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Multiple solutions with precise sign for nonlinear parametric Robin problems

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Abstract

We consider a parametric nonlinear Robin problem driven by the p-Laplacian. We show that if the parameter $\lambda > \hat{\lambda}_2 =$ the second eigenvalue of the Robin p-Laplacian, then the problem has at least three nontrivial solutions, two of constant sign and the third nodal. In the semilinear case ($p = 2$), we show that we can generate a second nodal solution. Our approach uses variational methods, truncation and perturbation techniques, and Morse theory. In the process we produce two useful remarks about the first two eigenvalues of the Robin p-Laplacian.

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Keywords: Nonlinear regularity; Nonlinear maximum principle; Robin p-Laplacian; Nodal and constant sign solutions; Extremal solutions; Morse theory

Contents

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem:

$$\begin{cases} -\Delta_p u(z) = \lambda |u(z)|^{p-2} u(z) - f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p}(z) + \beta(z)|u(z)|^{p-2} u(z) = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

In this problem, Δ_p $(1 < p < \infty)$ denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}\left(\|Du\|^{p-2} Du\right) \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also, $\frac{\partial u}{\partial n_p} = \|Du\|^{p-2}(Du, n)_{\mathbb{R}^N}$ with $n(z)$ being the outward unit normal at $z \in \partial\Omega$. In addition, $\lambda > 0$ is a parameter and $f(z, x)$ is a Carathéodory perturbation (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z, x)$ is continuous), which exhibits $(p-1)$-superlinear growth near $\pm\infty$.

Our aim in this paper is to prove a multiplicity theorem for problem (P_λ) for all $\lambda > 0$ big. More precisely, we show that, if $\hat{\lambda}_2$ is the second eigenvalue of $-\Delta_p$ with Robin boundary conditions (denoted by $-\Delta_p^R$) and $\lambda > \hat{\lambda}_2$ then problem (P_λ) admits at least three nontrivial solutions, two of constant sign (the first positive and the second negative) and the third solution is nodal (sign changing). Moreover, in the semilinear case ($p = 2$), we show the existence of a second nodal solution, for a total of four nontrivial solutions all with sign information. Our approach uses variational methods coupled with suitable truncation and perturbation techniques and Morse theory.

This kind of problem was studied for semilinear (that is, $p = 2$) Dirichlet equations by Ambrosetti and Lupo [2], Ambrosetti and Mancini [3] and Struwe [20], [21, p. 133]. Extensions to Dirichlet p-Laplacian equations can be found in Papageorgiou and Papageorgiou [18]. However, none of the aforementioned works produced nodal solutions and the hypotheses on the data of the problem are more restrictive. Another class of Robin eigenvalue problems was investigated by Duchateau [7], who proved multiplicity results producing two solutions with no sign information.

2. Mathematical background – auxiliary results

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the Palais–Smale condition (PS-condition for short), if the following is true

> "Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and
>
> $\varphi'(x_n) \to 0$ in X^*admits a strongly convergent subsequence."

This is a compactness type condition, which compensates for the fact that the underlying space X being in general infinite dimensional, need not be locally compact. It leads to the following minimax theorem, known in the literature as the "mountain pass theorem". It characterizes certain critical values of $\varphi \in C^1(X)$.

Theorem 1. *If $\varphi \in C^1(X)$ satisfies the PS-condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > \rho > 0$*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf[\varphi(x): \|x - x_0\| = \rho] = \eta_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$ where $\Gamma = \{\gamma \in C([0,1], X): \gamma(0) = x_0,\ \gamma(1) = x_1\}$, then $c \geqslant \eta_\rho$ and c is a critical value of φ.

In the analysis of problem (P_λ), in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\bar{\Omega})$, which is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\bar{\Omega}): u(z) \geqslant 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+: u(z) > 0 \text{ for all } z \in \bar{\Omega}\}.$$

In the sequel by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, that is,

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

To distinguish, by $|\cdot|$ we denote the norm in $\mathbb{R}^m$ $(m \geqslant 1)$. Also, given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \qquad |u| = u^+ + u^-, \qquad u = u^+ - u^-.$$

If on $\partial\Omega$ we employ the $(N-1)$-dimensional surface (Hausdorff) measure $\sigma(\cdot)$, we can define the Lebesgue space $L^p(\partial\Omega)$. Recall that there is a unique continuous, linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in C^1(\bar{\Omega})$. This map is known as the "trace map". Recall that $\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ and $\ker\gamma_0 = W_0^{1,p}(\Omega)$. In the sequel, for the sake of notational simplicity, we will drop the use of the map γ_0 to denote the restriction of a Sobolev function on $\partial\Omega$. All such restrictions are understood in the sense of traces.

If $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), then we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W^{1,p}(\Omega)$$

(the Nemytskii map corresponding to h).

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega |Du|^{p-2}(Du, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } u, y \in W^{1,p}(\Omega).$$

Proposition 2. *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone, too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W^{1,p}(\Omega)$.*

Suppose that $f_0(z, x)$ is a Carathéodory function with subcritical growth in $x \in \mathbb{R}$, that is,

$$\bigl|f_0(z,x)\bigr| \leqslant a(z)\bigl(1 + |x|^{r-1}\bigr) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p<N, \\ +\infty & \text{otherwise.}\end{cases}$

We set $F_0(z, x) = \int_0^x f_0(z, s)\,ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)\bigl|u(z)\bigr|^p\,d\sigma - \int_\Omega F_0\bigl(z,u(z)\bigr)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

As a consequence of the nonlinear regularity theory (see Lieberman [15]), we show that local $C^1(\bar\Omega)$-minimizers and local $W^{1,p}(\Omega)$-minimizers of φ_0 coincide. The first such result is due to Brezis and Nirenberg [5] for the space $H_0^1(\Omega)$. It was extended to the space $W^{1,p}(\Omega)$ with $\beta \equiv 0$, see Motreanu and Papageorgiou [16] (see also Garcia Azorero, Manfredi and Peral Alonso [11]).

We impose the following conditions on the boundary weight $\beta(\cdot)$:

$$H(\beta): \quad \beta \in C^{0,\tau}(\bar\Omega) \quad \text{with } \tau \in (0,1),\ \beta(z) \geqslant 0 \text{ for all } z \in \bar\Omega,\ \beta \neq 0.$$

Proposition 3. *Assume that $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\bar\Omega)$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in C^1(\bar\Omega) \textit{ with } \|h\|_{C^1(\bar\Omega)} \leqslant \rho_0.$$

Then $u_0 \in C^{1,\alpha}(\bar\Omega)$ for same $\alpha \in (0,1)$ and u_0 is a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\| \leqslant \rho_1.$$

Proof. Let $h \in C^1(\bar\Omega)$ and $t > 0$ small. Then by hypothesis we have

$$\begin{aligned} &\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \\ &\quad\Rightarrow\quad 0 \leqslant \bigl\langle \varphi_0'(u_0), h\bigr\rangle \quad \text{for all } h \in C^1(\bar\Omega) \\ &\quad\Rightarrow\quad \varphi_0'(u_0) = 0 \quad \bigl(\text{since } C^1(\bar\Omega) \text{ is dense in } W^{1,p}(\Omega)\bigr) \\ &\quad\Rightarrow\quad \bigl\langle A(u_0, h)\bigr\rangle + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h\,d\sigma = \int_\Omega f_0(z,u_0)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{2.1}$$

From the nonlinear Green's identity (see, for example, Gasinski and Papageorgiou [12, p. 210]), we have

$$\langle A(u_0), h\rangle = \langle -\Delta_p u_0, h\rangle + \left\langle \frac{\partial u_0}{\partial n_p}, h\right\rangle_{\partial\Omega} \quad \text{for all } h \in W^{1,p}(\Omega) \tag{2.2}$$

(see Gasinski and Papageorgiou [12, p. 211]). Here by $\langle\cdot,\cdot\rangle_{\partial\Omega}$ we denote the duality brackets for the pair $(W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p'},p}(\partial\Omega))(\frac{1}{p}+\frac{1}{p'}=1)$. From the representation theorem for the dual space $W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$ (see Gasinski and Papageorgiou [12, p. 212]), we have $\Delta_p u_0 \in W^{-1,p'}(\Omega)$. Then using $h \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega)$ in (2.1), we have

$$\langle -\Delta_p u_0, h\rangle = \int_\Omega f_0(z, u_0) h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega) \ \big(\text{see (2.2)}\big)$$
$$\Rightarrow \quad -\Delta_p u_0(z) = f_0\big(z, u_0(z)\big) \quad \text{a.e. in } \Omega.$$

Then from (2.1) and (2.2) we have

$$\left\langle \frac{\partial u_0}{\partial n_p} + \beta(z)|u_0|^{p-2}u_0, h\right\rangle_{\partial\Omega} = 0 \quad \text{for all } h \in W^{1,p}(\Omega).$$

Recall that the image of the trace map is $W^{\frac{1}{p'},p}(\partial\Omega)$. So, from this last equality it follows that

$$\frac{\partial u_0}{\partial n_p} + \beta(z)|u_0|^{p-2}u_0 = 0 \quad \text{on } \partial\Omega.$$

From Winkert [24], we know that $u_0 \in L^\infty(\Omega)$. So, we can apply Theorem 2 of Lieberman [15] and have that

$$u_0 \in C^{1,\alpha}(\bar{\Omega}) \quad \text{for some } \alpha \in (0,1).$$

Next we show that u_0 is a local $W^{1,p}(\Omega)$-minimizer of φ_0. We argue by contradiction. So, suppose that u_0 is not a local $W^{1,p}(\Omega)$-minimizer of φ_0. Let $\epsilon > 0$ and consider the set $\bar{B}_\epsilon^r = \{h \in W^{1,p}(\Omega)\colon \|h\|_r \leqslant \epsilon\}$. We have

$$-\infty < m_0^\epsilon = \inf\big[\varphi_0(u_0 + h)\colon h \in \bar{B}_\epsilon^r\big]. \tag{2.3}$$

By virtue of the contradiction hypothesis, we have

$$m_0^\epsilon < \varphi_0(u_0). \tag{2.4}$$

Let $\{h_n\}_{n\geqslant 1} \subseteq \bar{B}_\epsilon^r$ be a minimizing sequence for problem (2.3). Recalling that $u \to \|u\|_r + \|Du\|_p$ is an equivalent norm on $W^{1,p}(\Omega)$ (see [12, p. 227]), we see that $\{h_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$h_n \xrightarrow{w} h_\epsilon \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad h_n \to h_\epsilon \quad \text{in } L^r(\Omega).$$

Clearly φ_0 is sequentially weakly lower semicontinuous. So, we have

$$\begin{aligned}&\varphi_0(u_0+h_\epsilon)\leqslant\liminf_{n\to\infty}\varphi_0(u_0+h_n)\\&\quad\Rightarrow\quad \varphi_0(u_0+h_\epsilon)=m_0^\epsilon\quad\text{and}\quad h_\epsilon\neq 0\quad\big(\text{see (2.4)}\big).\end{aligned}$$

From the Lagrange multiplier rule, we can find $\lambda_\epsilon\leqslant 0$ such that

$$\begin{aligned}&\varphi_0'(u_0+h_\epsilon)=\lambda_\epsilon|h_\epsilon|^{r-2}h_\epsilon\\&\quad\Rightarrow\quad\big\langle A(u_0+h_\epsilon),v\big\rangle+\int_{\partial\Omega}\beta(z)|u_0+h_\epsilon|^{p-2}(u_0+h_\epsilon)v\,d\sigma\\&\qquad\qquad=\int_\Omega f_0(z,u_0+h_\epsilon)v\,dz+\lambda_\epsilon\int_\Omega|h_\epsilon|^{r-2}h_\epsilon v\,dz\quad\text{for all } v\in W^{1,p}(\Omega).\end{aligned}$$

From this equality, as in the first part of the proof, we infer that

$$\left\{\begin{array}{ll}-\Delta_p(u_0+h_\epsilon)(z)=f_0\big(z,(u_0+h_\epsilon)(z)\big)+\lambda_\epsilon\big|h_\epsilon(z)\big|^{r-2}h_\epsilon(z) & \text{a.e. in }\Omega\\ \dfrac{\partial(u_0+h_\epsilon)}{\partial n_p}+\beta(z)|u_0+h_\epsilon|^{p-2}(u_0+h_\epsilon)=0 & \text{on }\partial\Omega\end{array}\right\}. \tag{2.5}$$

Recall that

$$-\Delta_p u_0(z)=f_0\big(z,u_0(z)\big)\quad\text{a.e. in }\Omega,\qquad \frac{\partial u_0}{\partial n_p}+\beta(z)|u_0|^{p-2}u_0=0\quad\text{on }\partial\Omega. \tag{2.6}$$

From (2.5) and (2.6) it follows that

$$\left\{\begin{array}{ll}-\Delta_p(u_0+h_\epsilon)(z)+\Delta_p u_0(z)=f_0\big(z,(u_0+h_\epsilon)(z)\big) & \\ \qquad\qquad -f_0\big(z,u_0(z)\big)+\lambda_\epsilon\big|h_\epsilon(z)\big|^{r-2}h_\epsilon(z) & \text{a.e. in }\Omega\\ \dfrac{\partial(u_0+h_\epsilon)}{\partial n_p}-\dfrac{\partial u_0}{\partial n_p}+\beta(z)\big[|u_0+h_\epsilon|^{p-2}(u_0+h_\epsilon)-|u_0|^{p-2}u_0\big]=0 & \text{on }\partial\Omega\end{array}\right\}. \tag{2.7}$$

We consider two distinct cases:

Case 1. $\lambda_\epsilon\in[-1,1]$ *for all* $\epsilon\in(0,1]$.

