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Combined effects of singular and sublinear nonlinearities in some elliptic problems

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ABSTRACT

We consider a parametric singular Dirichlet equation, with the singular term $u^{-\gamma}$ appearing in the left-hand side. We establish the existence and nonexistence of positive solutions as the parameter $\lambda > 0$ and the exponent $\gamma > 0$ of the singularity vary. In particular, we show that for all $\lambda > 0$ and all $\gamma \geqslant 1$, the problem has no positive solution. Our approach combines truncation arguments with the method of upper and lower solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric singular Dirichlet elliptic problem

$$\begin{cases} -\Delta u + \dfrac{1}{u^{\gamma}} = \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases} \tag{P_λ}$$

Here $\gamma > 0$, $\lambda > 0$ and $f(x, u)$ is a Carathéodory function (that is, for all $u \in \mathbb{R}$ the mapping $x \longmapsto f(x, u)$ is measurable and for a.a. $x \in \Omega$, $u \longmapsto f(x, u)$ is continuous).

The aim of this work is to examine the existence and nonexistence of positive solutions as $\lambda > 0$ and $\gamma > 0$ vary. By a solution of problem (P_λ) we understand the following.

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Definition 1. A function $u(\cdot)$ is a solution of problem (P$_\lambda$) if $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$, $u(x) > 0$ for a.a. $x \in \Omega$, $u^{-\gamma} \in L^1(\Omega)$, $u \geqslant \hat{c}d$ for some $\hat{c} > 0$ with $d(x) = d(x, \partial\Omega)$ and

$$\int_\Omega (Du, Dh)_{\mathbb{R}^N}dx + \int_\Omega \frac{h}{u^\gamma}dx = \lambda \int_\Omega f(x, u)hdx \quad \text{for all } h \in H_0^1(\Omega).$$

This problem with a reaction (right-hand side) independent of u, was investigated by Diaz, Morel and Oswald [1]. Problem (P$_\lambda$) differs from the usual singular equations encountered in the literature, where the singular term $u^{-\gamma}$ appears in the right-hand side. So, the problem under consideration in these cases is the following:

$$\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} + \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases}$$

In fact, the perturbation term $f(x, u)$ has specific form, namely $f(x, u) = f(u) = u^{q-1}$ with $2 < q < 2^*$. For such problems it is proved that the equation exhibits bifurcation phenomena. Namely, there exists a critical parameter value $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ the problem has at least two positive solutions, for $\lambda = \lambda^*$ there is one positive solution and for $\lambda > \lambda^*$ there is no positive solution. We refer to Carl and Perera [2], Ghergu and Rădulescu [3, Chapter 7], Perera and Silva [4,5], and the references therein. In our problem (P$_\lambda$), the singular term appears in the reaction with a negative sign and this changes the geometry of the problem.

We introduce the following conditions on the reaction $f(x, u)$:

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.a. $x \in \Omega$ and
 (i) for every $\rho > 0$, there exists a function $a_\rho \in L^\infty(\Omega)_+$ such that
 $f(x, u) \leqslant a_\rho(x)$ for a.a. $x \in \Omega$, all $0 \leqslant u \leqslant \rho$;
 (ii) there exists $q \in (1, 2)$ and $c_1 > 0$ such that
 $f(x, u) \geqslant c_1 u^{q-1}$ for a.a. $x \in \Omega$, all $u \geqslant 0$;
 (iii) $\limsup_{u\to+\infty} \frac{f(x,u)}{u^{q-1}} \leqslant \beta < +\infty$ uniformly for a.a. $x \in \Omega$;
 (iv) for a.a. $x \in \Omega$, all $u \geqslant 0$ and all $t \geqslant 1$, we have
 $f(x, tu) \leqslant tf(x, u)$.

In this setting we show that positive solutions exist only for large values of the positive parameter λ. More precisely, we prove the following existence theorem.

