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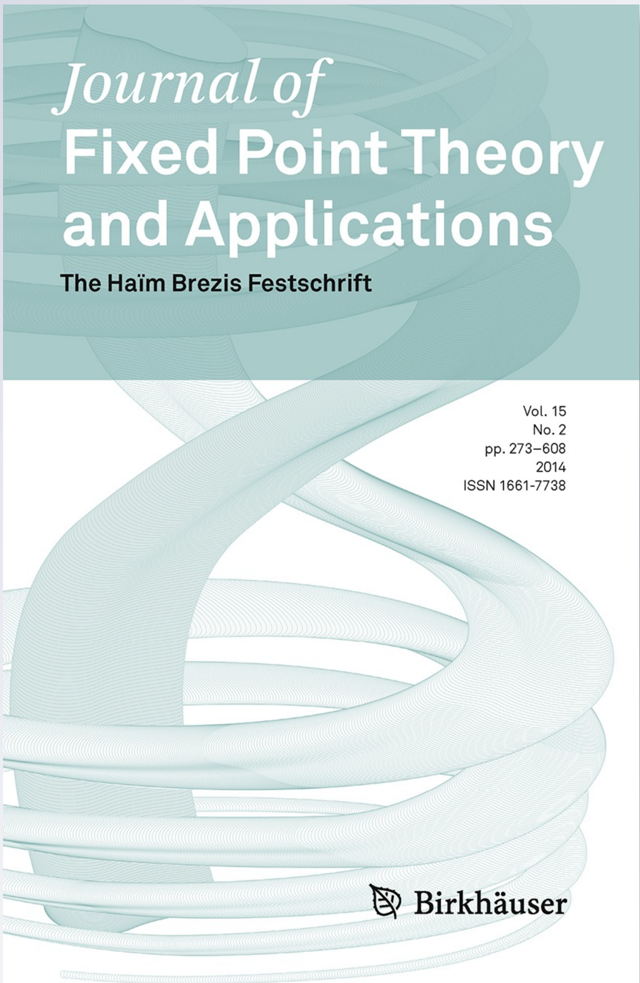

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Positive solutions for nonlinear nonhomogeneous Neumann equations of superdiffusive type

Nikolaos S. Papageorgiou and Vicenţiu D. Rădulescu

Dedicated with esteem to Professor Haim Brezis on his 70th anniversary

Abstract. We consider a nonlinear logistic-type equation, driven by a nonhomogeneous differential operator and with a reaction of superdiffusive type. Using variational methods together with suitable truncation and comparison techniques, we prove a bifurcation-type result describing the set of positive solutions as the parameter $\lambda > 0$ varies.

Mathematics Subject Classification. 35J25, 35J92.

Keywords. Logistic-type equation, nonhomogeneous differential operator, superdiffusive reaction, truncations, strong comparison.

1. Introduction

The aim of this paper is to study the existence, nonexistence and multiplicity of positive solutions for the following nonlinear, nonhomogeneous logistic-type equation:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \beta(z)u(z)^{p-1} = \lambda h(z,u(z)) - f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \qquad (P_\lambda)$$

where $\lambda > 0$.

In this problem, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$ and $a : \mathbb{R}^N \to \mathbb{R}^N$ is a strictly monotone, continuous map that satisfies certain other regularity and growth conditions. The precise assumptions on the map $a(\cdot)$ are given in hypotheses $H(a)$ below and incorporate as special cases important differential operators such as the p-Laplacian $(1 < p < \infty)$, the (p, q)-Laplacian $(1 < q < p < \infty)$ and the generalized p-mean curvature operator. In problem (P_λ), $\lambda > 0$ is a parameter and in the reaction the two terms $h(z,x)$ and $f(z,x)$ are both Carathéodory functions (that is, for all $x \in \mathbb{R}$, $z \mapsto h(z,x)$ and $f(z,x)$ are measurable and for a.a. $z \in \Omega$, $x \mapsto h(z,x)$ and $f(z,x)$ are continuous). The asymptotic growth conditions on $h(z,\cdot)$ and $f(z,\cdot)$ correspond to a reaction of superdiffusive type. Indeed, a very special case of our problem is when the differential operator is the p-Laplacian (that is, $a(y) = \|y\|^{p-2}y$ for all $y \in \mathbb{R}^N$ with $1 < p < \infty$) and the reaction is

$$x \mapsto \lambda x^{q-1} - x^{r-1} \quad \text{for all } x \geqslant 0,$$

where

$$1 < p < q < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leqslant p. \end{cases}$$

This is the superdiffusive p-logistic equation. Such equations, in contrast to the subdiffusive and equidiffusive cases, exhibit bifurcation phenomena as the parameter $\lambda > 0$ varies. Finally, we mention that $\beta \in L^\infty(\Omega)$, $\beta(z) \geqslant 0$ a.e. in Ω, $\beta \neq 0$ and in the boundary condition $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

Under natural assumptions (as described in hypotheses $H(a)$, H_0, H_1 and H_2 below), the main result in this paper (see Theorem 3.12) establishes that there exists $\lambda_* > 0$ such that

(a) for all $\lambda > \lambda_*$ problem (P_λ) has at least two positive solutions;
(b) for $\lambda = \lambda_*$ problem (P_{λ_*}) has at least one positive solution;
(c) for $\lambda \in (0, \lambda_*)$ problem (P_λ) has no positive solutions.

Superdiffusive p-logistic equations (that is, logistic-type equations driven by the p-Laplace operator) were studied by Dong [5], Filippakis, O'Regan and Papageorgiou [6], Takeuchi [14, 15] (Dirichlet equations) and Cardinali, Papageorgiou and Rubbioni [4] (Neumann equations). In all the aforementioned works, the reaction has a more restricted form than in (P_λ).

The nonhomogeneity of the differential operator in (P_λ) is the source of serious difficulties in establishing the bifurcation-type result and the methods used in the case of p-Laplacian equations do not work here (see Cardinali, Papageorgiou and Rubbioni [4]).

Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be an operator and let $\langle \cdot, \cdot \rangle$ denote the inner product in $\mathbb{R}^N$. We recall (see Brezis [1]) the following basic notions:

(i) a is *monotone* if

$$\langle a(x) - a(y), x - y \rangle \geqslant 0 \quad \text{for all } x, y \in \mathbb{R}^N;$$

(ii) a is *strictly monotone* if

$$\langle a(x) - a(y), x - y \rangle > 0 \quad \text{for all } x, y \in \mathbb{R}^N,\ x \neq y;$$

(iii) a is *maximal monotone* if it is monotone and

$$\left[\langle a(x) - y', x - y \rangle \geqslant 0\ \forall\, x \in \mathbb{R}^N\right] \Longrightarrow y' = a(y).$$

We refer the reader to the book by Brezis [2], which gives an excellent account of the interplay between functional analysis and partial differential equations.

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the Palais–Smale condition (PS condition for short), if the following is true:

> "Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(x_n) \to 0$ in X^* as $n \to \infty$ admits a strongly convergent subsequence."

This compactness-type condition on φ leads to the following minimax theorem for critical values of φ. The result is known in the literature as the "mountain pass theorem."

Theorem 2.1. *Let X be a Banach space, $\varphi \in C^1(X)$ satisfies the PS condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > r > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\left[\varphi(x) : \|x - x_0\| = r\right] = m_r,$$

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}.$$

Then $m_r \leqslant c$ and c is a critical value of φ.

Now, we will introduce the hypotheses on the map $a(\cdot)$.

So, let $\eta \in C^1(0, \infty)$ be a function such that $\eta(t) > 0$ for all $t > 0$ and

$$0 < \hat{c} \leqslant \frac{t\eta'(t)}{\eta(t)} \leqslant c_0 \qquad \text{for all } t > 0 \text{ and some } c_0, \hat{c} > 0, \tag{2.1}$$

$$c_1 t^{p-1} \leqslant \eta(t) \leqslant c_2\left(1 + t^{p-1}\right) \quad \text{for all } t > 0 \text{ and some } c_1, c_2 > 0. \tag{2.2}$$

Then the conditions imposed on the map $a(\cdot)$ are the following.

$H(a)$: $a(y) = a_0(\|y\|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \mapsto ta_0(t)$ is strictly increasing on $(0, +\infty)$, $ta_0(t) \to 0$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} = c > -1;$$

(ii) for every $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$\|\nabla a(y)\| \leqslant c_3 \frac{\eta(\|y\|)}{\|y\|} \quad \text{for some } c_3 > 0;$$

(iii) for every $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant \frac{\eta(\|y\|)}{\|y\|}\|\xi\|^2 \quad \text{for all } \xi \in \mathbb{R}^N;$$

(iv) if $G_0(t) = \int_0^t s a_0(s)\, ds$ for all $t > 0$, then

$$t^2 a_0(t) - G_0(t) \geqslant c_4 t^p \quad \text{for all } t > 0 \text{ and some } c_4 > 0.$$

Remark 2.2. Evidently $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(y) = G_0(\|y\|)$ for all $y \in \mathbb{R}^N$. Then $G(\cdot)$ is convex, $G(0) = 0$ and

$$\begin{aligned}\nabla G(y) &= G_0'(\|y\|)\frac{y}{\|y\|}\\ &= a_0(\|y\|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\} \text{ and } \nabla G(0) = 0.\end{aligned}$$

Therefore, $G(\cdot)$ is the primitive of $a(\cdot)$. The convexity of $G(\cdot)$ and the fact that $G(0) = 0$ imply that

$$G(y) \leqslant (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.3}$$

Hypotheses $H(a)$(i), (ii), (iii) and (2.1), (2.2) lead to the following lemma summarizing the main properties of the map $a(\cdot)$.

Lemma 2.3. *Assume that hypotheses $H(a)$(i), (ii), (iii) are fulfilled. Then*

(a) *the map $y \mapsto a(y)$ is continuous, strictly monotone, hence maximal monotone too;*
(b) *there exists $c_5 > 0$ such that $\|a(y)\| \leqslant c_5(1 + \|y\|^{p-1})$ for all $y \in \mathbb{R}^N$;*
(c) *$(a(y), y)_{\mathbb{R}^N} \geqslant \frac{c_1}{p-1}\|y\|^p$ for all $y \in \mathbb{R}^N$.*

Then Lemma 2.3, (2.3) and the integral form of the mean value theorem lead to the following growth properties of the primitive $G(\cdot)$.

Corollary 2.4. *If hypotheses $H(a)$(i), (ii), (iii) hold, then*

$$\frac{c_1}{p(p-1)}\|y\|^p \leqslant G(y) \leqslant c_6(1 + \|y\|^p) \quad \textit{for all } y \in \mathbb{R}^N \textit{ and some } c_6 > 0.$$

Example 2.5. The following maps satisfy hypotheses $H(a)$:

(a) $a(y) = \|y\|^{p-2}y$ with $1 < p < \infty$.

This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(\|Du\|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = \|y\|^{p-2}y + \|y\|^{q-2}y$ with $1 < q < p < \infty$.

This map corresponds to the (p, q)-Laplace differential operator defined by

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Such operators arise in mathematical physics. Recently the authors studied the existence and multiplicity of solutions for $(p, 2)$-equations under resonance conditions (see Papageorgiou and Rădulescu [12]).

(c) $a(y) = (1 + \|y\|^2)^{\frac{p-2}{2}}y$ with $1 < p < \infty$.

This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}\left(\left(1 + \|Du\|^2\right)^{\frac{p-2}{2}}Du\right) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = \|y\|^{p-2}y + \frac{\|y\|^{p-2}y}{1+\|y\|^p}$ with $1 < p < \infty$.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with subcritical growth in the $x \in \mathbb{R}$ variable; that is,

$$|f_0(z,x)| \leqslant a(z)(1+|x|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{all } x \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$ and $1 < r < p^*$. We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\psi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_0(u) = \int_\Omega G(Du(z))\,dz - \int_\Omega F_0(z,u(z))\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

The following result can be found in Motreanu and Papageorgiou [11] and it is an outgrowth of the nonlinear regularity theory (see Lieberman [10]). The first such result was proved by Brezis and Nirenberg [3] for $G(y) = \frac{1}{2}\|y\|^2$ for all $y \in \mathbb{R}^N$ and for space $H_0^1(\Omega)$.