Let $v_\epsilon(z)=(u_0+h_\epsilon)(z)$ and set $\sigma_\epsilon(z,y)=\|y\|^{p-2}y+\|Du_0(z)\|^{p-2}Du_0(z)$. Then we can rewrite (2.7) as follows

$$\left\{\begin{array}{ll}-\operatorname{div}\sigma_\epsilon\big(z,Dv_\epsilon(z)\big)=f_0\big(z,v_\epsilon(z)\big)-f_0\big(z,u_0(z)\big) & \\ \qquad\qquad +\lambda_\epsilon\big|(v_\epsilon-u_0)(z)\big|^{r-2}(v_\epsilon-u_0)(z) & \text{a.e. in }\Omega\\ \dfrac{\partial v_\epsilon}{\partial n_p}-\dfrac{\partial u_0}{\partial n_p}+\beta(z)\big[\big|v_\epsilon(z)\big|^{p-2}v_\epsilon(z)-\big|u_0(z)\big|^{p-2}u_0(z)\big]=0 & \end{array}\right\}.$$

Note that $\frac{\partial u_0}{\partial n_p} \in C^{0,\alpha}(\partial\Omega)$ (recall that $u_0 \in C^{1,\alpha}(\bar{\Omega})$) and also on $\mathbb{R}^m (m \geqslant 1)$ the map $y \to |y|^{p-2}y$ is locally Lipschitz if $p > 2$ and Hölder continuous if $1 < p < 2$. From Winkert [24], we know that we can find $M_1 > 0$ such that $\|v_\epsilon\|_\infty \leqslant M_1$ for all $\epsilon \in (0, 1]$. Therefore, we can apply Theorem 2 of Lieberman [15] and find $\gamma \in (0, 1)$, $M_2 > 0$ such that

$$v_\epsilon \in C^{1,\gamma}(\bar{\Omega}), \qquad \|v_\epsilon\|_{C^{1,\gamma}(\bar{\Omega})} \leqslant M_2 \quad \text{for all } \epsilon \in (0, 1].$$

Case 2. $\lambda_{\epsilon_n} < -1$ *for all* $n \geqslant 1$ *with* $\epsilon_n \downarrow 0$.

In this case, we set

$$\hat{\sigma}_{\epsilon_n}(z, y) = \frac{1}{|\lambda_{\epsilon_n}|}\left[|y|^{p-2}y - \left|Du_0(z)\right|^{p-2}Du_0(z)\right]$$

Then we can rewrite (2.7) as follows

$$\left\{\begin{array}{ll} -\operatorname{div}\hat{\sigma}_{\epsilon_n}\big(z, Dv_{\epsilon_n}(z)\big) = \frac{1}{|\lambda_{\epsilon_n}|}\left[f_0\big(z, v_{\epsilon_n}(z)\big) - f_0\big(z, u_0(z)\big)\right] & \\ \qquad\qquad - \left|(v_{\epsilon_n} - u_0)(z)\right|^{r-2}(v_{\epsilon_n} - u_0)(z) & \text{a.e. in } \Omega \\ \frac{\partial v_{\epsilon_n}}{\partial n_p} - \frac{\partial u_0}{\partial n_p} + \beta(z)\left[\left|v_{\epsilon_n}(z)\right|^{p-2}v_{\epsilon_n}(z) - \left|u_0(z)\right|^{p-2}u_0(z)\right] = 0 & \text{on } \partial\Omega \end{array}\right\}. \tag{2.8}$$

Recall that for all $y \in W^{1,p}(\Omega)$, we have

$$\big\langle A(u_0), y\big\rangle + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 y\, d\sigma = \int_{\Omega} f_0(z, u_0) y\, dz, \tag{2.9}$$

$$\begin{aligned} &\big\langle A(v_{\epsilon_n}), y\big\rangle + \int_{\partial\Omega} \beta(z)|v_{\epsilon_n}|^{p-2}v_{\epsilon_n} y\, d\sigma \\ &\quad = \int_{\Omega} f_0(z, v_{\epsilon_n}) y\, dz + \lambda_{\epsilon_n}\int_{\Omega} |v_{\epsilon_n} - u_0|^{r-2}(v_{\epsilon_n} - u_0)\, dz \quad \text{for all } n \geqslant 1. \end{aligned} \tag{2.10}$$

Let $\mu > 1$ and consider the function

$$|v_{\epsilon_n} - u_0|^{\mu}(v_{\epsilon_n} - u_0).$$

We have

$$\begin{aligned} &D\big(|v_{\epsilon_n} - u_0|^{\mu}(v_{\epsilon_n} - u_0)\big) \\ &\quad = |v_{\epsilon_n} - u_0|^{\mu} D(v_{\epsilon_n} - u_0) + \mu(v_{\epsilon_n} - u_0)\frac{v_{\epsilon_n} - u_0}{|v_{\epsilon_n} - u_0|}|v_{\epsilon_n} - u_0|^{\mu-1}D(v_{\epsilon_n} - u_0) \\ &\quad = (\mu + 1)|v_{\epsilon_n} - u_0|^{\mu} D(v_{\epsilon_n} - u_0) \\ &\Rightarrow \quad |v_{\epsilon_n} - u_0|^{\mu}(v_{\epsilon_n} - u_0) \in W^{1,p}(\Omega) \quad \big(\text{recall that } v_{\epsilon_n}, u_0 \in C^1(\bar{\Omega}),\ \text{see (2.7)}\big). \end{aligned}$$

So, we can use $|v_{\epsilon_n} - u_0|^{\mu}(v_{\epsilon_n}, u_0)$ as a test function in (2.9) and (2.10). We have

$$\begin{aligned} 0 &\leqslant \bigl\langle A(v_{\epsilon_n}) - A(u_0), |v_{\epsilon_n} - u_0|^{\mu}(v_{\epsilon_n} - u_0)\bigr\rangle \\ &\quad + \int_{\partial\Omega} \beta(z)\bigl[|v_{\epsilon_n}|^{p-2}v_{\epsilon_n} - |u_0|^{p-2}u_0\bigr]|v_{\epsilon_n} - u_0|^{\mu}(v_{\epsilon_n} - u_0)\,d\sigma \\ &= \int_{\Omega} \bigl[f_0(z, v_{\epsilon_n}) - f_0(z, u_0)\bigr]|v_{\epsilon_n} - u_0|^{\mu}(v_{\epsilon_n} - u_0)\,dz \\ &\quad + \lambda_{\epsilon_n} \int_{\Omega} |v_{\epsilon_n} - u_0|^{r+\mu}\,dz. \end{aligned} \tag{2.11}$$

As before, from Winkert [24], we have $\|v_{\epsilon_n}\|_\infty \leqslant M_3$ for some $M_3 > 0$, all $n \geqslant 1$. Hence

$$\begin{aligned} &\left|\int_{\Omega} \bigl[f_0(z, v_{\epsilon_n}) - f_0(z, u_0)\bigr]|v_{\epsilon_n} - u_0|^{\mu}(v_{\epsilon_n} - u_0)\,dz\right| \\ &\quad\leqslant M_4 \int_{\Omega} |v_{\epsilon_n} - u_0|^{\mu+1}\,dz \quad \text{for some } M_4 > 0,\ \text{all } n \geqslant 1 \\ &\quad\leqslant M_4\, |\Omega|_N^{\frac{r-1}{\mu+r}} \|v_{\epsilon_n} - u_0\|_{\mu+r}^{\mu+1} \end{aligned} \tag{2.12}$$

(here we have used Hölder's inequality with conjugate exponents $\frac{\mu+r}{\mu+1}$, $\frac{\mu+r}{r-1}$).

From (2.11) and (2.12) it follows that

$$\begin{aligned} &-\lambda_{\epsilon_n}\|v_{\epsilon_n} - u_0\|_{\mu+r}^{\mu+r} \leqslant M_4\, |\Omega|_N^{\frac{r-1}{\mu+r}} \|v_{\epsilon_n} - u_0\|_{\mu+r}^{\mu+1} \\ &\Rightarrow\quad |\lambda_{\epsilon_n}|\|v_{\epsilon_n} - u_0\|_{\mu+r}^{r-1} \leqslant M_4|\Omega|_N^{\frac{r-1}{\mu+r}} \quad \text{for all } n \geqslant 1. \end{aligned}$$

Recall that $\mu > 1$ is arbitrary. So, we let $\mu \to +\infty$ and obtain

$$\|h_{\epsilon_n}\|_\infty \leqslant \left[\frac{M_4}{|\lambda_{\epsilon_n}|}\right]^{\frac{1}{r-1}} \quad \text{for all } n \geqslant 1. \tag{2.13}$$

In (2.8) we denote the reaction (right hand side) by $\vartheta_{\epsilon_n}(z, x)$. Using (2.13) we have

$$\bigl|\vartheta_{\epsilon_n}\bigl(z, h_{\epsilon_n}(z)\bigr)\bigr| \leqslant \frac{M_5}{|\lambda_{\epsilon_n}|^{\frac{1}{r-1}}} \leqslant M_5 \quad \text{for all } n \geqslant 1 \text{ and some } M_5 > 0.$$

From this, as before, the nonlinear regularity theory (see [15]) implies the existence of $\gamma_0 \in (0, 1)$, $M_6 > 0$ such that

$$h_{\epsilon_n} \in C^{1,\gamma_0}(\bar{\Omega}) \quad \text{and} \quad \|h_{\epsilon_n}\|_{C^{1,\gamma_0}(\bar{\Omega})} \leqslant M_6 \quad \text{for all } n \geqslant 1.$$

So, in both Case 1 and Case 2, we reached similar uniform bounds for the sequence $\{h_{\epsilon_n}\}_{n\geqslant 1} \subseteq C^{1,\mu}(\bar{\Omega})$ for some $\mu \in (0,1)$. Therefore, exploiting the compact embedding of $C^{1,\mu}(\bar{\Omega})$ into $C^1(\bar{\Omega})$, we may assume that

$$u_0 + h_{\epsilon_n} \to u_0 \quad \text{in } C^1(\bar{\Omega}) \quad \big(\text{recall that } \|h_{\epsilon_n}\|_r \leqslant \epsilon_n \text{ for all } n \geqslant 1\big).$$

But by hypothesis u_0 is a local $C^1(\bar{\Omega})$-minimizer of φ_0. So, we can find $n_0 \geqslant 1$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h_{\epsilon_n}) \quad \text{for all } n \geqslant n_0.$$

On the other hand, from the choice of the $h_n's$ we have

$$\varphi_0(u_0 + h_{\epsilon_n}) < \varphi_0(u_0) \quad \text{for all } n \geqslant 1 \ \big(\text{see (2.4)}\big),$$

a contradiction. This proves that u_0 is also a local $W^{1,p}(\Omega)$ minimizer of φ_0. □

Finally, we recall some basic definitions and facts from Morse theory (critical groups). So, let X be a Banach space and let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Recall that $H_k(Y_1, Y_2) = 0$ for all integers $k < 0$.