Theorem A. *If hypotheses* H *hold and* $\gamma \in (0, 1)$, *then there exists* $\lambda_* > 0$ *such that for all* $\lambda > \lambda_*$ *problem* (P$_\lambda$) *admits a solution* $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *with* $u_\lambda{}^{-\gamma} \in L^1(\Omega)$ *and* $u_\lambda \geqslant \hat{c}d$ *for some* $\hat{c} > 0$; *moreover for* $\lambda \in (0, \lambda_*)$ *there is no positive solution.*

Moreover, we investigate also the case $\gamma \geqslant 1$ and prove the following nonexistence result.

Theorem B. *Assume that hypotheses* H *hold,* $\lambda > 0$ *and* $\gamma \geqslant 1$. *Then problem* (P$_\lambda$) *has no solution* $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Our approach uses the method of upper and lower solutions. For this reason we define what we mean by upper and lower solutions for problem (P$_\lambda$).

Definition 2. (a) A function $\bar{u}(\cdot)$ is an upper solution for problem (P$_\lambda$), if $\bar{u} \in H_0^1(\Omega)$, $\bar{u}(x) > 0$ for a.a. $x \in \Omega$ and

$$\int_\Omega (D\bar{u}, Dh)_{\mathbb{R}^N}dx + \int_\Omega \frac{h}{\bar{u}^\gamma}dx \geqslant \lambda \int_\Omega f(x, \bar{u})hdx \quad \text{for all } h \in H_0^1(\Omega),\ h \geqslant 0.$$

(b) A function $\underline{u}(\cdot)$ is a lower solution for problem (P$_\lambda$), if $\underline{u} \in H_0^1(\Omega)$, $\underline{u}(x) > 0$ for a.a. $x \in \Omega$ and

$$\int_\Omega (D\underline{u}, Dh)_{\mathbb{R}^N}dx + \int_\Omega \frac{h}{\underline{u}^\gamma}dx \leqslant \lambda \int_\Omega f(x, \underline{u})hdx \quad \text{for all } h \in H_0^1(\Omega),\ h \geqslant 0.$$

Remark 1. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that $f(x, u) = 0$ for a.a. $x \in \Omega$ and for all $u \leqslant 0$. Hypothesis H(iv) is equivalent to saying that for a.a. $x \in \Omega$ the function $u \longmapsto \frac{f(x,u)}{u}$ is nonincreasing. So, the reaction $f(x, \cdot)$ is sublinear.

In addition to the Sobolev space $H_0^1(\Omega)$, we will also use the Banach space

$$C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(x) \geqslant 0 \ \text{ in } \Omega\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega\right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

For $a \in \mathbb{R}$, we set $a^{\pm} = \max\{0, \pm a\}$. For $u \in H_0^1(\Omega)$, we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that $u^{\pm} \in H_0^1(\Omega)$. Also, for all $u \in H_0^1(\Omega)$, we denote $\|u\| := \|Du\|_2$.

2. Auxiliary results

We start by considering the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta u = \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases} \tag{Au$_\lambda$}$$

For this problem we have the following existence result valid for all values of the parameter $\lambda > 0$. To the best of our knowledge, this result is new and of independent interest. Note that in problem (Au$_\lambda$) there is no singular term and so the regularity of the solution is easier to establish.

Proposition 3. *If hypotheses* H *hold and* $\lambda > 0$, *then problem* (Au$_\lambda$) *admits a positive solution* $\tilde{u}_\lambda \in \operatorname{int} C_+$.