Proposition 2.6. *Assume that hypotheses* $H(a)$(i), (ii), (iii) *hold and* $u_0 \in W^{1,p}(\Omega)$ *is a local* $C^1(\overline{\Omega})$*-minimizer of* ψ_0*; that is, there exists* $\varrho_1 > 0$ *such that*

$$\psi_0(u_0) \leqslant \psi_0(u_0+h) \quad \textit{for all } h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leqslant \varrho_1.$$

Then $u_0 \in C^{1,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$ *and* u_0 *is also a local* $W^{1,p}(\Omega)$*-minimizer of* ψ_0*; that is, there exists* $\varrho_2 > 0$ *such that*

$$\psi_0(u_0) \leqslant \psi_0(u_0+h) \quad \textit{for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\| \leqslant \varrho_2.$$

Hereafter, by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

Note that the notation $\|\cdot\|$ is also used to denote the norm of $\mathbb{R}^N$. However, no confusion is possible since it will always be clear from the context which norm is used.

The Banach space $C^1(\overline{\Omega})$ used in the above proposition is an ordered Banach space with positive cone given by

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega (a(Du), Dv)_{\mathbb{R}^N}\,dz \quad \text{for all } u, v \in W^{1,p}(\Omega). \tag{2.4}$$

From Gasinski and Papageorgiou [8] we have the following result.

Proposition 2.7. *Assume that hypotheses* $H(a)$(i), (ii), (iii) *hold. Then the operator* $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *defined by* (2.4) *is bounded (that is, it maps bounded sets to bounded sets), demicontinuous, monotone (hence maximal monotone too) and of type* $(S)_+$*; that is,*

$$u_n \xrightarrow{w} u \textit{ in } W^{1,p}(\Omega) \textit{ and } \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0 \Longrightarrow u_n \to u \textit{ in } W^{1,p}(\Omega).$$

Finally, let us fix our notation in this paper. Given $x \in \mathbb{R}$, we define $x^{\pm} = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$, we set $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We have

$$u^{\pm} \in W^{1,p}(\Omega), \quad u = u^{+} - u^{-}, \quad |u| = u^{+} + u^{-}.$$

Also, given a measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we define

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \text{ in } W^{1,p}(\Omega)$$

(the Nemytskii map corresponding to $g(\cdot,\cdot)$). Moreover, by $|\cdot|_N$ we denote the Lebesgue measure $\mathbb{R}^N$.

3. Bifurcation-type theorem

The hypotheses on the other three data of (P_λ) (namely, the functions $\beta(z)$, $h(z,x)$ and $f(z,x)$) are the following.

H_0: $\beta \in L^\infty(\Omega)$, $\beta(z) \geqslant 0$ a.e. in Ω, $\beta \neq 0$.

H_1: $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $h(z,0) = 0$, $h(z,\cdot)$ is nondecreasing on $[0,\infty)$ and

(i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)_+$ such that $h(z,x) \leqslant a_\varrho(z)$ for a.a. $z \in \Omega$, all $x \in [0,\varrho]$;

(ii) there exists $q \in (p, p^*)$ such that

$$0 < c_7 \leqslant \liminf_{x \to +\infty} \frac{h(z,x)}{x^{q-1}} \leqslant \limsup_{x \to +\infty} \frac{h(z,x)}{x^{q-1}} \leqslant c_8$$

uniformly for a.a. $z \in \Omega$;

(iii) there exist $0 < c_9 < c_{10}$ such that

$$c_9 \leqslant \liminf_{x \to 0^+} \frac{h(z,x)}{x^{q-1}} \leqslant \limsup_{x \to 0^+} \frac{h(z,x)}{x^{q-1}} \leqslant c_{10}$$

uniformly for a.a. $z \in \Omega$;

(iv) for every $\mu > 0$, there exists $\vartheta_\mu > 0$ such that $f(z,x) \geqslant \vartheta_\mu$ for a.a. $z \in \Omega$, all $x \geqslant \mu$.

Remark 3.1. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality, we can set $h(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$.

Example 3.2. The following functions satisfy hypotheses H_1. For the sake of simplicity, we drop the z dependence:

$$h_1(x) = x^{q-1} \quad \text{for all } x \geqslant 0 \text{ with } 1 < p < q < p^*;$$

$$h_2(x) = \begin{cases} x^{q-1} - \xi x^{\tau-1} & \text{if } x \in [0,1], \\ (1-\xi)[x^{q-1} - \ln x] & \text{if } 1 < x \end{cases}$$

$$\text{with } 1 < p < q < \tau,\ q < p^*,\ \xi \in \left(0, \tfrac{q-1}{\tau-1}\right),\ q \geqslant 2.$$

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x) \geqslant 0$ for all $x \geqslant 0$ and

(i) $f(z,x) \leqslant a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$ with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;

(ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{q-1}} = +\infty$ uniformly for a.a. $z \in \Omega$ (here $q \in (p,p^*)$ is as in hypothesis H_1(ii));

(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(iv) for every $\varrho > 0$, there exists $\hat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$ the function $x \mapsto \hat{\xi}_\varrho x^{p-1} - f(z,x)$ is nondecreasing on $[0,\varrho]$.

Remark 3.3. As we did for $h(z,x)$, without any loss of generality, we assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$.

Example 3.4. The following functions satisfy hypotheses H_2. Again, for the sake of simplicity, we drop the z dependence:

$$f_1(x) = x^{r-1} \quad \text{for all } x \geqslant 0 \text{ with } q < r < p^*,$$

$$f_2(x) = \begin{cases} x^{\tau-1} - x^{q-1} & \text{if } x \in [0,1], \\ x^{q-1}\ln x & \text{if } 1 < x \end{cases} \quad \text{with } 1 < p < \tau < q < p^*.$$

Remark 3.5. If $a(y) = \|y\|^{p-2}y$ with $1 < p < \infty$ (that is, the differential operator is the p-Laplacian) and the reaction is $\lambda h_1(x) - f_1(x) = \lambda x^{q-1} - x^{r-1}$ with $p < q < r < p^*$, then problem (P_λ) recovers the classical p-logistic equation of superdiffusive type.

For $\lambda > 0$, let

$$S(\lambda) = \text{the set of positive solutions of problem } (P_\lambda).$$

Also, we introduce the set

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ admits a positive solution (that is, } S(\lambda) \neq \emptyset)\}.$$

We start with a simple lemma.