Let X be a Banach space and $\varphi \in C^1(X)$, $c \in \mathbb{R}$. We introduce the following sets

$$\varphi^c = \big\{x \in X\colon \varphi(x) \leqslant c\big\}, \qquad K_\varphi = \big\{x \in X\colon \varphi'(x) = 0\big\}, \qquad K_\varphi^c = \big\{x \in K_\varphi\colon \varphi(x) = c\big\}.$$

Then the critical groups of φ at an isolated critical point $x \in X$ with $\varphi(x) = c$, are defined by

$$C_k(\varphi, x) = H_k\big(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{x\}\big) \quad \text{for all } k \geqslant 0.$$

Here U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The excision property of the singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and $\inf \varphi(K_\varphi) > \infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k\big(X, \varphi^c\big) \quad \text{for all } k \geqslant 0.$$

The second deformation theorem (see, for example, Gasinski and Papageorgiou [12, p. 628]), implies that the above definition of critical groups, is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Assume that K_φ is finite and introduce the following items:

$$M(t, x) = \sum_{k\geqslant 0} \operatorname{rank} C_k(\varphi, x) t^k \quad \text{for all } t \in \mathbb{R},\ \text{all } x \in K_\varphi,$$

$$P(t, \infty) = \sum_{k\geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t)Q(t), \tag{2.14}$$

where $Q(t) = \sum_{k \geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Suppose that $X = H$ is a Hilbert space, $x \in H$, U a neighborhood of x and $\varphi \in C^2(U)$. Suppose that $x \in K_\varphi$. Then, the Morse index of $x \in K_\varphi$ is defined to be the supremum of the dimensions of the subspaces of H on which $\varphi''(x)$ is negative definite. The nullity of $x \in K_\varphi$, is the dimension of $\ker \varphi''(x)$. We say that $x \in K_\varphi$ is nondegenerate, if $\varphi''(x)$ is invertible, that is, the nullity of x is zero. If $x \in K_\varphi$ is nondegenerate with Morse index m, then $C_k(\varphi, x) = \delta_{k,m}\,\mathbb{Z}$ for all $k \geqslant 0$, where

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m \end{cases} \quad \text{(the Kronecker symbol).}$$

Suppose that $H = Y \oplus V$ with $\dim Y < +\infty$ and $\varphi \in C^1(H)$. We say that φ admits a local linking at the origin with respect to the decomposition (Y, V), if there exists $\rho > 0$ such that

$$\begin{aligned} \varphi(u) &\leqslant \varphi(0) \quad \text{for all } u \in Y,\ \|u\| \leqslant \rho, \\ \varphi(u) &\geqslant \varphi(0) \quad \text{for all } u \in V,\ \|u\| \leqslant \rho. \end{aligned}$$

3. Some remarks on the spectrum of $-\Delta_p^R$

We consider the following nonlinear eigenvalue problem

$$\left\{ \begin{array}{ll} -\Delta_p u(z) = \lambda \left|u(z)\right|^{p-2} u(z) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p}(z) + \beta(z)\left|u(z)\right|^{p-2} u(z) = 0 & \text{on } \partial\Omega \end{array} \right\}. \tag{3.1}$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\Delta_p^R$, if problem (3.1) admits a nontrivial solution. This eigenvalue problem, was investigated by Le [14], who proved many important facts concerning the first two eigenvalues of $-\Delta_p^R$. Here, we prove two additional results concerning the first two eigenvalues of $-\Delta_p^R$.

We introduce the following quantity

$$\hat{\lambda}_1 = \inf\left[\frac{\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p\, d\sigma}{\|u\|_p^p}:\ u \in W^{1,p}(\Omega),\ u \neq 0\right]. \tag{3.2}$$

This is the first eigenvalue of $-\Delta_p^R$ (see [14]). We also have:

Proposition 4. *If $\beta \in L^\infty(\partial\Omega)\backslash\{0\}$ and $\beta(z) \geqslant 0$ σ-a.e. on $\partial\Omega$, then $\hat{\lambda}_1 > 0$.*

Proof. Evidently $\hat{\lambda}_1 \geqslant 0$. Suppose that $\hat{\lambda}_1 = 0$ and let $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ be such that

$$\|Du_n\|_p^p + \int_{\partial\Omega} \beta(z)|u_n|^p \, d\sigma \to 0^+ \quad \text{as } n \to \infty, \text{ with } \|u_n\|_p = 1 \text{ for all } n \geqslant 1. \tag{3.3}$$

Clearly $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we any assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega). \tag{3.4}$$

From the weak lower semicontinuity of the norm functional in a Banach space, we have

$$\|Du\|_p^p \leqslant \liminf_{n\to\infty} \|Du_n\|_p^p.$$

Moreover, the continuity of the trace map and (3.4), imply that

$$\int_{\partial\Omega} \beta(z)|u_n|^p \, d\sigma \to \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma.$$

Therefore in the limit as $n \to \infty$, we have

$$\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma \leqslant 0 \quad \Rightarrow \quad u \equiv 0,$$

a contradiction to the fact that $\|u\|_p = 1$ (see (3.3), (3.4)). $\square$

From Le [14], we know that $\hat{\lambda}_1 > 0$ is a simple eigenvalue (that is, if u, y are eigenfunctions corresponding to $\hat{\lambda}_1$, then $u = \vartheta y$ for some $\vartheta \in \mathbb{R}\backslash\{0\}$) and it isolated (that is, if $\sigma_R(p)$ denotes the set of eigenvalues of $-\Delta_p^R$, then there exists $\epsilon > 0$ such that $(\hat{\lambda}_1, \hat{\lambda}_1 + \epsilon) \cap \sigma_R(p) = \varnothing$). Let $\hat{u}_1 \in W^{1,p}(\Omega)$ be the L^p-normalized (that is, $\|\hat{u}_1\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1 > 0$. It is clear from (3.2) that $\hat{u}_1$ does not change sign and so we may assume that $\hat{u}_1 \geqslant 0$. If hypothesis $H(\beta)$ holds, then Theorem 2 of Lieberman [15] implies that $\hat{u}_1 \in C_+\backslash\{0\}$. Finally the nonlinear strong maximum principle of Vazquez [23] implies that $\hat{u}_1 \in \operatorname{int} C_+$.

The Ljusternik–Schnirelmann minimax scheme, implies that $-\Delta_p^R$ admits a whole strictly increasing sequence of eigenvalues $\{\hat{\lambda}_k\}_{k\geqslant 1}$ such that $\hat{\lambda}_k \to +\infty$. These eigenvalues are known as the LS-eigenvalues (or variational eigenvalues) of $-\Delta_p^R$. If $p = 2$ (linear eigenvalue problem), then $\sigma_R(p) = \{\hat{\lambda}_k\}_{k\geqslant 1}$. If $p \neq 2$ (nonlinear eigenvalue problem), then we do not know if this is the case. We can easily see that $\sigma_R(p)$ is closed. Since $\hat{\lambda}_1 > 0$ is isolated, we can define

$$\hat{\lambda}_2^* = \inf\bigl[\lambda \in \sigma_R(p) \colon \lambda > \hat{\lambda}_1\bigr].$$

The closedness of $\sigma_R(p)$ implies that $\hat{\lambda}_2^*$ is the second eigenvalue of $-\Delta_p^R$. We have (see Le [14])

$$\hat{\lambda}_2^* = \hat{\lambda}_2,$$

that is, the second eigenvalue and the second LS-eigenvalue of $-\Delta_p^R$ coincide. For $\hat{\lambda}_2$ we have the minimax characterization provided by the Ljusternik–Schnirelmann theory. In the next proposition, we produce an alternative minimax characterization of $\hat{\lambda}_2$, which is more suitable to our purposes. Analogous characterizations for the Dirichlet and Neumann p-Laplacians, were produced by Cuesta, de Figueiredo and Gossez [6] and by Aizicovici, Papageorgiou and Staicu [1] respectively.

Proposition 5. *Assume that hypotheses $H(\beta)$ hold. Then $\hat{\lambda}_2 = \inf_{\hat{\gamma}\in\hat{\Gamma}} \max_{-1\leqslant t\leqslant 1} \varphi(\hat{\gamma}(t))$, where*

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M):\ \hat{\gamma}(-1) = -\hat{u}_1,\ \hat{\gamma}(1) = \hat{u}_1\},$$
$$M = W^{1,p}(\Omega) \cap \partial B_1^{L^p}, \partial B_1^{L^p} = \{u \in L^p(\Omega):\ \|u\|_p = 1\}$$

and

$$\varphi(u) = \|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p\, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proof. By Ljusternik's theorem (see, for example, Papageorgiou and Kyritsi [17, p. 74]), we know that M is a C^1-Banach manifold and

$$T_u M = \left\{h \in W^{1,p}(\Omega):\ \int_\Omega |u|^{p-2}uh\, dz = 0\right\} \quad \text{for all } u \in M$$

(the tangent space to M at u).

Claim 1. *$\varphi|_M$ satisfies the PS-condition.*

Let $\{u_n\}_{n\geqslant 1} \subseteq M$ such that

$$|\varphi(u_n)| \leqslant M_1 \quad \text{for some } M_1 > 0,\ \text{all } n \geqslant 1, \quad \text{and} \tag{3.5}$$

$$\left|\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta|u_n|^{p-2}u_n h\, d\sigma\right| \leqslant \epsilon_n\|h\| \quad \text{for all } h \in T_{u_n}M \text{ with } \epsilon_n \to 0^+. \tag{3.6}$$

Given any $y \in W^{1,p}(\Omega)$, we define

$$h = y - \left(\int_\Omega |u_n|^{p-2}u_n y\, dz\right)u_n.$$

Evidently $h \in T_{u_n}M$ and so we can use it as a test function in (3.6). We have

$$\left|\langle A(u_n), y\rangle - \left(\int_\Omega |u_n|^{p-2}u_n y\, dz\right)\|Du\|_p^p + \int_{\partial\Omega} \beta|u_n|^{p-2}u_n y\, d\sigma\right.$$

$$-\left(\int_{\Omega}|u_n|^{p-2}u_n y\,dz\right)\int_{\partial\Omega}\beta|u_n|^p\,d\sigma\Bigg|\leqslant\epsilon_n\|h\|\quad\text{for all } n\geqslant 1$$

$$\Rightarrow\quad\left|\langle A(u_n),y\rangle+\int_{\partial\Omega}\beta|u_n|^{p-2}u_n y\,d\sigma-\left(\int_{\Omega}|u_n|^{p-2}u_n y\,dz\right)\varphi(u_n)\right|$$

$$\leqslant\epsilon_n c_1\|y\|\text{ for some } c_1>0,\ \text{all } n\geqslant 1\ \big(\text{see Goldberg [13, p. 48]}\big)$$

$$\Rightarrow\quad\left|\langle A(u_n),y\rangle+\int_{\partial\Omega}\beta|u_n|^{p-2}u_n y\,d\sigma\right|\leqslant c_2\|y\|\quad\text{for some } c_2>0,\ \text{all } n\geqslant 1$$

$$\big(\text{see (3.5) and recall } \|u_n\|_p=1 \text{ for all } n\geqslant 1\big).\tag{3.7}$$

From (3.5) and since $\int_{\partial\Omega}\beta|u_n|^p\,d\sigma\geqslant 0$ for all $n\geqslant 1$, we have that $\{Du_n\}_{n\geqslant 1}\subseteq L^p(\Omega,\mathbb{R}^N)$ is bounded. Recall that $\{u_n\}_{n\geqslant 1}\subseteq M$, hence $\|u_n\|_p=1$ for all $n\geqslant 1$. Therefore $\{u_n\}_{n\geqslant 1}\subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n\xrightarrow{w}u\quad\text{in } W^{1,p}(\Omega).$$

Since $y\in W^{1,p}(\Omega)$ is arbitrary, in (3.7) we may choose $y=u_n-u$. We pass to the limit and exploit the continuity of the trace map. We obtain

$$\lim_{n\to\infty}\langle A(y_n),y_n-y\rangle=0\quad\Rightarrow\quad u_n\to u\quad\text{in } W^{1,p}(\Omega).$$

This proves the claim.

Note that

$$\varphi(\pm\hat{u}_1)=\hat{\lambda}_1\quad\text{and both }\pm\hat{u}_1\text{ are local minimizers of }\varphi.$$

From Filippakis, Kristaly and Papageorgiou [10] (see the proof of Proposition 3.2) or from de Figueiredo [9, p. 42], we know that we can find $\rho_\pm\in(0,1)$ such that

$$\varphi(\pm\hat{u}_1)<\inf\big[\varphi(u):\ u\in M,\ \big\|u-(\pm\hat{u}_1)\big\|=\rho_\pm\big],\qquad\rho_\pm<2\|\hat{u}_1\|.\tag{3.8}$$

Let

$$\hat{\lambda}=\inf_{\hat{\gamma}\in\hat{\Gamma}}\max_{-1\leqslant t\leqslant 1}\varphi\big(\hat{\gamma}(t)\big).\tag{3.9}$$

Every path connecting $-\hat{u}_1$ and $\hat{u}_1$ crosses $\partial B_{\rho_\pm}(\pm\hat{u}_1)$ (see (3.8)) and $\varphi(\pm\hat{u}_1)=\hat{\lambda}_1$, from (3.9) we see that $\hat{\lambda}>\hat{\lambda}_1$. It is well-known that $\hat{\lambda}$ is a critical value of $\varphi|_M$, hence an eigenvalue of $-\Delta_p^R$ distinct from $\hat{\lambda}_1$.