Proof. Let $F(x, u) = \int_0^u f(x, s)ds$ and consider the C^1-functional $\psi_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{2}\|Du\|_2^2 - \lambda \int_\Omega F(x, u(x))dx \quad \text{for all } u \in H_0^1(\Omega).$$

By virtue of hypotheses H(i) and (iii), we have

$$F(x, u) \leqslant a(x)(1 + u^q) \quad \text{for a.a. } x \in \Omega,\ \text{all } u \geqslant 0,\ \text{with } a \in L^\infty(\Omega)_+.$$

Then for all $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} \psi_\lambda(u) &\geqslant \frac{1}{2}\|Du\|_2^2 - c_2(\|u\|_q^q + 1) \quad \text{for some } c_2 > 0 \\ &\geqslant \frac{1}{2}\|u\|^2 - c_3\|u\|^q - c_3 \quad \text{for some } c_3 > 0. \end{aligned} \tag{1}$$

Since $q < 2$, from (1) it follows that ψ_λ is coercive. Also, using the Sobolev embedding theorem, we check that ψ_λ is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in H_0^1(\Omega)$ such that

$$\psi_\lambda(\tilde{u}_\lambda) = \inf\left\{\psi_\lambda(u) : u \in H_0^1(\Omega)\right\}. \tag{2}$$

From H(ii) we have

$$F(x, u) \geqslant \frac{c_1}{q} u^q \quad \text{for a.a. } x \in \Omega,\ \text{all } u \geqslant 0.$$

For $u \in \operatorname{int} C_+$ and $t > 0$, we have

$$\psi_\lambda(tu) \leqslant \frac{t^2}{2}\|u\|^2 - \frac{c_1 t^q \lambda}{q}\|u\|_q^q.$$

Since $q < 2$, choosing $t \in (0, 1)$ small, we obtain

$$\begin{aligned} &\psi_\lambda(tu) < 0 \\ \Rightarrow\quad &\psi_\lambda(\tilde{u}_\lambda) < 0 = \psi_\lambda(0) \quad (\text{see } (2)),\ \text{hence } \tilde{u}_\lambda \neq 0. \end{aligned}$$

From (2) we have

$$\begin{aligned} &\psi_\lambda'(\tilde{u}_\lambda) = 0 \\ \Rightarrow\quad &A(\tilde{u}_\lambda) = \lambda N_f(\tilde{u}_\lambda), \end{aligned} \tag{3}$$

where $A \in \mathcal{L}(H_0^1(\Omega),\ H^{-1}(\Omega))$ is defined by $\langle A(u), y\rangle = \int_\Omega (Du, Dy)_{\mathbb{R}^N} dx$ for all $u, y \in H_0^1(\Omega)$ and $N_f(\cdot)$ is the Nemitsky map corresponding to f, that is,

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \quad \text{for all } u \in H_0^1(\Omega).$$

On (3) we act with $-\tilde{u}_\lambda^- \in H_0^1(\Omega)$ and obtain $\|D\tilde{u}_\lambda^-\|_2^2 = 0$, hence $\tilde{u}_\lambda \geqslant 0,\ \tilde{u}_\lambda \neq 0$. Then from (3) we have

$$\begin{cases} -\Delta\tilde{u}_\lambda = \lambda f(x, \tilde{u}_\lambda) & \text{for a.a. } x \in \Omega \\ \tilde{u}_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

From this equation and using the Moser iteration technique (see Ladyzhenskaya and Uraltseva [6, p. 286] and Gasinski and Papageorgiou [7, p. 737]), we have that $\tilde{u}_\lambda \in L^\infty(\Omega)$. Then the result of Lieberman [8, Theorem 1] (see also Gasinski and

Papageorgiou [7, p. 738]) implies $\tilde{u}_\lambda \in C_0^1(\overline{\Omega})$. In fact, this regularity of the solution can also be obtained using the Agmon–Douglis–Nirenberg theorem (see Brezis [9, p. 316]), which implies that $\tilde{u}_\lambda \in W^{2,q}(\Omega) \cap H_0^1(\Omega)$ $(q > N)$ and this, by virtue of the Sobolev embedding theorem, implies that $\tilde{u}_\lambda \in C_0^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0, 1)$. Finally note that hypothesis H(ii) implies that

$$\begin{aligned} & -\Delta u(x) \geqslant 0 \quad \text{for a.a. } x \in \Omega \Longrightarrow \Delta u(x) \leqslant 0 \quad \text{for a.a. } x \in \Omega \\ \Longrightarrow \quad & \tilde{u}_\lambda \in \operatorname{int} C_+ \quad \text{see, for example, Gasinski and Papageorgiou [7, p. 738].} \end{aligned}$$