Lemma 3.6. *Assume that $\beta \in L^\infty(\Omega)$, $\beta \geqslant 0$ a.e. in Ω and $\beta \neq 0$. Then there exists $\xi_0 > 0$ such that*

$$\xi_0\|u\|^p \leqslant \psi(u) = \frac{c_1}{p-1}\|Du\|_p^p + \int_\Omega \beta(z)|u(z)|^p\,dz \quad \textit{for all } u \in W^{1,p}(\Omega).$$

Proof. Suppose the lemma is not true. Then exploiting the p-homogeneity of the functional $\psi(\cdot)$, we can find $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\| = 1 \quad \text{for all } n \geqslant 1 \qquad \text{and} \qquad \psi(u_n) \downarrow 0 \quad \text{as } n \to \infty.$$

By passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad u_n \to u \quad \text{in } L^p(\Omega).$$

Since the norm in the Banach space $L^p(\Omega, \mathbb{R}^N)$ is weakly lower semicontinuous, in the limit as $n \to \infty$, we have

$$0 \leqslant \frac{c_1}{p-1}\|Du\|_p^p \leqslant -\int_\Omega \beta(z)|u|^p\,dz \leqslant 0 \Longrightarrow u \equiv \xi \in \mathbb{R}.$$

If $\xi = 0$, then $Du_n \to 0$ in $L^p(\Omega, \mathbb{R}^N)$ and so $u_n \to 0$ in $W^{1,p}(\Omega)$, a contradiction to the fact that $\|u_n\| = 1$ for all $n \geqslant 1$.

So, $\xi \neq 0$ and we have

$$0 \leqslant -|\xi|^p \int_\Omega \beta(z)\,dz < 0,$$

a contradiction again. □

Using the previous lemma, we have the following result.

Proposition 3.7. *Assume that hypotheses $H(a)$, H_0, H_1 and H_2 hold. Then* $\inf \mathcal{L} > 0$ *(if $\mathcal{L} = \emptyset$, then* $\inf \mathcal{L} = +\infty$*).*

Proof. Let $\xi_0 > 0$ be as postulated by Lemma 3.6 and let $\xi \in (0, \xi_0)$. Hypotheses H_1 and H_2 imply that we can find $\hat{\lambda} = \hat{\lambda}(\xi) > 0$ small such that

$$\hat{\lambda} h(z,x) - f(z,x) \leqslant \xi x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3.1}$$

Let $\lambda \in (0, \hat{\lambda}]$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u \in S(\lambda)$ and so we have

$$A(u) + \beta u^{p-1} = \lambda_n N_h(u) - N_f(u). \tag{3.2}$$

On (3.2) we act with $u \geqslant 0$ and using the nonlinear Green's identity (see Gasinski and Papageorgiou [7, p. 210]), we have

$$\begin{aligned}
&\int_\Omega (a(Du), Du)_{\mathbb{R}^N}\,dz + \int_\Omega \beta(z)|u|^p\,dz = \lambda \int_\Omega h(z,u)u\,dz - \int_\Omega f(z,u)u\,dz \\
&\quad\Longrightarrow \frac{c_1}{p-1}\|Du\|_p^p + \int_\Omega \beta(z)|u|^p\,dz \\
&\qquad\qquad \leqslant \xi\|u\|_p^p \leqslant \xi\|u\|^p \quad \text{(see Lemma 2.3(c) and (3.1))} \\
&\quad\Longrightarrow (\xi_0 - \xi)\|u\|^p \leqslant 0 \qquad\qquad \text{(see Lemma 3.6)},
\end{aligned}$$

a contradiction since $\xi \in (0, \xi_0)$.

Therefore, $\lambda \notin \mathcal{L}$ and so $\inf \mathcal{L} \geqslant \hat{\lambda} > 0$. □

Next we establish that $\mathcal{L} \neq \emptyset$ (hence $\inf \mathcal{L} \in (0, +\infty)$).

Proposition 3.8. *Assume that hypotheses $H(a)$, H_0, H_1 and H_2 hold. Then $\mathcal{L} \neq \emptyset$ and for every $\lambda \in \mathcal{L}$ we have $S(\lambda) \subseteq \operatorname{int} C_+$.*

Proof. Let $H(z,x) = \int_0^x h(z,s)\,ds$ and $F(z,x) = \int_0^x f(z,s)\,ds$ and, for $\lambda > 0$, we consider the C^1-functional $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}
\varphi_\lambda(u) = &\int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \beta(z)|u|^p\,dz \\
&- \lambda \int_\Omega H(z,u)\,dz + \int_\Omega F(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).
\end{aligned}$$

By virtue of hypotheses H_1(i), (ii), we can find $c_{11} > 0$ such that

$$H(z,x) \leqslant c_{11}(1 + x^q) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3.3}$$

Moreover, hypotheses H_2(i), (ii) imply that, given any $\xi > 0$, we can find $c_{12} = c_{12}(\xi) > 0$ such that

$$F(z,x) \geqslant \xi x^q - c_{12} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3.4}$$

Therefore, for all $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}
\varphi_\lambda(u) &= \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \beta(z)|u|^p\,dz - \lambda\int_\Omega H(z,u)\,dz + \int_\Omega F(z,u)\,dz \\
&\geqslant \frac{1}{p}\left[\frac{c_1}{p-1}\|Du\|_p^p + \int_\Omega \beta(z)|u|^p\,dz\right] - \lambda c_{11}\|u\|_q^q \\
&\quad + \xi\|u\|_q^q - (\lambda c_{11} + c_{12})|\Omega|_N \quad \text{(see Corollary 2.4 and (3.3), (3.4))} \\
&\geqslant \frac{\xi_0}{p}\|u\|^p + (\xi - \lambda c_{11})\|u\|_q^q - (\lambda c_{11} + c_{12})|\Omega|_N \quad \text{(see Lemma 3.6).}
\end{aligned}$$

Choosing $\xi > \lambda c_{11}$, we see that φ_λ is coercive. Also, using the Sobolev embedding theorem, we can easily check that φ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \inf\left[\varphi_\lambda(u) : \ u \in W^{1,p}(\Omega)\right]. \tag{3.5}$$