Suppose that $\lambda\in(\hat{\lambda}_1,\hat{\lambda})$ is an eigenvalue of $-\Delta_p^R$ with $\hat{u}\in M$ a corresponding eigenfunction. From Le [14], we know that $\hat{u}$ must be nodal (sign changing) and so, we have $\hat{u}^+\neq 0$, $\hat{u}^-\neq 0$. We consider the following two paths in the manifold M

$$\gamma_1(t)=\frac{\hat{u}^+-t\hat{u}^-}{\|\hat{u}^+-t\hat{u}^-\|_p}\quad\text{and}\quad\gamma_2(t)=\frac{-\hat{u}^-+(1-t)\hat{u}^+}{\|-\hat{u}^-+(1-t)\hat{u}^+\|_p}\quad\text{for all } t\in[0,1].\tag{3.10}$$

Note that γ_1 connects $\frac{\hat{u}^+}{\|\hat{u}^+\|_p}$ with $\hat{u}$, while γ_2 connects $\hat{u}$ with $\frac{-\hat{u}^-}{\|\hat{u}^-\|_p}$. So, if concatenate the two paths, we produce a path γ in M connecting $\frac{\hat{u}^+}{\|\hat{u}^+\|_p}$ with $\frac{-\hat{u}^-}{\|\hat{u}^-\|_p}$.

Recall that

$$-\Delta_p \hat{u}(z) = \lambda \left|\hat{u}(z)\right|^{p-2}\hat{u}(z) \quad \text{a.e. in } \Omega, \qquad \frac{\partial \hat{u}}{\partial n_p} + \beta(z)|\hat{u}|^{p-2}\hat{u} = 0 \quad \text{on } \partial\Omega. \tag{3.11}$$

On (3.11) we act with $\hat{u}^+$. Using the nonlinear Green's identity (see, for example Gasinski and Papageorgiou [12, p. 211]), we have

$$\begin{aligned} &\int_\Omega |D\hat{u}|^{p-2}\left(D\hat{u}, D\hat{u}^+\right)_{\mathbb{R}^N} dz - \int_{\partial\Omega} \frac{\partial \hat{u}}{\partial n_p}\hat{u}^+ \, d\sigma = \lambda \left\|\hat{u}^+\right\|_p^p \\ \Rightarrow \quad &\left\|D\hat{u}^+\right\|_p^p + \int_{\partial\Omega} \beta\left(\hat{u}^+\right)^p d\sigma = \lambda\left\|\hat{u}^+\right\|_p^p. \end{aligned} \tag{3.12}$$

Similarly, acting on (3.11) with $-\hat{u}^- \in W^{1,p}(\Omega)$, we obtain

$$\left\|D\hat{u}^-\right\|_p^p + \int_{\partial\Omega} \beta\left(\hat{u}^-\right)^p d\sigma = \lambda\left\|\hat{u}^-\right\|_p^p. \tag{3.13}$$

From (3.10), (3.12), (3.13) and since $\hat{u}^+$ and $\hat{u}^-$ have disjoint interior supports, we have

$$\varphi\big(\gamma_1(t)\big) = \varphi\big(\gamma_2(t)\big) = \lambda \quad \text{for all } t \in [0,1]. \tag{3.14}$$

Let $\hat{L} = \{u \in M\colon\ \varphi(u) < \hat{\lambda}\}$. Since $\hat{u}_1,\ -\hat{u}_1 \in \hat{L}$, this set cannot be path connected or otherwise we violate relation (3.9). Moreover, using the Ekeland variational principle and the fact that $\varphi|_M$ satisfies the PS-condition (see the claim), we see that every path component of $\hat{L}$ contains a critical point of $\varphi|_M$. Since $\pm\hat{u}_1$, are the only critical points of $\varphi|_M$ in $\hat{L}$, we infer that $\hat{L}$ has two path components.

Since $\frac{\hat{u}^+}{\|\hat{u}^+\|_p} \in M \cap (\operatorname{int} C_+)$ and $\varphi(\frac{\hat{u}^+}{\|\hat{u}^+\|_p}) = \lambda$ (see (3.12)), we see that $\frac{\hat{u}^+}{\|\hat{u}^+\|_p}$ cannot be a critical point of $\varphi|_M$. Hence we can find a path $s : [-\epsilon, \epsilon] \to M$ such that

$$s(0) = \frac{\hat{u}^+}{\|\hat{u}^+\|_p} \quad \text{and} \quad \frac{d}{dt}(\varphi|_M)\,\big(s(t)\big) \neq 0 \quad \text{for all } t \in [-\epsilon, \epsilon].$$

Moving along this path, we can start from $\frac{\hat{u}^+}{\|\hat{u}^+\|_p}$ and reach a point $y \in M$ staying in the set $\hat{L}$ with the exception of the starting point $\frac{\hat{u}^+}{\|\hat{u}^+\|_p}$. Let U_1 be the path-component of $\hat{L}$ containing y. Without any loss of generality, we may assume that $\hat{u}_1 \in U_1$. Then y and $\hat{u}_1$ can be connected by a path which stays in U_1. Concatenating this path with s introduced above, we have a path $\gamma_+ : [0,1] \to U_1$ such that

$$\gamma_+(0) = \hat{u}_1, \qquad \gamma_+(1) = \frac{\hat{u}^+}{\|\hat{u}^+\|_p} \quad \text{and} \quad \gamma_+(t) \in \hat{L} \quad \text{for all } t \in [0,1).$$

Similarly, if U_2 is the other path component of $\hat{L}$ containing $-\hat{u}_1$, then we produce a path $\gamma_- : [0,1] \to U_2$ such that

$$\gamma_-(0) = \frac{-\hat{u}^-}{\|\hat{u}^-\|_p}, \qquad \gamma_-(1) = -\hat{u}_1 \quad \text{and} \quad \gamma_-(t) \in \hat{L} \quad \text{for all } t \in (0,1].$$

Finally, we concatenate γ_-, γ, γ_+ and have $\hat{\gamma}_* \in \hat{\Gamma}$ such that

$$\varphi\big(\gamma_*(t)\big) \leqslant \lambda \quad \text{for all } t \in [-1,1] \quad \Rightarrow \quad \hat{\lambda} \leqslant \lambda \quad \big(\text{see (3.9)}\big), \text{ a contradiction.}$$

This means that $(\hat{\lambda}_1, \hat{\lambda}) \cap \sigma_R(\rho) = \varnothing$ and so we conclude that $\hat{\lambda} = \hat{\lambda}_2$. $\square$

4. Nonlinear equations

We introduce the following conditions on the perturbation $f(z,x)$:

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and
- (i) for every $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that $|f(z,x)| \leqslant a_\rho(z)$ for a.a. $z \in \Omega$, all $|x| \leqslant \rho$;
- (ii) $\lim_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) $\lim_{x \to 0} \frac{f(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in \Omega$.

Remark 6. We stress that no global growth restriction is imposed on $f(z,\cdot)$. So, the function $x \mapsto f(z,x)$ can have any growth faster than $|x|^{p-2}x$ near $\pm\infty$.

First we produce two nontrivial constant sign solutions.

Proposition 7. *Assume that hypotheses $H(\beta)$ and H_1 hold and $\lambda > \hat{\lambda}_1$. Then problem (P_λ) has at least two nontrivial constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

Proof. First we produce a nontrivial positive solution.

By virtue of hypothesis H_1(ii), given $\xi > 0$, we can find $M_7 = M_7(\xi) > 0$ such that

$$f(z,x) \geqslant \xi x^{p-1} \quad \text{for a.a. } z \in \Omega \text{ all } x \geqslant M.$$

Since $\hat{u}_1 \in \operatorname{int} C_+$, we can find $t > 0$ big such that $t\hat{u}_1 \geqslant M_7$. Then we have

$$f\big(z, t\hat{u}_1(z)\big) \geqslant \xi\big(t\hat{u}_1(z)\big)^{p-1} \quad \text{a.e. in } \Omega. \tag{4.1}$$

Also, we have

$$-\Delta_p(t\hat{u}_1)(z) = \hat{\lambda}_1(t\hat{u}_1)(z)^{p-1} \quad \text{a.e. in } \Omega, \qquad \frac{\partial(t\hat{u}_1)}{\partial n_p} + \beta(z)(t\hat{u}_1)^p = 0 \quad \text{on } \partial\Omega.$$

Then for every $h \in W^{1,p}(\Omega)$, $h \geqslant 0$, we have

$$\begin{aligned}
&\langle -\Delta_p(t\hat{u}_1), h\rangle = \int_\Omega \hat{\lambda}_1 (t\hat{u}_1)^{p-1} h\,dz \\
&\Rightarrow \quad \langle A(t\hat{u}_1), h\rangle - \left\langle \frac{\partial(t\hat{u}_1)}{\partial n_p}, h\right\rangle_{\partial\Omega} = \int_\Omega \hat{\lambda}_1 (t\hat{u}_1)^{p-1} h\,dz \\
&\qquad \big(\text{by the nonlinear Green's identity, see [12, p. 211]}\big) \\
&\Rightarrow \quad \langle A(t\hat{u}_1), h\rangle + \int_{\partial\Omega} \beta(z)(t\hat{u}_1)^{p-1} h\,d\sigma = \int_\Omega \hat{\lambda}_1 (t\hat{u}_1)^{p-1} h\,dz.
\end{aligned} \tag{4.2}$$

Choosing $\xi = \lambda - \hat{\lambda}_1 > 0$, from (4.1) and (4.2), we have

$$\begin{aligned}
&\int_\Omega \big[\lambda (t\hat{u}_1)^{p-1} - f(z, t\hat{u}_1)\big] h\,dz \\
&\quad \leqslant \int_\Omega \hat{\lambda}_1 (t\hat{u}_1)^{p-1} h\,dz \\
&\quad = \langle A(t\hat{u}_1), h\rangle + \int_{\partial\Omega} \beta(z)(t\hat{u}_1)^{p-1} h\,d\sigma \quad \text{for all } h \in W^{1,p}(\Omega),\ h \geqslant 0.
\end{aligned} \tag{4.3}$$

Setting $\bar{u} = t\hat{u}_1 \in \operatorname{int} C_+$, we introduce the following truncation–perturbation of the reaction in problem (P_λ)

$$h_\lambda^+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ (\lambda+1)x^{p-1} - f(z,x) & \text{if } 0 \leqslant x \leqslant \bar{u}(z), \\ (\lambda+1)\bar{u}(z)^{p-1} - f(z,\bar{u}(z)) & \text{if } \bar{u}(z) < x. \end{cases} \tag{4.4}$$

This is a Carathéodory function. We set $H_\lambda^+(z,x) = \int_0^x h_\lambda^+(z,s)\,ds$ and consider the C^1-functional $\Psi_\lambda^+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi_\lambda^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)\big|u(z)\big|^p\,d\sigma - \int_\Omega H_\lambda^+\big(z, u(z)\big)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (4.4) it is clear that Ψ_λ^+ is coercive. Also, using the Sobolev embedding theorem and the continuity of the trace map, we see that Ψ_λ^+ is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\Psi_\lambda^+(u_0) = \inf\big[\Psi_\lambda^+(u):\ u \in W^{1,p}(\Omega)\big]. \tag{4.5}$$

By virtue of hypothesis H_1(iii) given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) \in (0, \min_{\bar{\Omega}} \bar{u}]$ such that

$$F(z,x) \leqslant \frac{\epsilon}{p}|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{4.6}$$

Choose $\vartheta \in (0,1)$ small such that $\vartheta \hat{u}_1(z) \in (0,\delta]$ for all $z \in \bar{\Omega}$. Then

$$\begin{aligned}
\Psi_\lambda^+(\vartheta \hat{u}_1) &= \frac{\vartheta^p}{p}\|D\hat{u}_1\|_p^p + \frac{\vartheta^p}{p}\int_{\partial\Omega} \beta(z)|\hat{u}_1|^p d\sigma - \frac{\lambda\vartheta^p}{p}\|\hat{u}_1\|_p^p + \int_\Omega F(z, t\hat{u}_1)\,dz \quad \big(\text{see (4.4)}\big) \\
&\leqslant \frac{\vartheta^p}{p}\big[(\hat{\lambda}_1 + \epsilon) - \lambda\big] \quad \big(\text{see (4.6) and recall that } \|\hat{u}_1\|_p = 1\big).
\end{aligned}$$

Choosing $\epsilon \in (0, \lambda - \hat{\lambda}_1)$ (recall $\lambda > \hat{\lambda}_1$), we have

$$\Psi_\lambda^+(\vartheta \hat{u}_1) < 0 \quad \Rightarrow \quad \Psi_\lambda^+(u_0) < 0 = \Psi_\lambda^+(0) \quad \big(\text{see (4.5)}\big), \text{ hence } u_0 \neq 0.$$

From (4.5) we have

$$\begin{aligned}
&\big(\Psi_\lambda^+\big)'(u_0) = 0 \\
&\quad\Rightarrow \quad \big\langle A(u_0), v\big\rangle + \int_\Omega |u_0|^{p-2}u_0 v\,dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0\, v\,d\sigma \\
&\qquad\qquad = \int_\Omega h_\lambda^+(z,u_0)\, v\,dz \quad \text{for all } v \in W^{1,p}(\Omega).
\end{aligned} \tag{4.7}$$