This completes the proof. □

Let $\hat{\lambda}_1 > 0$ be the principal eigenvalue of $(-\Delta,\ H_0^1(\Omega))$ and $\hat{u}_1$ the corresponding positive, L^2–normalized (that is, $\|\hat{u}_1\|_2 = 1$) eigenfunction. It is well known that $\hat{u}_1 \in \operatorname{int} C_+$. Inspired by Diaz, Morel and Oswald [1], we introduce

$$\underline{u} = \xi \hat{u}_1^{\frac{2}{1+\gamma}} \in \operatorname{int} C_+ \quad (\xi > 0).$$

Proposition 4. *Assume that hypotheses* H *hold. Then for $\lambda > 0$ big enough, $\underline{u} \in \operatorname{int} C_+$ is a lower solution for problem* (P_λ)

Proof. We have

$$\begin{aligned} -\Delta \underline{u} + \frac{1}{\underline{u}^\gamma} &= -\operatorname{div}(D\underline{u}) + \frac{1}{\underline{u}^\gamma} \\ &= -\frac{2\xi}{1+\gamma} \operatorname{div}\left(\hat{u}_1^{\frac{1-\gamma}{1+\gamma}} D\hat{u}_1\right) + \frac{1}{\underline{u}^\gamma} \\ &= -\frac{2\xi}{1+\gamma} \hat{u}_1^{\frac{1-\gamma}{1+\gamma}} \operatorname{div}(D\hat{u}_1) - \frac{2\xi}{1+\gamma} \left(D\hat{u}_1^{\frac{1-\gamma}{1+\gamma}}, D\hat{u}_1\right)_{\mathbb{R}^N} + \frac{1}{\underline{u}^\gamma} \\ &= \frac{2\xi\hat{\lambda}_1}{1+\gamma} \hat{u}_1^{\frac{2}{1+\gamma}} - \frac{2\xi(1-\gamma)}{(1+\gamma)^2} \frac{1}{\hat{u}_1^{\frac{2\gamma}{1+\gamma}}} \|D\hat{u}_1\|_{\mathbb{R}^N}^2 + \frac{1}{\underline{u}^\gamma}. \end{aligned} \tag{4}$$

Recall that $\hat{u}_1 \in \operatorname{int} C_+$. So, given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$\|D\hat{u}_1(x)\|_{\mathbb{R}^N} \geqslant \epsilon \quad \text{for all } x \in \Omega_\delta^c = \{x \in \Omega : d(x, \partial\Omega) < \delta\} \tag{5}$$

$$\hat{u}_1(x) \geqslant \epsilon \quad \text{for all } x \in \Omega \backslash \Omega_\delta^c. \tag{6}$$

Let $\hat{\xi}_1 = \max\left\{\left[\frac{(1+\gamma)^2}{2(1-\gamma)\epsilon^2}\right]^{1/\gamma}, 1\right\}$. Then for $\xi \geqslant \hat{\xi}_1$ we have

$$\begin{aligned} \frac{1}{\underline{u}^\gamma} = \frac{1}{\xi^\gamma \hat{u}_1^{\frac{2\gamma}{1+\gamma}}} &\leqslant \frac{2(1-\gamma)\epsilon^2}{(1+\gamma)^2} \frac{1}{\hat{u}_1^{\frac{2\gamma}{1+\gamma}}} \\ &\leqslant \frac{2(1-\gamma)\xi \|D\hat{u}_1(x)\|_{\mathbb{R}^N}^2}{(1+\gamma)^2 \hat{u}_1(x)^{\frac{2\gamma}{1+\gamma}}} \quad \text{for all } x \in \Omega_\delta^c \text{ (see (5) and recall } \xi \geqslant \hat{\xi}_1 \geqslant 1). \end{aligned} \tag{7}$$