By hypotheses H_1(iii) and H_2(iii), we can find $\delta > 0$ and $c_{13} > 0$ such that

$$H(z,x) \geqslant \frac{c_{13}}{q}x^q \quad \text{and} \quad F(z,x) \leqslant \frac{1}{p}x^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{3.6}$$

So, if $\xi \in (0,\delta]$, then

$$\varphi_\lambda(\xi) \leqslant \frac{\xi^p}{p}\left[\|\beta\|_1 + |\Omega|_N\right] - \frac{\lambda c_{13}}{q}\xi^q|\Omega|_N \quad \text{(see (3.6)).}$$

Choosing $\lambda > 0$ big, we infer that

$$\begin{aligned}
&\varphi_\lambda(\xi) < 0 = \varphi_\lambda(0) \\
&\qquad \Longrightarrow \varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0) \quad \text{(see (3.5)) and so } u_\lambda \neq 0.
\end{aligned}$$

From (3.5) we have

$$\begin{aligned}
&\varphi_\lambda'(u_\lambda) = 0 \\
&\qquad \Longrightarrow A(u_\lambda) + \beta|u_\lambda|^{p-2}u_\lambda = \lambda N_h(u_\lambda) - N_f(u_\lambda).
\end{aligned} \tag{3.7}$$

On (3.7) we act with $-u_\lambda^- \in W^{1,p}(\Omega)$. Using Lemma 2.3(c) and recalling that for a.a. $z \in \Omega$ and all $x \leqslant 0$, we have $h(z,x) = f(z,x) = 0$, we obtain

$$\begin{aligned}
&\frac{c_1}{p-1}\|Du_\lambda^-\|_p^p + \int_\Omega \beta(u_\lambda^-)^p\,dz \leqslant 0, \\
&\qquad \Longrightarrow \xi_0\|u_\lambda^-\|^p \leqslant 0 \quad \text{(see Lemma 3.6), hence } u_\lambda \geqslant 0, \ u_\lambda \neq 0.
\end{aligned}$$

Then from (3.7) and using the nonlinear Green's identity, as in Gasinski and Papageorgiou [8] (see the proof of Theorem 3.9), we have

$$\begin{cases} -\operatorname{div} a(Du_\lambda(z)) + \beta(z)u_\lambda(z)^{p-1} = \lambda h(z,u_\lambda(z)) - f(z,u_\lambda(z)) & \text{a.e. in } \Omega, \\ \dfrac{\partial u_\lambda}{\partial n} = 0 & \text{on } \partial\Omega, \\ u_\lambda \geqslant 0, \ u_\lambda \neq 0 & \text{in } \Omega, \end{cases} \tag{3.8}$$

which implies that $u_\lambda \in S(\lambda)$ and so $\mathcal{L} \neq \emptyset$.

Let $\lambda \in \mathcal{L}$ and let $u_\lambda \in S(\lambda)$. Then (3.8) holds. From Hu and Papageorgiou [9] and Winkert [16], we have $u_\lambda \in L^\infty(\Omega)$. Then we can apply the nonlinear regularity result of Lieberman [10, p. 320] and infer that $u_\lambda \in C_+ \setminus \{0\}$. Let $\varrho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\varrho > 0$ be as postulated by hypothesis H_2(iv). Then

$$\begin{aligned} -\operatorname{div} a(Du_\lambda(z)) &+ (\beta(z) + \hat{\xi}_\varrho)u_\lambda(z)^{p-1} \\ &= \lambda h(z,u_\lambda(z)) - f(z,u_\lambda(z)) + \hat{\xi}_\varrho u_\lambda(z)^{p-1} \\ &\geqslant 0 \quad \text{a.e. in } \Omega \text{ (see } H_2\text{(iv) and recall that } h \geqslant 0) \\ &\Longrightarrow \operatorname{div} a(Du_\lambda(z)) \leqslant (\|\beta\|_\infty + \hat{\xi}_\varrho)u_\lambda(z)^{p-1} \quad \text{a.e. in } \Omega. \end{aligned}$$

Let $\gamma(t) = ta_0(t)$ for all $t > 0$. Then

$$\begin{aligned} t\gamma'(t) &= t^2 a_0'(t) + ta_0(t) \\ \Longrightarrow \int_0^t s\gamma'(s)\,ds &= t\gamma(t) - \int_0^t \gamma(s)\,ds \quad \text{(by integration by parts)} \\ &= t^2 a_0(t) - G_0(t) \geqslant c_4 t^p \quad \text{for all } t > 0. \end{aligned}$$

So, we can apply the results of Pucci and Serrin [13, pp. 111, 120] and conclude that $u_\lambda \in \operatorname{int} C_+$. Therefore $S(\lambda) \subseteq \operatorname{int} C_+$. □

Proposition 3.9. *Assume that hypotheses $H(a)$, H_0, H_1 and H_2 hold, $\lambda \in \mathcal{L}$ and $\mu > \lambda$. Then $\mu \in \mathcal{L}$.*

Proof. Let $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 3.8). We have

$$A(u_\lambda) + \beta u_\lambda^{p-1} = \lambda N_h(u_\lambda) - N_f(u_\lambda). \tag{3.9}$$

Using $u_\lambda \in \operatorname{int} C_+$, we introduce the following truncation of the reaction in problem (P_λ):

$$k_\mu(z,x) = \begin{cases} \mu h(z,u_\lambda(z)) - f(z,u_\lambda(z)) & \text{if } x \leqslant u_\lambda(z), \\ \mu h(z,x) - f(z,x) & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.10}$$

Evidently $k_\mu(z,x)$ is a Carathéodory function. We set

$$K_\mu(z,x) = \int_0^x k_\mu(z,s)\,ds$$

and consider the C^1-functional $\psi_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\mu(u) = \int_\Omega G(Du(z))\,dz + \frac{1}{p}\int_\Omega \beta(z)|u(z)|^p\,dz \\ - \int_\Omega K_\mu(z,u(z))\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

As we did for the functional φ_λ (see the proof of Proposition 3.8), we can check that ψ_μ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \psi_\mu(u_\mu) &= \inf\left[\psi_\mu(u) : \ u \in W^{1,p}(\Omega)\right] \\ &\Longrightarrow \psi_\mu'(u_\mu) = 0 \\ &\Longrightarrow A(u_\mu) + \beta|u_\mu|^{p-2}u_\mu = N_{k_\mu}(u_\mu). \end{aligned} \tag{3.11}$$