In (4.7) first we choose $v = -u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\big\|Du_0^-\big\|_p^p + \big\|u_0^-\big\|_p^p \leqslant 0 \quad \big(\text{see (4.4) and } H(\beta)\big) \\
&\quad\Rightarrow \quad u_0 \geqslant 0, \ u_0 \neq 0.
\end{aligned}$$

Next in (4.7) we choose $v = (u_0 - \bar{u})^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\big\langle A(u_0), (u_0-\bar{u})^+\big\rangle + \int_\Omega u_0^{p-1}(u_0-\bar{u})^+ dz + \int_{\partial\Omega} \beta(z) u_0^{p-1}(u_0-\bar{u})^+\,d\sigma \\
&\quad = \int_\Omega \big[(\lambda+1)\bar{u}^{p-1} - f(z,\bar{u})\big](u_0-\bar{u})^+ dz \quad \big(\text{see (4.4)}\big) \\
&\quad \leqslant \big\langle A(\bar{u}), (v_0-\bar{u})^+\big\rangle + \int_\Omega \bar{u}^{p-1}(u_0-\bar{u})^+\,dz + \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}(u_0-\bar{u})^+\,d\sigma \\
&\quad \big(\text{see (4.3)}\big) \\
&\Rightarrow \quad \big\langle A(u_0) - A(\bar{u}), (u_0-\bar{u})^+\big\rangle + \int_\Omega \big(u_0^{p-1} - \bar{u}^{p-1}\big)(u_0-\bar{u})^+ dz
\end{aligned}$$

$$+\int_{\partial\Omega} \beta(z)\big[u_0^{p-1} - \bar{u}^{p-1}\big](u_0 - \bar{u})^+ \, d\sigma \leqslant 0$$
$$\Rightarrow \quad \big|\{u_0 > \bar{u}\}\big|_N = 0, \quad \text{hence } u_0 \leqslant \bar{u}.$$

So, we have proved that

$$u_0 \in [0, \bar{u}] = \big\{u \in W^{1,p}(\Omega)\colon\ 0 \leqslant u(z) \leqslant \bar{u}(z) \text{ a.e. in } \Omega\big\}, \quad u_0 \neq 0.$$

Therefore from (4.4) and (4.7), we have

$$\big\langle A(u_0), h\big\rangle + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h \, d\sigma = \int_{\Omega} \big[\lambda u_0^{p-1} - f(z, u_0)\big] h \, dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

As before (see the proof of Proposition 3), via the nonlinear Green's identity we have

$$-\Delta_p u_0(z) = \lambda u_0(z)^{p-1} - f\big(z, u_0(z)\big) \quad \text{a.e. in } \Omega, \qquad \frac{\partial u_0}{\partial n_p} + \beta(z)u_0^{p-1} = 0 \quad \text{on } \partial\Omega$$
$$\Rightarrow \quad u_0 \quad \text{is a nontrivial positive solution of problem } (P_\lambda).$$

The nonlinear regularity theory, implies that $u_0 \in C_+ \backslash \{0\}$. Hypotheses H_1(i), (iii) imply that we can find $c_3 > 0$ such that

$$f(z, x) \leqslant c_3 x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \big[0, \|\bar{u}\|_\infty\big].$$

Then

$$-\Delta_p u_0(z) \geqslant -f\big(z, u_0(z)\big) \geqslant -c_3 u_0(z)^{p-1} \quad \text{a.e. in } \Omega$$
$$\Rightarrow \quad \Delta_p u_0(z) \leqslant c_3 u_0(z)^{p-1} \quad \text{a.e. in } \Omega$$
$$\Rightarrow \quad u_0 \in \operatorname{int} C_+ \quad \big(\text{see Vazquez [23]}\big).$$

Similarly, we produce a nontrivial negative solution $v_0 \in -\operatorname{int} C_+$. Using this time $\bar{v} = -\hat{t}\hat{u}_1$ for $\hat{t} > 0$ big, for which we have

$$\big\langle A(\bar{v}), h\big\rangle + \int_{\partial\Omega} \beta(z)|\bar{v}|^{p-2}\bar{v} h \, d\sigma \leqslant \int_{\Omega} \big[\lambda|\bar{v}|^{p-2}\bar{v} - f(z, \bar{v})\big] h \, dz$$
$$\text{for all } h \in W^{1,p}(\Omega), \ h \geqslant 0.$$

Truncating and perturbing the reaction of (P_λ) at $\{\bar{v}(z), 0\}$, as above we produce $v_0 \in [\bar{v}, 0] \cap (-\operatorname{int} C_+)$, a solution of (P_λ), $\lambda > \hat{\lambda}_1$. □

In fact, we can produce extremal nontrivial constant sign solutions for problem (P_λ), $\lambda > \hat{\lambda}_1$, that is, there exist $u_* \in \operatorname{int} C_+$ the smallest nontrivial positive solution of (P_λ) and $v_* \in -\operatorname{int} C_+$ the biggest nontrivial negative solution of (P_λ).

For $\lambda > \hat{\lambda}_1$ we define

$$S_+(\lambda) = \{u \in W^{1,p}(\Omega)\colon\ u \neq 0,\ u \in [0, \bar{u}],\ u \text{ is a solution of } (P_\lambda)\},$$
$$S_-(\lambda) = \{v \in W^{1,p}(\Omega)\colon\ v \neq 0,\ v \in [\bar{v}, 0],\ v \text{ is a solution of } (P_\lambda)\}.$$

From Proposition 7 and its proof we have

$$\varnothing \neq S_+(\lambda) \subseteq \operatorname{int} C_+ \quad \text{and} \quad \varnothing \neq S_-(\lambda) \subseteq \operatorname{int} C_+.$$

Moreover, as in Filippakis, Kristaly and Papageorgiou [10], we have that the set of nontrivial positive (resp. negative) solutions of (P_λ) is downward directed, that is, if $u, \hat{u}$ are nontrivial positive solutions of (P_λ), then there exists y a nontrivial positive solution of (P_λ) such that $y \leqslant u,\ y \leqslant \hat{u}$ (resp. upward directed, that is, if $v, \hat{v}$ are nontrivial negative solutions of (P_λ), there exists w a nontrivial negative solution of (P_λ), such that $v \leqslant w,\ \hat{v} \leqslant w$).

Proposition 8. *Assume that hypotheses $H(\beta)$ and H_1 hold and $\lambda > \hat{\lambda}_1$. Then problem (P_λ) admits smallest nontrivial positive solution $u_*^\lambda \in \operatorname{int} C_+$ and a biggest nontrivial negative solution $v_*^\lambda \in \operatorname{int} C_+$.*

Proof. We consider a chain $C \subseteq S_+(\lambda)$ (that is, a totally ordered subset of $S_+(\lambda)$). Then from Dunford and Schwartz [8, p. 336], we know that we can find $\{u_n\}_{n\geqslant 1} \subseteq C$ such that $\inf C = \inf_{n\geqslant 1} u_n$. We have

$$-\Delta_p u_n(z) = \lambda u_n(z)^{p-1} - f(z, u_n(z)) \quad \text{a.e. in } \Omega, \qquad \frac{\partial u_n}{\partial n_p} + \beta(z) u_n^{p-1} = 0 \quad \text{on } \partial\Omega.$$

Using the nonlinear Green's identify, we obtain

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\, d\sigma = \int_\Omega \lambda u_n^{p-1} h\, dz - \int_\Omega f(z, u_n) h\, dz$$
$$\text{for all } h \in W^{1,p}(\Omega). \tag{4.8}$$

We choose $h = u_n \in W^{1,p}(\Omega)$. Then

$$\|Du_n\|_p^p \leqslant M_8 \quad \text{for some } M_8 > 0,\ \text{all } n \geqslant 1$$
$$\Rightarrow \quad \{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded (recall } 0 \leqslant u_n \leqslant \bar{u} \text{ for all } n \geqslant 1).$$

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega).$$

In (4.8) we choose $h = u_n - u \in W^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$. Using the continuity of the trace map (hence $u_n|_{\partial\Omega} \xrightarrow{w} u|_{\partial\Omega}$ in $L^p(\partial\Omega)$), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\langle A(u_n), u_n - u\rangle = 0\\
&\quad\Rightarrow\quad u_n \to u \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2)}\\
&\quad\Rightarrow\quad \langle A(u), h\rangle + \int_{\partial\Omega} \beta(z)u^{p-1}h\,d\sigma = \int_{\Omega} \lambda u^{p-1}h\,dz - \int_{\Omega} f(z,u)h\,d\sigma\\
&\qquad\qquad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned}$$

Hence u is a positive solution of (P_λ) So, if we show that $u \neq 0$, then $u \in S_+(\lambda)$. Arguing by contradiction, suppose that $u = 0$ and let $y_n = \frac{u_n}{\|u_n\|} n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega).$$

From (4.8) we have

$$\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z)y_n^{p-1}h\,d\sigma = \int_{\Omega} \lambda y_n^{p-1}h\,dz - \int_{\Omega} \frac{N_f(u_n)}{\|u_n\|^{p-1}}h\,dz. \tag{4.9}$$

By virtue of hypotheses H_1(i), (iii), we have that

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} 0 \quad \text{in } L^{p'}(\Omega) \left(\frac{1}{p} + \frac{1}{p'} = 1\right). \tag{4.10}$$

So, if in (4.9) we choose $h = y_n - y$ and pass to the limit as $n \to \infty$, then

$$\begin{aligned}
&\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0\\
&\quad\Rightarrow\quad y_n \to y \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2)}, \text{ hence } \|y\| = 1,\ y \geqslant 0.
\end{aligned} \tag{4.11}$$

If in (4.9) we pass to the limit as $n \to \infty$ and use (4.10) and (4.11), then

$$\begin{aligned}
&\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z)y^{p-1}h\,d\sigma = \int_{\Omega} \lambda y^{p-1}h\,dz \quad \text{for all } h \in W^{1,p}(\Omega)\\
&\Rightarrow\quad -\Delta_p y(z) = \lambda y(z)^{p-1} \quad \text{a.e. in } \Omega, \qquad \frac{\partial y}{\partial n_p} + \beta(z)y^{p-1} = 0 \quad \text{on } \partial\Omega\\
&\qquad \text{(see the proof of Proposition 3)}.
\end{aligned}$$

Since $\lambda > \hat{\lambda}_1$, $y = 0$ or y is a nodal, a contradiction to (4.11). Therefore $u \neq 0$ and so

$$u \in C_+(\lambda) \quad \text{and} \quad u = \inf C.$$

Because $C \subseteq S_+(\lambda)$ is an arbitrary chain, invoking the Kuratowski–Zorn lemma, we can find $u_*^\lambda \in S_+(\lambda) \subseteq \operatorname{int} C_+$ a minimal element. If u is a nontrivial positive solution, then we know that we can find $\tilde{u} \in S_+(\lambda)$ such that $\tilde{u} \leqslant u_*^\lambda$, $\tilde{u} \leqslant u$. The minimality of u_*^λ implies that $\tilde{u} = u_*^\lambda$ and so $u_*^\lambda \in \operatorname{int} C_+$ is the smallest nontrivial positive solution of (P_λ).