Let $\hat{\xi}_2 = \max\left\{\left[\frac{1+\gamma}{\hat{\lambda}\epsilon^2}\right]^{1/\gamma}, 1\right\}$. Then for $\xi \geqslant \hat{\xi}_2$ we have

$$\frac{1}{\underline{u}^\gamma} = \frac{1}{\xi^\gamma \hat{u}_1^{\frac{2\gamma}{1+\gamma}}} \leqslant \frac{\hat{\lambda}_1 \epsilon^2}{1+\gamma} \frac{1}{\hat{u}_1^{\frac{2\gamma}{1+\gamma}}} \leqslant \frac{\hat{\lambda}_1 \xi \hat{u}_1^{\frac{2}{1+\gamma}}}{1+\gamma} \quad \text{for all } x \in \Omega \backslash \Omega_\delta^c \text{ (see (6) and recall } \xi \geqslant \hat{\xi}_2 \geqslant 1). \tag{8}$$

We return to (4) and use (7) and (6). We see that if $\hat{\xi} = \max\{\hat{\xi}_1, \hat{\xi}_2\}$ and $\xi \geqslant \hat{\xi}$, then

$$-\Delta \underline{u} + \frac{1}{\underline{u}^\gamma} \leqslant \frac{3}{1+\gamma} \underline{u} \quad \text{for a.a. } x \in \Omega. \tag{9}$$

Let $\lambda_1^* = \frac{3\xi^{2-q}}{(1+\gamma)c_1} \|\hat{u}_1\|_\infty^{\frac{2(2-q)}{1+\gamma}}$. Then for $\lambda \geqslant \lambda_1^*$ we have

$$\begin{aligned} \lambda f(x, \underline{u}) &\geqslant \lambda c_1 \underline{u}^{q-1} \quad (\text{see H(iii)}) \geqslant \frac{3\xi^{2-q}}{1+\gamma} \hat{u}_1^{\frac{2(2-q)}{1+\gamma}} \xi^{q-1} \hat{u}_1^{\frac{2(q-1)}{1+\gamma}} = \frac{3\xi}{1+\gamma} \hat{u}_1^{\frac{2}{1+\gamma}} = \frac{3}{1+\gamma} \underline{u} \\ &\geqslant -\Delta \underline{u} + \frac{1}{\underline{u}^\gamma} \quad \text{a.e. in } \Omega \text{ (see (9))}. \end{aligned} \tag{10}$$

This shows that $\underline{u} \in \operatorname{int} C_+$ is a lower solution for problem (P_λ) for $\lambda \geqslant \lambda_1^*$, in the sense of Definition 2 (in fact, $\underline{u}$ is a strong lower solution for problem (P_λ) for $\lambda \geqslant \lambda_1$). This concludes the proof. □

Next we will produce an upper solution $\bar{u}_\lambda \in \operatorname{int} C_+$ such that $\underline{u} \leqslant \bar{u}_\lambda$.

Proposition 5. *If hypotheses* H *hold and* $\lambda > 0$, *then problem* (P_λ) *admits an upper solution* $\bar{u}_\lambda \in \operatorname{int} C_+$ *such that* $\underline{u} \leqslant \bar{u}_\lambda$.

Proof. Let $\tilde{u}_\lambda \in \operatorname{int} C_+$ be as in Proposition 3. By virtue of Lemma 3.3 of Filippakis, Kristaly and Papageorgiou [10], we can find $\mu \geqslant 1$ such that

$$\underline{u} \leqslant \mu \tilde{u}_\lambda = \bar{u}_\lambda \in \operatorname{int} C_+.$$