On (3.11) we act with $(u_\lambda - u_\mu)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\left\langle A(u_\mu), (u_\lambda - u_\mu)^+\right\rangle + \int_\Omega \beta|u_\mu|^{p-2}u_\mu(u_\lambda - u_\mu)^+\,dz \\ &\quad = \int_\Omega k_\mu(z,u_\mu)(u_\lambda - u_\mu)^+\,dz \\ &\quad = \int_\Omega \left[\mu h(z,u_\lambda) - f(z,u_\lambda)\right](u_\lambda - u_\mu)^+\,dz \quad \text{(see (3.10))} \\ &\quad \geqslant \int_\Omega \left[\lambda h(z,u_\lambda) - f(z,u_\lambda)\right](u_\lambda - u_\mu)^+\,dz \quad (\text{since } h \geqslant 0 \text{ and } \lambda < \mu) \\ &\quad = \left\langle A(u_\lambda), (u_\lambda - u_\mu)^+\right\rangle \\ &\qquad + \int_\Omega \beta(z)|u_\lambda|^{p-1}(u_\lambda - u_\mu)^+\,dz \quad (\text{since } u_\lambda \in S(\lambda)) \\ &\quad \Longrightarrow \left\langle A(u_\lambda) - A(u_\mu), (u_\lambda - u_\mu)^+\right\rangle \\ &\qquad\quad + \int_\Omega \beta(z)(u_\lambda^{p-1} - |u_\mu|^{p-2}u_\mu)(u_\lambda - u_\mu)^+\,dz \leqslant 0 \\ &\quad \Longrightarrow \int_{\{u_\lambda > u_\mu\}} (a(Du_\lambda) - a(Du_\mu), Du_\lambda - Du_\mu)_{\mathbb{R}^N}\,dz \leqslant 0 \quad (\text{see } H_0) \\ &\quad \Longrightarrow |\{u_\lambda > u_\mu\}|_N = 0 \quad (\text{see Lemma 2.3(a) and hypothesis } H_0) \\ &\quad \Longrightarrow u_\lambda \leqslant u_\mu. \end{aligned}$$

Therefore, (3.11) becomes

$$\begin{aligned} A(u_\mu) + \beta u_\mu^{p-1} &= \mu N_h(u_\mu) - N_f(u_\mu) \quad \text{(see (3.10))} \\ &\Longrightarrow u_\mu \in S(\mu) \subseteq \operatorname{int} C_+ \quad \text{and so } \mu \in \mathcal{L}. \end{aligned}$$ □

Let $\lambda_* = \inf \mathcal{L}$. From Proposition 3.7 we know that $\lambda_* > 0$.

Proposition 3.10. *Assume that hypotheses $H(a)$, H_0, H_1 and H_2 hold and $\lambda > \lambda_*$. Then problem (P_λ) admits at least two positive solutions:*

$$u_0,\ \hat{u} \in \operatorname{int} C_+.$$

Proof. Let $\eta \in (\lambda_*, \lambda)$ and let $u_\eta \in S(\eta) \subseteq \operatorname{int} C_+$ (see Proposition 3.8). We consider the reaction $x \mapsto \lambda h(z,x) - f(z,x)$ of problem (P_λ) and truncate it at $u_\eta(z)$ as we did in the proof of Proposition 3.8 (see (3.10)). Then reasoning as in that proof, using the direct method, we get

$$u_0 \in S(\lambda) \subseteq \operatorname{int} C_+$$

such that $u_\eta \leqslant u_0$.

Let $\varrho = \|u_0\|_\infty$ and let $\hat{\xi}_\varrho > 0$ be as postulated by hypothesis H_2(iv). Let $u_\eta^\delta = u_\eta + \delta \in \operatorname{int} C_+$. Then we have

$$\begin{aligned}
&-\operatorname{div} a(Du_\eta^\delta) + (\beta(z) + \hat{\xi}_\varrho)(u_\eta^\delta)^{p-1} \\
&\quad \leqslant -\operatorname{div} a(Du_\eta) + (\beta(z) + \hat{\xi}_\varrho)u_\eta^{p-1} + \tau(\delta) \quad \text{with } \tau(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
&\quad = \eta h(z,u_\eta) - f(z,u_\eta) + \hat{\xi}_\varrho u_\eta^{p-1} + \tau(\delta) \\
&\quad = \lambda h(z,u_\eta) - (\lambda - \eta)h(z,u_\eta) - f(z,u_\eta) + \hat{\xi}_\varrho u_\eta^{p-1} + \tau(\delta).
\end{aligned} \tag{3.12}$$

Since $h(z,\cdot)$ is nondecreasing (see hypotheses H_1) and $u_\eta \leqslant u_0$, we have

$$\lambda h(z,u_\eta) \leqslant \lambda h(z,u_0). \tag{3.13}$$

Since $u_\eta \in \operatorname{int} C_+$, we have $\mu = \min_{\overline{\Omega}} u_\eta > 0$ and so by virtue of hypothesis H_1(iv) we can find $\vartheta_\mu > 0$ such that

$$(\lambda - \eta)h(z,u_\eta) \geqslant (\lambda - \eta)\vartheta_\mu > 0. \tag{3.14}$$

Moreover, hypothesis H_1(iv) implies that

$$\hat{\xi}_\varrho u_\eta^{p-1} - f(z,u_\eta) \leqslant \hat{\xi}_\varrho u_0^{p-1} - f(z,u_0) \quad \text{(recall that } u_\eta \leqslant u_0\text{).} \tag{3.15}$$

We return to (3.12) and use (3.13), (3.14) and (3.15). Then

$$\begin{aligned}
&-\operatorname{div} a(Du_\eta^\delta) + (\beta(z) + \hat{\xi}_\varrho)(u_\eta^\delta)^{p-1} \\
&\qquad\qquad \leqslant \lambda h(z,u_0) - f(z,u_0) + \hat{\xi}_\varrho u_0^{p-1} - (\lambda - \eta)\vartheta_\mu + \tau(\delta).
\end{aligned}$$