Similarly, working with $S_-(\lambda) \subseteq -\operatorname{int} C_+$ and using again the Kuratowski–Zorn lemma, we produce $v_*^\lambda \in -\operatorname{int} C_+$ the biggest nontrivial negative solution of (P_λ). $\square$

These extremal nontrivial constant sign solutions, will lead to a nodal (sign changing solution). To this end, fix $\lambda > \hat{\lambda}_1$ and let

$$\eta = \max\{\|u_*^\lambda\|_\infty, \|v_*^\lambda\|_\infty\}.$$

Hypotheses H_1(i), (iii) imply that we can find $\xi > 0$ such that

$$(\lambda + \xi)x^{p-1} \geqslant f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\eta], \tag{4.12}$$

$$f(z,x) \geqslant (\lambda + \xi)|x|^{p-2}x \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [-\eta, 0]. \tag{4.13}$$

From (4.12) and (4.13), after integration, we obtain

$$F(z,x) \geqslant \frac{\lambda + \xi}{p}|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leqslant \eta. \tag{4.14}$$

We introduce the following Carathéodory functions

$$k_\lambda^+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ (\lambda+\xi)x^{p-1} - f(z,x) & \text{if } 0 \leqslant x \leqslant u_*^\lambda(z), \\ (\lambda+\xi)u_*^\lambda(z)^{p-1} - f(z,u_*^\lambda(z)) & \text{if } u_*^\lambda(z) < x, \end{cases} \tag{4.15}$$

$$k_\lambda^-(z,x) = \begin{cases} (\lambda+\xi)|v_*^\lambda(z)|^{p-2}v_*^\lambda(z) - f(z,v_*^\lambda(z)) & \text{if } x < v_*^\lambda(z), \\ (\lambda+\xi)|x|^{p-2}x - f(z,x) & \text{if } v_*^\lambda(z) \leqslant x \leqslant 0, \\ 0 & \text{if } 0 < x, \end{cases} \tag{4.16}$$

$$k_\lambda(z,x) = \begin{cases} (\lambda+\xi)|v_*^\lambda(z)|^{p-2}v_*^\lambda(z) - f(z,v_*^\lambda(z)) & \text{if } x < v_*^\lambda(z), \\ (\lambda+\xi)|x|^{p-2}x - f(z,x) & \text{if } v_*^\lambda(z) \leqslant x \leqslant u_*^\lambda(z), \\ (\lambda+\xi)u_*^\lambda(z)^{p-1} - f(z,u_*^\lambda(z)) & \text{if } u_*^\lambda(z) < x. \end{cases} \tag{4.17}$$

We set $K_\lambda^\pm(z,x) = \int_0^x k_\lambda^\pm(z,s)\,ds$, $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\,ds$ and consider the C^1-functionals $\hat{\varphi}_\lambda^\pm$, $\hat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda^\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{\xi}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \int_\Omega K_\lambda^\pm(z,u)\,dz,$$

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{\xi}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \int_\Omega K_\lambda(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 9. *Assume that hypotheses $H(\beta)$ and H_1 hold and $\lambda > \hat{\lambda}_1$. Then $K_{\hat{\varphi}_\lambda} \subseteq [v_*^\lambda, u_*^\lambda]$, $K_{\hat{\varphi}_\lambda^+} = \{0, u_\lambda^*\}$, $K_{\hat{\varphi}_\lambda^-} = \{v_\lambda^*, 0\}$.*

Proof. Let $u \in K_{\hat{\varphi}_\lambda}$. Then

$$\hat{\varphi}_\lambda'(u) = 0 \quad \Rightarrow \quad \langle A(u), h\rangle + \int_\Omega \xi |u|^{p-2} uh\,dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh\,d\sigma = \int_\Omega k_\lambda(z,u)h\,dz$$

for all $h \in W^{1,p}(\Omega)$.

First we choose $h = (u - u_*^\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\bigl\langle A(u), \bigl(u-u_*^\lambda\bigr)^+\bigr\rangle + \int_\Omega \xi u^{p-1}\bigl(u-u_*^\lambda\bigr)^+\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}\bigl(u-u_*^\lambda\bigr)^+\,d\sigma$$
$$= \int_\Omega \bigl[(\lambda+\xi)\bigl(u_*^\lambda\bigr)^{p-1} - f\bigl(z,u_*^\lambda\bigr)\bigr]\bigl(u-u_*^\lambda\bigr)^+\,dz \quad \bigl(\text{see (4.17)}\bigr). \tag{4.18}$$

Recall that

$$-\Delta_p u_*^\lambda(z) = \lambda u_*^\lambda(z)^{p-1} - f\bigl(z,u_*^\lambda(z)\bigr) \quad \text{a.e. in } \Omega, \qquad \frac{\partial u_*^\lambda}{\partial n_p} + \beta(z)\bigl(u_*^\lambda\bigr)^{p-1} = 0 \quad \text{on } \partial\Omega$$
$$\Rightarrow \quad \bigl\langle A\bigl(u_*^\lambda\bigr),\ \bigl(u-u_*^\lambda\bigr)^+\bigr\rangle + \int_{\partial\Omega} \beta(z)\bigl(u_*^\lambda\bigr)^{p-1}\bigl(u-u_*^\lambda\bigr)^+\,d\sigma$$
$$= \int_\Omega \bigl[\lambda\bigl(u_*^\lambda\bigr)^{p-1} - f\bigl(z,u_*^\lambda\bigr)\bigr]\bigl(u-u_*^\lambda\bigr)^+ dz. \tag{4.19}$$

From (4.18) and (4.19) it follows that

$$\bigl\langle A(u) - A\bigl(u_*^\lambda\bigr),\ \bigl(u-u_*^\lambda\bigr)^+\bigr\rangle + \xi\int_\Omega \bigl(u^{p-1} - \bigl(u_*^\lambda\bigr)^{p-1}\bigr)\bigl(u-u_*^\lambda\bigr)^+ dz$$
$$+ \int_{\partial\Omega} \beta(z)\bigl(u^{p-1} - \bigl(u_*^\lambda\bigr)^{p-1}\bigr)\bigl(u-u_*^\lambda\bigr)^+ d\sigma = 0$$
$$\Rightarrow \quad \bigl|\bigl\{u > u_*^\lambda\bigr\}\bigr|_N = 0, \quad \text{hence } u \leqslant u_*^\lambda.$$

Similarly, using the test function $(v_*^\lambda - u)^+ \in W^{1,p}(\Omega)$, we show that $v_*^\lambda \leqslant u$. So, we have proved that

$$u \in \bigl[v_*^\lambda, u_*^\lambda\bigr] = \bigl\{y \in W^{1,p}(\Omega)\colon\ v_*^\lambda(z) \leqslant y(z) \leqslant u_*^\lambda(z) \text{ a.e. in } \Omega\bigr\}$$
$$\Rightarrow \quad K_{\hat{\psi}_\lambda} \subseteq \bigl[v_*^\lambda, u_*^\lambda\bigr].$$

In a similar fashion, using (4.15) (resp. (4.16)), we show that

$$K_{\hat{\varphi}_\lambda^+} \subseteq \bigl[0, u_*^\lambda\bigr] = \bigl\{y \in W^{1,p}(\Omega)\colon\ 0 \leqslant y(z) \leqslant u_*^\lambda(z) \text{ a.e. in } \Omega\bigr\}$$
$$\bigl(\text{resp. } K_{\hat{\varphi}_\lambda^-} \subseteq \bigl[v_*^\lambda, 0\bigr] = \bigl\{y \in W^{1,p}(\Omega)\colon\ v_*^\lambda(z) \leqslant y(z) \leqslant 0 \text{ a.e. in } \Omega\bigr\}\bigr).$$

The extremality of $u_*^\lambda \in \operatorname{int} C_+$ and $v_*^\lambda \in -\operatorname{int} C_+$ (see Proposition 8), implies that

$$K_{\hat{\varphi}_\lambda^+} = \{0, u_*^\lambda\} \quad \text{and} \quad K_{\hat{\varphi}_\lambda^-} = \{v_*^\lambda, 0\}. \qquad \square$$

Proposition 10. *Assume that hypotheses $H(\beta)$ and H_1 hold and $\lambda > \hat{\lambda}_1$. Then $u_*^\lambda \in \operatorname{int} C_+$ and $v_*^\lambda \in -\operatorname{int} C_+$ are both local minimizers of $\hat{\varphi}_\lambda$.*

Proof. It is clear from (4.15) that $\hat{\varphi}_\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_*^\lambda \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda^+\big(\tilde{u}_*^\lambda\big) = \inf\big[\hat{\varphi}_\lambda^+(u):\ u \in W^{1,p}(\Omega)\big]. \tag{4.20}$$

As before (see the proof of Proposition 7), for $\vartheta \in (0,1)$ small we have

$$\hat{\varphi}_\lambda^+(\vartheta \hat{u}_1) < 0 \quad \Rightarrow \quad \hat{\varphi}_\lambda^+\big(\tilde{u}_*^\lambda\big) < 0 = \hat{\varphi}_\lambda^+(0) \quad \big(\text{see (4.20)}\big), \ \text{hence } \tilde{u}_*^\lambda \neq 0.$$

Since $\tilde{u}_*^\lambda \in K_{\hat{\varphi}_\lambda^+} \setminus \{0\}$, from Proposition 9 it follows $\tilde{u}_*^\lambda = u_*^\lambda \in \operatorname{int} C_+$. Note that

$$\begin{aligned}
&\hat{\varphi}_\lambda|_{C_+} = \hat{\varphi}_\lambda^+|_{C_+} \quad \big(\text{see (4.15), (4.17)}\big)\\
&\quad \Rightarrow \quad u_*^\lambda \in \operatorname{int} C_+ \text{ is a local } C^1(\bar{\Omega})\text{-minimizer } \hat{\varphi}_\lambda\\
&\quad \Rightarrow \quad u_*^\lambda \in \operatorname{int} C_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer } \hat{\varphi}_\lambda \quad (\text{see Proposition 3}).
\end{aligned}$$

Similarly for $v_*^\lambda \in -\operatorname{int} C_+$, using this time the functional $\hat{\varphi}_\lambda^-$ and (4.16). $\square$

To produce a nodal solution, we need to restrict further the range of the parameter λ.

Proposition 11. *Assume that hypotheses $H(\beta)$ and H_1 hold and $\lambda > \hat{\lambda}_2$. Then problem (P_λ) admits solution $y_\lambda \in [v_*^\lambda, u_*^\lambda] \cap C^1(\bar{\Omega})$.*

Proof. Let $u_*^\lambda \in \operatorname{int} C_+$ and $v_*^\lambda \in \operatorname{int} C_+$ be the two extremal nontrivial constant sign solutions of problem (P_λ) produced in Proposition 8. Without any loss of generality, we may assume that $\hat{\varphi}_\lambda(v_*^\lambda) \leqslant \hat{\varphi}_\lambda(u_*^\lambda)$ (the analysis is similar if the opposite inequality holds). From Proposition 10, we know that $u_*^\lambda \in \operatorname{int} C_+$ is a local minimizer of $\hat{\varphi}_\lambda$. So, we can find $\rho \in (0,1)$ small such that

$$\hat{\varphi}_\lambda\big(v_*^\lambda\big) \leqslant \hat{\varphi}_\lambda\big(u_*^\lambda\big) < \inf\big[\hat{\varphi}_\lambda(u):\ \big\|u - u_*^\lambda\big\| = \rho\big] = \eta_\rho^\lambda, \qquad \big\|v_*^\lambda - u_*^\lambda\big\| > \rho. \tag{4.21}$$

Recall that $\hat{\varphi}_\lambda$ is coercive (see (4.17)), hence it satisfies the PS-condition. This fact and (4.21) permit the use of Theorem 1 (the mountain pass theorem). So, there exists $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\hat{\varphi}_\lambda} \subseteq \big[v_*^\lambda, u_*^\lambda\big] \quad (\text{see Proposition 9}) \quad \text{and} \quad \eta_\rho^\lambda \leqslant \hat{\varphi}_\lambda(y_\lambda) \quad \big(\text{see (4.21)}\big). \tag{4.22}$$

From (4.21) and (4.22) it follows that $y_\lambda \notin \{v_*^\lambda, u_*^\lambda\}$ and it solves problem (P_λ) see (4.17), hence $y_\lambda \in C^1(\bar{\Omega})$ (nonlinear regularity theory).

We need to show that $y_\lambda \neq 0$ and then by virtue of the extremality of the solutions u_*^λ and v_*^λ, we will have that y_λ is nodal. From the mountain pass theorem, we have

$$\hat{\varphi}_\lambda(y_0) = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \hat{\varphi}_\lambda\big(\gamma(t)\big),$$

where $\Gamma = \{\gamma \in C([0,1], W^{1,p}(\Omega)):\ \gamma(0) = v_*^\lambda,\ \gamma(1) = u_*^\lambda\}$.