Then we have

$$\begin{aligned} -\Delta \bar{u}_\lambda = \mu(-\Delta \tilde{u}_\lambda) &= \lambda \mu f(x, \tilde{u}_\lambda) \quad (\text{see Proposition 3}) \\ &\geqslant \lambda f(x, \mu \tilde{u}_\lambda) \quad (\text{see H(iv)}) \\ &= \lambda f(x, \bar{u}_\lambda) \quad \text{a.e. in } \Omega, \end{aligned}$$

$$\Rightarrow -\Delta \bar{u}_\lambda + \frac{1}{\bar{u}_\lambda^\gamma} \geqslant \lambda f(x, \bar{u}_\lambda) \quad \text{a.e. in } \Omega,$$

$$\Rightarrow \bar{u}_\lambda \in \operatorname{int} C_+ \text{ is an upper solution for } (P_\lambda) \text{ and } \underline{u} \leqslant \bar{u}_\lambda.$$

This completes the proof. □

3. Case $\gamma \in (0, 1)$

We point out that all the results in this section remain valid if the nonlinear term $u^{-\gamma}$ in the left-hand side of problem (P_λ) is replaced with $g \in C^{0,\alpha}[0, \infty)$ $(0 < \alpha < 1)$ such that $g \geqslant 0$, g is nonincreasing on $(0, \infty)$ with $\lim_{s \to 0^+} g(s) = +\infty$ and there exist $C_0, \eta_0 > 0$ and $\gamma \in (0, 1)$ such that

$$g(s) \leqslant C_0 s^{-\gamma} \quad \text{for all } s \in (0, \eta_0). \tag{11}$$

As proved by Bénilan, Brezis and Crandall in [11], condition (11) is equivalent to the *property of compact support*, that is, for any $h \in L^1(\mathbb{R}^N)$ with compact support, there is a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u + g(u) = h \quad \text{a.e. in } \mathbb{R}^N.$$

The proof of the next proposition uses suitable truncation and comparison techniques going back to the seminal work of Rabinowitz [12]. However, the presence of the singular term $-u^{-\gamma}$ makes our case a little more complicated and some facts such as the differentiability and the weak lower semicontinuity of the truncated energy functional are not immediate and require a more careful analysis (see Claims 1 and 2 of the proof which follows).

Proposition 6. *Assume that hypotheses* H *hold,* $\lambda \geqslant \lambda_1^*$ *and* $\gamma \in (0, 1)$. *Then problem* (P_λ) *admits a solution* u_λ.

Proof. We consider the ordered pair $\{\bar{u}_\lambda, \underline{u}\}$ of upper–lower solutions of (P_λ) $(\lambda \geqslant \lambda_1^*)$ produced in Section 2. Then we introduce the following Carathéodory function

$$g_\lambda(x, u) = \begin{cases} \lambda f(x, \underline{u}(x)) - \underline{u}(x)^{-\gamma} & \text{if } u < \underline{u}(x) \\ \lambda f(x, u) - u^{-\gamma} & \text{if } \underline{u}(x) \leqslant u \leqslant \bar{u}_\lambda(x) \\ \lambda f(x, \bar{u}_\lambda(x)) - \bar{u}_\lambda(x)^{-\gamma} & \text{if } \bar{u}_\lambda(x) < u. \end{cases} \tag{12}$$

Since $\bar{u}_\lambda \in \operatorname{int} C_+$ (see Proposition 5), we can find $\vartheta > 0$ small such that

$$\begin{aligned} & \vartheta \hat{u}_1 \leqslant \bar{u}_\lambda \quad (\text{see Filippakis, Kristaly and Papageorgiou [10]}), \\ \Rightarrow \quad & \bar{u}_\lambda^{-\gamma} \leqslant (\vartheta \hat{u}_1)^{-\gamma} \\ \Rightarrow \quad & \frac{1}{\bar{u}_\lambda^\gamma} \in L^1(\Omega) \quad (\text{see Lazer and McKenna [13] and recall } \gamma \in (0, 1)). \end{aligned}$$

Since $\underline{u} \in \operatorname{int} C_+$, in a similar fashion we can show that $\frac{1}{\underline{u}^\gamma} \in L^1(\Omega)$.