Since $\tau(\delta) \to 0^+$ as $\delta \to 0^+$, for $\delta > 0$ small we have

$$(\lambda - \eta)\vartheta_\mu \geqslant \tau(\delta).$$

Therefore, finally we have

$$\begin{aligned}
&-\operatorname{div} a(Du_\eta^\delta) + (\beta(z) + \hat{\xi}_\varrho)(u_\eta^\delta)^{p-1} \\
&\leqslant -\operatorname{div} a(Du_0) + (\beta(z) + \hat{\xi}_\varrho)u_0^{p-1} \quad \text{(recall that } u_0 \in S(\lambda)\text{)} \\
&\Longrightarrow u_\eta^\delta \leqslant u_0 \text{ (acting with } (u_\eta^\delta - u_0)^+ \in W^{1,p}(\Omega) \text{ and using Lemma 2.3(a))} \\
&\Longrightarrow u_0 - u_\eta \in \operatorname{int} C_+.
\end{aligned} \tag{3.16}$$

Let $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional introduced in the first part of this proof. Recall that the solution $u_0 \in \operatorname{int} C_+$ was obtained as a minimizer of the functional ψ_λ. We introduce the set

$$[u_0) = \{u \in W^{1,p}(\Omega) : u_0(z) \leqslant u(z) \text{ a.e. in } \Omega\}.$$

Recall that $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ is a C^1 energy functional for problem (P_λ) (see the proof of Proposition 3.7). We have

$$\varphi_\lambda\big|_{[u_0)} = \psi_\lambda\big|_{[u_0)} + \xi_\lambda^* \quad \text{with } \xi_\lambda^* \in \mathbb{R}$$

(see (3.10) with u_λ replaced by u_η).

Because of (3.16), it follows that $u_0 \in \operatorname{int} C_+$ is a local $C^1(\overline{\Omega})$-minimizer of the functional φ_λ. Hence we can use Proposition 2.6 and have that u_0 is a local $W^{1,p}(\Omega)$-minimizer of φ_λ.

Hypothesis H_1(iii) implies that we can find $\delta > 0$ and $c_{14} > c_{10} > 0$ such that

$$\begin{aligned} &h(z,x) \leqslant c_{14} x^{q-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta] \\ &\qquad \Longrightarrow H(z,x) \leqslant \frac{c_{14}}{q} x^q \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \end{aligned} \tag{3.17}$$

Then for $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \leqslant \delta$, we have

$$\begin{aligned} \varphi_\lambda(u) &\geqslant \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \beta(z)|u|^p\,dz - \lambda \int_\Omega H(z,u)\,dz \quad (\text{since } F \geqslant 0) \\ &\geqslant \frac{c_1}{p(p-1)}\|Du\|_p^p + \frac{1}{p}\int_\Omega \beta(z)|u|^p\,dz - \frac{\lambda c_{14}}{q}\|u\|_q^q \quad (\text{see } (3.17)) \\ &\geqslant \frac{\xi_0}{p}\|u\|^p - \lambda c_{15}\|u\|^q \quad \text{for some } c_{15} > 0 \text{ (see Lemma 3.6).} \end{aligned} \tag{3.18}$$

Since $q > p$, from (3.18) we see that we can find $\delta_0 \in (0,\delta]$ such that

$$\begin{aligned} &\varphi_\lambda(u) \geqslant 0 = \varphi_\lambda(0) \quad \text{for all } u \in C^1(\overline{\Omega}) \text{ with } \|u\|_{C^1(\overline{\Omega})} \leqslant \delta_0 \\ &\quad \Longrightarrow u = 0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda \\ &\quad \Longrightarrow u = 0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \text{ (see Proposition 2.6).} \end{aligned}$$

Without any loss of generality, we may assume that

$$0 = \varphi_\lambda(0) \leqslant \varphi_\lambda(u_0).$$

The analysis is similar if the opposite inequality holds. Since u_0 is a local minimizer of φ_λ, we can find $\varrho \in (0,1)$ small such that

$$0 = \varphi_\lambda(0) \leqslant \varphi_\lambda(u_0) < \inf[\varphi_\lambda(u) : \ \|u - u_0\| = \varrho] = m_\varrho \tag{3.19}$$

(see Papageorgiou and Rădulescu [12], proof of Proposition 3.5, Claim 2). Recall that φ_λ is coercive (see the proof of Proposition 3.8). So, φ_λ satisfies the PS condition. This fact and (3.19) permit the use of Theorem 2.1. We can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\varphi_\lambda} \quad \text{and} \quad m_\varrho \leqslant \varphi_\lambda(\hat{u}).$$

Then $\hat{u} \notin \{0, u_0\}$ (see (3.19)) and it solves problem (P_λ); that is,

$$\hat{u} \in S(\lambda) \subseteq \operatorname{int} C_+. \qquad \square$$

Proposition 3.11. *If hypotheses $H(a)$, H_0, H_1 and H_2 hold, then $\lambda_* \in \mathcal{L}$.*

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L}$ such that

$$\lambda_n > \lambda_* \quad \text{for all } n \geqslant 1 \qquad \text{and} \qquad \lambda_n \downarrow \lambda_* \quad \text{as } n \to \infty.$$

Then we can find $u_n \in S(\lambda_n) \subseteq \operatorname{int} C_+$ for all $n \geqslant 1$. We have

$$A(u_n) + \beta u_n^{p-1} = \lambda N_h(u_n) - N_f(u_n) \quad \text{for all } n \geqslant 1. \tag{3.20}$$

Hypotheses H_1(i), (ii) imply that we can find $c_{16} > 0$ such that

$$h(z, x) \leqslant c_{16}\left(1 + x^{q-1}\right) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3.21}$$

Moreover, hypotheses H_2(i), (ii) imply that, given any $\xi > 0$, we can find $c_{17} = c_{17}(\xi) > 0$ such that

$$f(z, x) \geqslant \xi x^{q-1} - c_{17} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3.22}$$

