(f(z,-u_n^-) + \lambda_n \hat{\theta}(z)\right)(\tilde{u}-u_n)^+ \mathrm{d}z \\
&\quad \text{(see (4) and recall that } \hat{\theta} \geq 0) \\
\leq &\int_\Omega \left(f(z,u_n) + \lambda_n \hat{\theta}(z)\right)(\tilde{u}-u_n)^+ \mathrm{d}z \\
&\quad \text{(on account of the sign condition, see } H_1(\text{iv})) \\
=&\langle \gamma_p'(u_n), (\tilde{u}-u_n)^+\rangle + \langle A_q(u_n), (\tilde{u}-u_n)^+\rangle, \\
\Rightarrow &\ \tilde{u} \leq u_n.
\end{aligned} \tag{61}$$

Also choosing $h = \tilde{u}^+ \in W^{1,p}(\Omega)$ in (60), we obtain

$$\begin{aligned}
&\gamma_p(\tilde{u}^+) + \|D\tilde{u}^+\|_q^q \leq 0, \quad \text{(see (59))} \\
\Rightarrow &\ \tilde{u} \leq 0, \tilde{u} \neq 0.
\end{aligned}$$

From (61), we have $u_n^- \leq \tilde{u}^- = -\tilde{u} \Rightarrow \tilde{u} \leq -u_n^-$. Therefore, from (59) and Proposition 4, we infer that

$$\tilde{u} = \bar{u} \leq u_n \quad \text{(see (61))}.$$

This proves (58).

Consider the functional $\hat{k}_{\lambda_n}(\cdot)$ from the proof Proposition 10 but with $u_\mu(\cdot)$ replaced by $\bar{u}(\cdot)$. Then from the proof of Proposition 10, we have

$$\hat{k}_{\lambda_n}(u_n) \leq \hat{k}_{\lambda_n}(\bar{u}) \leq \hat{k}_{\lambda_1}(\bar{u}) = m^*, \tag{62}$$

$$\gamma_p'(u_n) + A_q(u_n) = N_{\hat{k}_{\lambda_n}}(u_n) + h_n^* \tag{63}$$

with $h_n^*(z) \in \partial_x T_\lambda(z, u_n(z))$ for a.a. $z \in \Omega$.

Using (62) and (63) and reasoning as in the Claim in the proof of Proposition 10, we obtain that at least for a subsequence, we have

$$\begin{aligned}
&u_n \to u^* \text{ in } W^{1,p}(\Omega), \\
\Rightarrow &\ u^* \in S_{\lambda^*} \text{ and so } \lambda^* \in \mathscr{L}.
\end{aligned}$$

The proof is now complete. ☐

Finally, we can state the following global existence and multiplicity result for the Ambrosetti–Prodi problem (P_λ).

Theorem 12. *If hypotheses H_0 and H_1 hold, then there exists $\lambda^* > 0$ such that*

(a) for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least two solutions $u_\lambda, \hat{u}_\lambda \in C^1(\bar{\Omega})\backslash\{0\}$, $u_\lambda \neq \hat{u}_\lambda$;
(b) for all $\lambda = \lambda^*$, problem (P_λ) has at least one solution $u^* \in C^1(\bar{\Omega})\backslash\{0\}$;
(c) for all $\lambda > \lambda^*$, problem (P_λ) has no solution.

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