Recall that $M = W^{1,p}(\Omega) \cap \partial B^{L^p}$ (see Proposition 5). Let $M_c = M \cap C^1(\bar{\Omega})$. We consider the following two sets of paths

$$\hat{\Gamma} = \big\{\hat{\gamma} \in C\big([-1,1], M\big):\ \hat{\gamma}(-1) = -\hat{u}_1,\ \hat{\gamma}(1) = \hat{u}_1\big\} \quad \text{(see Proposition 5)},$$
$$\hat{\Gamma}_c = \big\{\gamma \in C\big([-1,1], M_c\big):\ \hat{\gamma}(-1) = -\hat{u}_1,\ \hat{\gamma}(1) = \hat{u}_1\big\}.$$

From Papageorgiou and Rădulescu [19], we know that $\hat{\Gamma}_c$ is dense in $\hat{\Gamma}$. Since $u_*^\lambda \in \operatorname{int} C_+,\ v_*^\lambda \in -\operatorname{int} C_+$, we have

$$m_0 = \min\Big\{\min_{\bar{\Omega}} u_*^\lambda,\ \min_{\bar{\Omega}}\big(-v_*^\lambda\big)\Big\} > 0.$$

Hypothesis H(iii) implies that given $\epsilon > 0$, we can find $\delta \in (0, m_0)$ such that

$$\big|F(z,x)\big| \leqslant \frac{\epsilon}{p}|x|^p \quad \text{for a.a. } z \in \Omega,\ \text{all } |x| \leqslant \delta \tag{4.23}$$

(recall $F(z,x) = \int_0^x f(z,s)\,ds$). From (4.17) and (4.23) we have

$$K_\lambda(z,x) = \frac{\lambda+\xi}{p}|x|^p - F(z,x) \geqslant \frac{\lambda+\xi-\epsilon}{p}|x|^p \quad \text{for a.a. } z \in \Omega,\ \text{all } |x| \leqslant \delta.$$

From Proposition 5 and the density of $\hat{\Gamma}_c$ in $\hat{\Gamma}$, we see that given $\epsilon \in (0, \frac{\lambda-\hat{\lambda}_2}{2})$ (recall that $\lambda > \hat{\lambda}_2$), we can find $\hat{\gamma}_0 \in \hat{\Gamma}_c$ such that

$$\varphi\big(\hat{\gamma}_0(t)\big) \leqslant \hat{\lambda}_2 + \epsilon \quad \text{for all } t \in [-1,1]. \tag{4.24}$$

Recall that $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ is defined by

$$\varphi(u) = \|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma \quad \text{for all } u \in W^{1,p}(\Omega)$$

(see Proposition 5). Evidently $\hat{\gamma}_0([-1,1]) \subseteq C^1(\bar{\Omega})$ and recall that $u_*^\lambda \in \operatorname{int} C_+$, $v_*^\lambda \in -\operatorname{int} C_+$. So, we can find $\tau \in (0,1)$ small such that for all $u \in \hat{\gamma}_0([-1,1])$ we have

$$\big|\tau u(z)\big| \leqslant \delta \quad \text{for all } z \in \bar{\Omega} \text{ and } \tau u \in \big[v_*^\lambda, u_*^\lambda\big]. \tag{4.25}$$

Then for every $u \in \hat{\gamma}_0([-1,1])$ we have

$$\begin{aligned}\hat{\varphi}_\lambda(\tau u) &= \frac{\tau^p}{p}\|Du\|_p^p + \frac{\xi\tau^p}{p}\|u\|_p^p + \frac{\tau p}{p}\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma - \int_\Omega K_\lambda(z,\tau u)\,dz \\ &\leqslant \frac{\tau^p}{p}[\hat{\lambda}_2+\epsilon] - \frac{\tau^p}{p}[\lambda-\epsilon] \quad \big(\text{see (4.17), (4.23), (4.24)}\big) \\ &= \frac{\tau^p}{p}[\hat{\lambda}_2+2\epsilon-\lambda] < 0 \quad \left(\text{recall that } \epsilon < \frac{\lambda-\hat{\lambda}_2}{2}\right).\end{aligned}$$

So, if we set $\gamma_0 = \tau\hat{\gamma}_0$, then γ_0 is a continuous path in $W^{1,p}(\Omega)$ which connects $-\tau\hat{u}_1$ and $\tau\hat{u}_1$ and such that

$$\hat{\varphi}_\lambda|_{\gamma_0} < 0. \tag{4.26}$$

Recall that $\hat{\varphi}_\lambda^+(u_*^\lambda) < 0 = \hat{\varphi}_\lambda^+(0)$ and $K_{\hat{\varphi}_\lambda^+} = \{0, u_*^\lambda\}$ (see Propositions 9, 10 and the proof of the latter). Applying the second deformation theorem (see, for example, Gasinski and Papageorgiou [12, p. 628]), we produce a deformation $h : [0,1] \times ((\hat{\varphi}_\lambda^+)^0 \backslash \{0\}) \to (\hat{\varphi}_\lambda^+)^0$ such that

$$h\big(1, \big(\hat{\varphi}_\lambda^+\big)^0 \backslash \{0\}\big) = u_*^\lambda, \tag{4.27}$$

$$\hat{\varphi}_\lambda^+\big(h(t,u)\big) \leqslant \hat{\varphi}_\lambda^+(u) \quad \text{for all } t \in [0,1]. \tag{4.28}$$

Let $\gamma_+(t) = h(t, \tau\hat{u}_1)^+$ for all $t \in [0,1]$. Then γ_+ is a continuous path in $W^{1,p}(\Omega)$ such that $\gamma_+(0) = \tau\hat{u}_1$ (h is a deformation), $\gamma_+(1) = u_*^\lambda$ (see (4.27) and recall $u_*^\lambda \in \operatorname{int} C_+$) and

$$\hat{\varphi}_\lambda^+|_{\gamma_+} < 0 \quad \big(\text{see (4.28) and (4.26)}\big).$$

Since $\operatorname{im}\gamma_+ \subseteq W_+ = \{u \in W^{1,p}(\Omega)\colon\ u(z) \geqslant 0 \text{ a.e. in } \Omega\}$ and $\hat{\varphi}_\lambda^+|_{W_+} = \hat{\varphi}_\lambda|_{W_+}$ (see (4.15), (4.17)), we have

$$\hat{\varphi}_\lambda|_{\gamma_+} < 0. \tag{4.29}$$

In a similar fashion, using this time the functional $\hat{\varphi}_\lambda^-$, we produce another continuous path γ_- in $W^{1,p}(\Omega)$ which connects $-\tau\hat{u}_1$ and v_*^λ and such that

$$\hat{\varphi}|_{\gamma_-} < 0. \tag{4.30}$$

Concatenating $\gamma_-,\ \gamma_0,\ \gamma_+$, we produce a path $\gamma_* \in \Gamma$ such that

$$\begin{aligned}&\hat{\varphi}_\lambda|_{\gamma_*} < 0 \quad \big(\text{see (4.26), (4.29), (4.30)}\big) \\ &\quad \Rightarrow \quad y_\lambda \neq 0 \quad \text{and so } y_\lambda \in C^1(\bar{\Omega}) \text{ is nodal solution of } (P_\lambda). \qquad \square\end{aligned} \tag{4.31}$$

So, we can state the following multiplicity theorem for problem (P_λ) $(\lambda > \hat{\lambda}_2)$.

Theorem 12. *Assume that hypotheses $H(\beta)$ and H_1 hold. Then for every $\lambda > \hat{\lambda}_2$ problem (P_λ) has at least three nontrivial solutions $u_0^\lambda \in \operatorname{int} C_+$, $v_0^\lambda \in -\operatorname{int} C_+$, and $y_\lambda \in [v_0^\lambda, u_0^\lambda] \cap C^1(\bar{\Omega})$ nodal.*

5. Semilinear problems

In this section, we deal with the semilinear problem (that is, $p = 2$). So, the problem under consideration is the following:

$$-\Delta u(z) = \lambda u(z) - f\big(z, u(z)\big) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \tag{S_λ}$$

For this problem, under additional regularity conditions on $f(z, \cdot)$ and with a global growth restriction this time, we show that for all $\lambda > \hat{\lambda}_2$ problem (S_λ) admits a second nodal solution, for a total of four nontrivial solutions all with sign information.

The new hypotheses on the perturbation $f(z, x)$ are the following:

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and
- (i) $|f_x'(z, x)| \leqslant a(z)(1 + |x|^{r-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $2 < r < 2^*$;
- (ii) $\lim_{x \to \pm\infty} \frac{f(z,x)}{x} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) $f_x'(z, 0) = \lim_{x \to 0} \frac{f(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$;
- (iv) there exists $\delta > 0$ such that $f(z, x)x \geqslant 0$ for a.a. $z \in \Omega$, all $|x| \leqslant \delta$.

Remark 13. It is clear that hypothesis H_2(i) implies that given $\rho > 0$, we can find $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto (\lambda + \xi)x - f(z, x)$ is nondecreasing on $[-\rho, \rho]$.

We have the following multiplicity theorem for problem (S_λ).

Theorem 14. *Assume that hypotheses $H(\beta)$ and H_2 hold. Then for every $\lambda > \hat{\lambda}_2$ problem (S_λ) has at least four nontrivial solutions*

$$u_0^\lambda \in \operatorname{int} C_+, \qquad v_0^\lambda \in -\operatorname{int} C_+$$

and $y_\lambda, \hat{y}_\lambda \in \operatorname{int}_{C^1(\bar{\Omega})}[v_0^\lambda, u_0^\lambda]$ nodal.

Proof. From Theorem 12, we already have three nontrivial solutions

$$u_0^\lambda \in \operatorname{int} C_+, \qquad v_0^\lambda \in -\operatorname{int} C_+ \quad \text{and} \quad y_\lambda \in \big[v_0^\lambda, u_0^\lambda\big] \cap C^1(\bar{\Omega}) \text{ nodal}.$$

Without any loss of generality, we may assume that u_0^λ and v_0^λ are extremal (see Proposition 8), that is, $u_0^\lambda = u_*^\lambda \in \operatorname{int} C_+$. Let $\rho = \max\{\|u_0^\lambda\|_\infty, \|v_0^\lambda\|_\infty\}$ and let $\xi_\rho > 0$ be such that for a.a. $z \in \Omega$ $x \to (\lambda + \xi_\rho)x - f(z, x)$ is nondecreasing on $[-\rho, \rho]$. Then

$$-\Delta y_\lambda(z) + \xi_\rho y_\lambda(z) = (\lambda + \xi_\rho) y_\lambda(z) - f\big(z, y_\lambda(z)\big)$$

$$\begin{aligned}
&\leqslant (\lambda+\xi_\rho)u_0^\lambda(z) - f\big(z, u_0^\lambda(z)\big) \quad \big(\text{since } y_\lambda \leqslant u_0^\lambda\big) \\
&= -\Delta u_0^\lambda(z) + \xi_\rho u_0^\lambda(z) \quad \text{a.e. in } \Omega \\
\Rightarrow \quad & \Delta\big(u_0^\lambda - y_\lambda\big)(z) \leqslant \xi_\rho\big(u_0^\lambda - y_\lambda\big)(z) \quad \text{a.e. in } \Omega \\
\Rightarrow \quad & u_0^\lambda - y_\lambda \in \operatorname{int} C_+ \quad \big(\text{see Vazquez [23]}\big).
\end{aligned}$$

Similarly, we show that

$$y_\lambda - v_0^\lambda \in \operatorname{int} C_+.$$

Therefore, we have

$$y_\lambda \in \operatorname{int}_{C^1(\bar{\Omega})}\big[v_0^\lambda, u_0^\lambda\big]. \tag{5.1}$$

Next let $\sigma_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the functional defined by

$$\sigma_\lambda(u) = \frac{1}{2}\|Du\|_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)u^2\,d\sigma - \frac{\lambda}{2}\|u\|_2^2 + \int_\Omega F(z,u)\,dz \quad \text{for all } u \in H^1(\Omega)$$

(recall $F(z,x) = \int_0^x f(z,s)\,ds$). Evidently $\sigma_\lambda \in C^2(H^1(\Omega))$.

We consider the following orthogonal direct sum decomposition of $H^1(\Omega)$

$$H^1(\Omega) = \bar{H} \oplus E(\hat{\lambda}_k) \oplus \hat{H}$$

with $k \geqslant 3$, $\bar{H} = \bigoplus_{i=1}^{k-1} E(\hat{\lambda}_i)$, $\hat{H} = \overline{\bigoplus_{i\geqslant k+1} E(\hat{\lambda}_i)}$. Set $Y = E(\hat{\lambda}_k) \oplus \hat{H}$.

Recall that $\lambda > \hat{\lambda}_2$. First we assume that $\lambda \in \sigma_R$ (2) (problem resonant at zero). Then $\lambda = \hat{\lambda}_k$ for some $k \geqslant 3$.