On (3.20) we act with $u_n \in \operatorname{int} C_+$. Then

$$\int_\Omega (a(Du_n), Du_n)_{\mathbb{R}^N}\, dz + \int_\Omega \beta(z) u_n^p\, dz \tag{3.23}$$
$$+ \int_\Omega f(z, u_n) u_n\, dz = \lambda_n \int_\Omega h(z, u_n) u_n\, dz,$$

$$\Longrightarrow \frac{c_1}{p-1}\|Du_n\|_p^p + \int_\Omega \beta(z) u_n^p\, dz + (\xi - \lambda_n c_{16})\|u_n\|_q^q \tag{3.24}$$
$$\leqslant (c_{16} + c_{17})|\Omega|_N \quad \text{(see Lemma 2.3(c) and (3.21), (3.22))}$$

$$\Longrightarrow \xi_0 \|u_n\|^p + (\xi - \lambda_n c_{16})\|u_n\|_q^q \leqslant (c_{16} + c_{17})|\Omega|_N. \tag{3.25}$$

Choosing $\xi > \lambda_n c_{16}$, from (3.23) we see that

$$\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

By passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W^{1,p}(\Omega) \qquad \text{and} \qquad u_n \to u_* \quad \text{in } L^r(\Omega). \tag{3.26}$$

On (3.20) we act with $u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit $n \to \infty$ and use (3.26). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0$$
$$\Longrightarrow u_n \to u_* \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2.7).} \tag{3.27}$$

So, if in (3.20) we pass to the limit as $n \to \infty$ and use (3.27), then

$$A(u_*) + \beta u_*^{p-1} = \lambda_* N_h(u_*) - N_f(u_*).$$

Therefore u_* is a solution of (P_{λ_*}). We need to show that $u_* \neq 0$.

As before, from Hu and Papageorgiou [9] and Winkert [16], we know that we can find $M > 0$ such that $\|u_n\|_\infty \leqslant M$ for all $n \geqslant 1$. Then the nonlinear regularity result of Lieberman [10, p. 320] implies that we can find $\vartheta \in (0,1)$ and $\hat{M} > 0$ such that

$$u_n \in C^{1,\vartheta}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\vartheta}(\overline{\Omega})} \leqslant \hat{M} \quad \text{for all } n \geqslant 1.$$

Exploiting the compact embedding of $C^{1,\vartheta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and using (3.27), we have

$$u_n \to u_* \quad \text{in } C^1(\overline{\Omega}). \tag{3.28}$$

Suppose that $u_* = 0$. Note that

$$\lim_{x\to 0^+} \frac{h(z,x)}{x^{p-1}} = \lim_{x\to 0^+} \frac{h(z,x)}{x^{q-1}} x^{q-p} = 0 \quad \text{uniformly for a.a. } z \in \Omega$$

(see hypothesis H_1(iii)).

So, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$h(z,x) \leqslant \varepsilon x^{p-1} \quad \text{for a.a } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{3.29}$$

Then from (3.28) and since we have assumed that $u_* = 0$, we see that we can find $n_0 \geqslant 1$ such that

$$0 < u_n(z) \leqslant \delta \quad \text{for all } z \in \overline{\Omega}, \text{ all } n \geqslant n_0. \tag{3.30}$$

From (3.20), as in Gasinski and Papageorgiou [8], using the nonlinear Green's identity, we have for all $n \geqslant n_0$,

$$\begin{aligned} -\operatorname{div} a(Du_n(z)) &+ \beta(z)u_n(z)^{p-1} \\ &= \lambda_n h(z,u_n(z)) - f(z,u_n(z)) \\ &\leqslant \lambda_n h(z,u_n(z)) \quad (\text{since } f \geqslant 0, \text{ see } H_2) \\ &\leqslant \lambda_n \varepsilon u_n(z)^{p-1} \quad \text{for a.a. } z \in \Omega \text{ (see (3.29), (3.30))}. \end{aligned} \tag{3.31}$$

Acting on (3.31) with u_n, using the nonlinear Green's identity (see Gasinski and Papageorgiou [7, p. 210] and recall that $\partial u_n/\partial n = 0$ on $\partial\Omega$) and applying Lemma 2.3(c), we have

$$\begin{aligned} &\frac{c_1}{p-1}\|Du_n\|_p^p + \int_\Omega \beta(z)u_n^p\,dz \leqslant \lambda_n\varepsilon\|u_n\|_p^p \leqslant \lambda_n\varepsilon\|u_n\|^p \quad \text{for all } n \geqslant n_0 \\ &\implies \xi_0\|u_n\|^p \leqslant \lambda_n\varepsilon\|u_n\|^p \quad \text{for all } n \geqslant n_0 \text{ (see Lemma 3.6)} \\ &\implies \frac{\xi_0}{\varepsilon} \leqslant \lambda_n \leqslant \lambda_1 \quad \text{for all } n \geqslant n_0. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \to 0^+$ to reach a contradiction. Hence $u_* \neq 0$ and so $u_* \in S(\lambda_*) \subseteq \operatorname{int} C_+$, which means that $\lambda_* \in \mathcal{L}$. □

Summarizing the situation for problem (P_λ), we can state the following bifurcation-type theorem.

Theorem 3.12. *Assume that hypotheses $H(a)$, H_0, H_1 and H_2 hold. Then there exists $\lambda_* > 0$ such that*

(a) *for all $\lambda > \lambda_*$ problem (P_λ) has at least two positive solutions:*

$$u_0,\ \hat{u} \in \operatorname{int} C_+;$$

(b) *for* $\lambda = \lambda_*$ *problem* (P_{λ_*}) *has at least one positive solution:*

$$u_* \in \operatorname{int} C_+;$$

(c) *for* $\lambda \in (0, \lambda_*)$ *problem* (P_λ) *has no positive solutions.*

Remark 3.13. When $a(y) = \|y\|^{p-2}y$ with $1 < p < \infty$ (the p-Laplace differential operator) and $h(z, x) = x^{q-1}$ for all $x \geqslant 0$ with $q \in (p, p^*)$, then Theorem 3.12 improves Theorem 3.6 of Cardinali, Papageorgiou and Rubbioni [4], since our hypotheses on $f(z, x)$ (see H_2) are less restrictive than those used in [4] (see hypotheses H). For example, the function $f(x) = x^q lnx$ for $x \geqslant 1$ is excluded from the hypotheses in [4], while it is admissible here. It is interesting to know that Theorem 3.12 remains valid if $\beta \equiv 0$ (noncoercive differential operator).

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