Therefore we can define the primitive $G_\lambda(x, u) = \int_0^u g_\lambda(x, s)ds$ and consider the functional $\sigma_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega G_\lambda(x, u(x))dx \quad \text{for all } u \in H_0^1(\Omega).$$

Claim 1. *We have* $\sigma_\lambda \in C^1(H_0^1(\Omega))$ *and* $\sigma_\lambda'(u) = A(u) - N_{g_\lambda}(u)$ *for all* $u \in H_0^1(\Omega)$.

We introduce the following Carathéodory function

$$h_\lambda(x, u) = \begin{cases} \underline{u}(x)^{-\gamma} & \text{if } u < \underline{u}(x) \\ u^{-\gamma} & \text{if } \underline{u}(x) \leqslant u \leqslant \bar{u}_\lambda(x) \\ \bar{u}_\lambda(x)^{-\gamma} & \text{if } \bar{u}_\lambda(x) < u. \end{cases} \tag{13}$$

From the previous considerations, we know that we can define the primitive $H_\lambda(x,u) = \int_0^u h_\lambda(x,s)ds$ and then consider the functional $\xi_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\xi_\lambda(u) = \int_\Omega H_\lambda(x, u(x))dx \quad \text{for all } u \in H_0^1(\Omega).$$

Evidently, to prove Claim 1, it suffices to show that $\xi_\lambda \in C^1(H_0^1(\Omega))$. To this end, let $u, v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$. We have

$$\frac{1}{t}\left[\xi_\lambda(u+tv) - \xi_\lambda(u)\right] = \int_\Omega \left(\int_0^1 h_\lambda(x, u+\vartheta tv)d\vartheta\right) v\, dx. \tag{14}$$

Note that

$$\int_0^1 h_\lambda(x, u+\vartheta tv)d\vartheta \longrightarrow h_\lambda(x,v) \quad \text{for a.a. } x \in \Omega,\ \text{as } t \to 0^+.$$

From (13) we have

$$\begin{aligned}
\int_0^1 h_\lambda(x, u+\vartheta tv)d\vartheta &\leqslant 2\underline{u}(x)^{-\gamma} + \int_0^1 |u+\vartheta tv|^{-\gamma}d\vartheta \quad (\text{since } \underline{u} \leqslant \bar{u}_\lambda)\\
&\leqslant 2\underline{u}(x)^{-\gamma} + c_4 \max_{0\leqslant\vartheta\leqslant 1} |u+\vartheta tv|^{-\gamma}\chi_{\{\underline{u}\leqslant u+\vartheta tv\leqslant \bar{u}_\lambda\}}(x) \quad \text{a.e in } \Omega\\
&\qquad \text{for some } c_4 > 0 \text{ (see Takac [14, p. 233])}\\
&\leqslant c_5\underline{u}(x)^{-\gamma} \quad \text{for a.a. } x \in \Omega \text{ and some } c_5 > 0.
\end{aligned} \tag{15}$$

Since $\underline{u} \in \operatorname{int} C_+$, we can find $c_6 > 0$ such that

$$\underline{u}(x) \geqslant c_6 d(x) \quad \text{for all } x \in \bar{\Omega}.$$

Then from (15) we have

$$v\int_0^1 h(x, u+\vartheta tv)d\vartheta \leqslant c_7 d^{-\gamma} v \quad \text{for a.a. } x \in \Omega \text{ and some } c_7 > 0. \tag{16}$$

We have

$$c_7 d^{-\gamma} v = c_7 d^{1-\gamma}\frac{v}{d} \leqslant c_8 \frac{v}{d} \quad \text{for a.a. } x \in \Omega \text{ and some } c_8 > 0 \text{ (recall that } \gamma \in (0,1)).$$