Claim 2. *The energy functional σ_λ admits a local linking at $u = 0$, with respect to the orthogonal direct sum*

$$H^1(\Omega) = \bar{H} \oplus Y.$$

By virtue of hypothesis H_2(iii), given $\epsilon > 0$, we can find $\delta_0 = \delta_0(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\epsilon}{2}x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta_0. \tag{5.2}$$

Since $\bar{H}$ is finite dimensional, all norms are equivalent and so we can find $\rho_1 = \rho_1(\epsilon) > 0$ such that

$$\text{“}\|u\| \leqslant \rho_1 \quad \Rightarrow \quad \big|u(z)\big| \leqslant \delta_0 \quad \text{for all } z \in \bar{\Omega}\text{”} \quad \text{for all } u \in \bar{H}. \tag{5.3}$$

Therefore, for $u \in \bar{H}$ with $\|u\| \leqslant \rho_1$, we have

$$\begin{aligned}\sigma_\lambda(u) &= \frac{1}{2}\|Du\|_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)u^2\,d\sigma - \frac{\hat{\lambda}_k}{2}\|u\|_2^2 + \int_\Omega F(z,u)\,dz \\ &\leqslant \frac{\hat{\lambda}_{k-1} - \hat{\lambda}_k}{2}\|u\|_2^2 + \frac{\epsilon}{2}\|u\|_2^2 \quad \big(\text{see (5.2), (5.3)}\big) \\ &\leqslant -c_4\|u\|^2 \quad \text{for some } c_4 > 0 \ \big(\text{choosing } \epsilon \in (0, \hat{\lambda}_k - \hat{\lambda}_{k-1})\big).\end{aligned}$$

So, we have proved that

$$\sigma_\lambda(u) \leqslant 0 \quad \text{for all } u \in \bar{H} \text{ with } \|u\| \leqslant \rho_1. \tag{5.4}$$

Next let $u \in Y$. Then we have $u = u^0 + \hat{u}$ with $u^0 \in E(\hat{\lambda}_k)$, $\hat{u} \in \hat{H}$. Hence

$$\begin{aligned}\sigma_\lambda(u) &= \frac{1}{2}\|Du\|_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)u^2\,d\sigma - \frac{\hat{\lambda}_k}{2}\|u\|_2^2 + \int_\Omega F(z,u)\,dz \\ &\geqslant \frac{c_5}{2}\|\hat{u}\|^2 + \int_{\{|u|\leqslant\delta\}} F(z,u)\,dz + \int_{\{|u|>\delta\}} F(z,u)\,dz \quad \text{for some } c_5 > 0 \\ &\qquad \big(\text{exploiting the orthogonality of the component spaces } E(\hat{\lambda}_k), \hat{H}\big) \\ &\geqslant \frac{c_5}{2}\|\hat{u}\|^2 + \int_{\{|u|>\delta\}} F(z,u)\,dz \quad \big(\text{see } H_2(\text{iv})\big).\end{aligned} \tag{5.5}$$

Since $E(\hat{\lambda}_k)$ is finite dimensional, we can find $\rho_2 > 0$ small such that

$$\text{“}\big\|u^0\big\| \leqslant \rho_2 \quad \Rightarrow \quad \big|u^0(z)\big| \leqslant \frac{\delta}{2} \quad \text{for all } z \in \bar{\Omega}\text{”} \quad \text{for all } u^0 \in E(\hat{\lambda}_k).$$

So, if $C_\delta = \{z \in \Omega\colon |u(z)| \leqslant \delta\}$, then for $u \in Y$ with $\|u\| \leqslant \rho_2$, we have $\|u^0\| \leqslant \rho_2$ hence $|u^0(z)| \leqslant \frac{\delta}{2}$ for all $z \in \bar{\Omega}$. Therefore, for $u \in Y$ with $\|u\| \leqslant \rho_2$, we have

$$\big|\hat{u}(z)\big| \geqslant \big|u(z)\big| - \big|u^0(z)\big| \geqslant \big|u(z)\big| - \frac{\delta}{2} \geqslant \frac{1}{2}\big|u(z)\big| \quad \text{a.e. on } C_\delta. \tag{5.6}$$

Moreover, it is clear from hypotheses H_2(i), (iii) that gives $\epsilon > 0$, we can find $c_6 = c_6(\epsilon) > 0$ such that

$$F(z,x) \geqslant -\frac{\epsilon}{2}x^2 - c_6|x|^r \quad \text{for a.a } z \in \Omega, \ \text{all } x \in \mathbb{R}, \ \text{with } 2 < r. \tag{5.7}$$

Then

$$\int_{C_\delta} F(z,u)\,dz \geqslant -\frac{\epsilon}{2}\int_{C_\delta} u^2\,dz - c_6\int_{C_\delta} |u|^r\,dz \quad \big(\text{see (5.7)}\big)$$

$$\begin{aligned} &\geqslant -\epsilon \int_{C_\delta} \hat{u}^2\,dz - c_7\|\hat{u}\|^r \quad \text{for some } c_7 > 0 \ \big(\text{see (5.6)}\big) \\ &\geqslant -\epsilon\|\hat{u}\|^2 - c_7\|\hat{u}\|^r. \end{aligned} \tag{5.8}$$

We return to (5.5) and use (5.8). Then

$$\sigma_\lambda(u) \geqslant \left(\frac{c_5}{2} - \epsilon\right)\|\hat{u}\|^2 - c_7\|\hat{u}\|^r.$$

Choosing $\epsilon \in (0, \frac{c_5}{2})$, we have

$$\sigma_\lambda(u) \geqslant c_8\|\hat{u}\|^2 - c_7\|\hat{u}\|^r \quad \text{for some } c_8 > 0.$$

Since $r > 2$, choosing $\rho_2 \in (0,1)$ small, for $u \in Y$ with $\|u\| \leqslant \rho_2$, we have $\|\hat{u}\| \leqslant \rho_2$ and so

$$\sigma_\lambda(u) \geqslant 0 \quad \text{for all } u \in Y \text{ with } \|u\| \leqslant \rho_2. \tag{5.9}$$

Let $\rho = \min\{\rho_1, \rho_2\}$. Then from (5.4), (5.9) we infer that σ_λ admits a local linking at $u = 0$ with respect to the orthogonal direct sum decomposition $H^1(\Omega) = \bar{H} \oplus Y$. This proves the claim.

Then by virtue of the claim and Proposition 2.3 of Su [22], we have

$$C_i(\sigma_\lambda, 0) = \delta_{i,d_k}\mathbb{Z} \quad \text{for all } i \geqslant 0,\ \text{with } d_k = \dim \bar{H} \geqslant 2. \tag{5.10}$$

Note that $\hat{\varphi}_\lambda|_{[v_0^\lambda, u_0^\lambda]} = \sigma_\lambda|_{[v_0^\lambda, u_0^\lambda]}$ (see (4.17)) and recall that $u_0^\lambda \in \operatorname{int} C_+$, $v_0^\lambda \in -\operatorname{int} C_+$. Hence

$$\begin{aligned} & C_i(\hat{\varphi}_\lambda|_{C^1(\bar{\Omega})}, 0) = C_i(\sigma_\lambda|_{C^1(\bar{\Omega})}, 0) \quad \text{for all } i \geqslant 0 \\ \Rightarrow\quad & C_i(\hat{\varphi}_\lambda, 0) = C_i(\sigma_\lambda, 0) = \delta_{i,d_k}\mathbb{Z}, \quad \text{for all } i \geqslant 0 \ \big(\text{see (5.10) and Bartsch [4]}\big). \end{aligned} \tag{5.11}$$

From Proposition 9, we have

$$c_i\big(\hat{\varphi}_\lambda, u_0^\lambda\big) = c_i\big(\hat{\varphi}_\lambda, v_0^\lambda\big) = \delta_{i,0}\,\mathbb{Z} \quad \text{for all } i \geqslant 0. \tag{5.12}$$

From the proof of Proposition 10, we know that y_λ is a critical point of mountain pass type the functional $\hat{\varphi}_\lambda$. Hence

$$\begin{aligned} & C_1(\hat{\varphi}_\lambda, y_\lambda) \neq 0 \\ \Rightarrow\quad & C_1(\sigma_\lambda, y_\lambda) \neq 0 \quad \big(\text{see (5.1) and recall } \hat{\varphi}_\lambda|_{[v_0^\lambda, u_0^\lambda]} = \sigma_\lambda|_{[v_0^\lambda, u_0^\lambda]}\big) \\ \Rightarrow\quad & C_i(\sigma_\lambda, y_\lambda) = \delta_{i,1}\mathbb{Z} \quad \text{for all } i \geqslant 0 \ \big(\text{see Bartsch [4]}\big) \\ \Rightarrow\quad & C_i(\varphi_\lambda, y_\lambda) = \delta_{i,1}\mathbb{Z} \quad \text{for all } i \geqslant 0 \ \big(\text{see (5.11)}\big). \end{aligned} \tag{5.13}$$

Finally, recall that $\hat{\varphi}_\lambda$ is coercive (see (4.17)). So, we have

$$C_i(\hat{\varphi}_\lambda, \infty) = \delta_{i,0}\mathbb{Z} \quad \text{for all } i \geqslant 0. \tag{5.14}$$

Suppose that $K_{\hat{\varphi}_\lambda} = \{0, u_0^\lambda, v_0^\lambda, y_\lambda\}$. Then from (5.11), (5.12), (5.13), (5.14) and the Morse relation with $t = -1$ (see (2.14)), we have

$$(-1)^{d_k} + 2(-1)^0 + (-1)^1 = (-1)^0 \quad \Rightarrow \quad (-1)^{d_k} = 0, \quad \text{a contradiction.}$$

So, there exists $\hat{y}_\lambda \in K_{\hat{\varphi}_\lambda} \setminus \{0, u_0^\lambda, v_0^\lambda, y_\lambda\}$. From Proposition 8 and (4.17) we see that $\hat{y}_\lambda \in C^1(\bar{\Omega})$ is a second nodal solution of (S_λ) and $\hat{y}_\lambda \in \operatorname{int}_{C^1(\bar{\Omega})}[v_0^\lambda, u_0^\lambda]$.

Next suppose $\lambda \notin \sigma_R$ (2). Then $\lambda \in (\hat{\lambda}_k, \hat{\lambda}_{k+1})$ for some $k \geqslant 2$. In this case $u = 0$ is a nondegenerate critical point of σ_λ with Morse index $d_k = \dim \bigoplus_{i=1}^k E(\hat{\lambda}_i) \geqslant 2$. Therefore

$$c_i(\sigma_\lambda, 0) = \delta_{i,d_k}\mathbb{Z} \quad \text{for all } i \geqslant 0.$$

Then reasoning as above, we produce a second nodal solution $\hat{y}_\lambda \in \operatorname{int}_{c^1(\Omega)}[v_0^\lambda, u_0^\lambda]$ for problem (S_λ) $(\lambda > \hat{\lambda}_2)$. $\square$

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References

[1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, The spectrum and an index formula for the Neumann p-Laplacian and multiple solutions for problems with crossing nonlinearity, Discrete Contin. Dyn. Syst. 25 (2009) 431–456.
[2] A. Ambrosetti, D. Lupo, One class of nonlinear Dirichlet problems with multiple solutions, Nonlinear Anal. 8 (1984) 1145–1150.
[3] A. Ambrosetti, G. Mancini, Sharp nonuniqueness results for some nonlinear problems, Nonlinear Anal. 3 (1979) 635–645.
[4] T. Bartsch, Critical point theory on partially ordered Hilbert spaces, J. Funct. Anal. 186 (2001) 117–152.
[5] H. Brezis, L. Nirenberg, H^1 versus C^1 local minimizers, C. R. Acad. Sci. Paris Sér. I 317 (1993) 465–472.
[6] M. Cuesta, D. de Figueiredo, J.-P. Gossez, The beginning of the Fucik spectrum for the p-Laplacian, J. Differential Equations 159 (1999) 212–238.
[7] X. Duchateau, On some quasilinear equations involving the p-Laplacian with Robin boundary conditions, Appl. Anal. 92 (2013) 270–307.
[8] N. Dunford, J. Schwartz, Linear Operators, vol. I, Wiley-Interscience, New York, 1958.
[9] D. de Figueiredo, The Ekeland Variation Principle with Applications and Detaurs, Tata Institute of Fundamental Research, Bombay, Springer-Verlag, Berlin, 1989.
[10] M. Filippakis, A. Kristaly, N.S. Papageorgiou, Existence of five nonzero solutions with exact sign for a p-Laplacian equation, Discrete Contin. Dyn. Syst. 24 (2009) 405–440.
[11] J.P. Garcia Azorero, J. Manfredi, I. Peral Alonso, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, Commun. Contemp. Math. 2 (2000) 385–404.
[12] L. Gasinski, N.S. Papageorgiou, Nonlinear Analysis, Chapman & Hall / CRC, Boca Raton, 2006.
[13] S. Goldberg, Unbounded Linear Operators: Theory and Applications, McGraw–Hill Book Co., New York, 1966.
[14] A. Le, Eigenvalue problems for the p-Laplacian, Nonlinear Anal. 64 (2006) 1057–1099.
[15] G. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.
[16] D. Motreanu, N.S. Papageorgiou, Multiple solution for nonlinear Neumann problems driven by a nonhomogeneous differential operator, Proc. Amer. Math. Soc. 139 (2011) 3527–3535.

[17] N.S. Papageorgiou, S. Kyritsi, Handbook of Applied Analysis, Springer, New York, 2009.
[18] E. Papageorgiou, N.S. Papageorgiou, A multiplicity theorem for problems with the p-Laplacian, J. Funct. Anal. 244 (2007) 63–77.
[19] N.S. Papageorgiou, V.D. Rădulescu, Solutions with sign information for nonlinear nonhomogeneous elliptic equations, submitted for publication.
[20] M. Struwe, A note on a result of Ambrosetti and Mancini, Ann. Mat. Pura Appl. 131 (1982) 107–115.
[21] M. Struwe, Variational Methods, Springer, Berlin, 1990.
[22] J. Su, Semilinear elliptic boundary value problems with double resonance between two consecutive eigenvalues, Nonlinear Anal. 48 (2002) 881–895.
[23] J. Vazquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.
[24] P. Winkert, L^∞-estimates for nonlinear elliptic Neumann boundary value problems, NoDEA Nonlinear Differential Equations Appl. 17 (2010) 289–302.