Since $y \in H_0^1(\Omega)$, using Hardy's inequality (see Brezis [9, p. 313]), we deduce that

$$c_7 d^{-\gamma} v \in L^2(\Omega).$$

Because of (16), we can apply the dominated convergence theorem and then from (14) we have

$$\begin{aligned}
&\left\langle \xi_\lambda'(u), v\right\rangle = \int_\Omega h_\lambda(x,u)v\, dx \quad \text{for all } v \in H_0^1(\Omega),\\
\Rightarrow\quad &\xi_\lambda'(u) = N_{h_\lambda}(u) \text{ and so } \xi_\lambda \in C^1(H_0^1(\Omega)).
\end{aligned}$$

Therefore we conclude that $\sigma_\lambda \in C^1(H_0^1(\Omega))$ and $\sigma_\lambda'(u) = A(u) - N_{g_\lambda}(u)$ for all $u \in H_0^1(\Omega)$. This proves Claim 1.

Claim 2. *The functional σ_λ is sequentially weakly lower semicontinuous.*

Again, if we can show the sequential weak continuity of the functional ξ_λ, we would have proved Claim 2. To this end, let $u_n \xrightarrow{w} u$ in $H_0^1(\Omega)$. By passing to a subsequence if necessary, we deduce that

$$u_n \to u \quad \text{in } L^2(\Omega) \quad \text{and} \quad u_n(x) \to u(x) \quad \text{a.e. in } \Omega.$$

Hence we have

$$h_\lambda(x, u_n(x)) \to h_\lambda(x, u(x)) \quad \text{a.e. in } \Omega.$$

Note that

$$0 \leqslant h_\lambda(x, u_n(x)) \leqslant \frac{1}{\underline{u}(x)^\gamma} \quad \text{for a.a. } x \in \Omega,\ \text{all } n \geqslant 1 \text{ and } \underline{u}^{-\gamma} \in L^1(\Omega).$$

So, by virtue of the dominated convergence theorem, we have

$$\xi_\lambda(u_n) \to \xi_\lambda(u).$$

This proves Claim 2.

It is clear from (12) that σ_λ is coercive. Therefore by the Weierstrass theorem (see Claim 2), we can find $u_\lambda \in H_0^1(\Omega)$ such that

$$\begin{aligned} & \sigma_\lambda(u_\lambda) = \inf\left\{\sigma_\lambda(u) : u \in H_0^1(\Omega)\right\} \\ \Rightarrow\quad & \sigma_\lambda'(u_\lambda) = 0 \quad \text{(see Claim 1)} \\ \Rightarrow\quad & A(u_\lambda) = N_{g_\lambda}(u_\lambda) \quad \text{(see Claim 1)}. \end{aligned} \tag{17}$$

On (17) first we act with $(u_\lambda - \bar{u}_\lambda)^+ \in H_0^1(\Omega)$. We have

$$\begin{aligned} \left\langle A(u_\lambda), (u_\lambda - \bar{u}_\lambda)^+\right\rangle &= \int_\Omega g_\lambda(x, u_\lambda)(u_\lambda - \bar{u}_\lambda)^+ dx \\ &= \int_\Omega \left[f(x, \bar{u}_\lambda) - \bar{u}_\lambda^{-\gamma}\right](u_\lambda - \bar{u}_\lambda)^+ dx \quad \text{(see (12))} \\ &\leqslant \left\langle A(\bar{u}_\lambda), (u_\lambda - \bar{u}_\lambda)^+\right\rangle \quad \text{(see Proposition 5 and Definition 2)} \\ \Rightarrow \|D(u_\lambda - \bar{u}_\lambda)^+\|_2^2 \leqslant 0, &\quad \text{hence } u_\lambda \leqslant \bar{u}_\lambda. \end{aligned}$$

In a similar fashion, acting on (17) with $(\underline{u} - u_\lambda)^+ \in H_0^1(\Omega)$ and using Proposition 4, we show that $\underline{u} \leqslant u_\lambda$. So, we have proved that

$$u_\lambda \in [\underline{u}, \bar{u}_\lambda] = \left\{u \in H_0^1(\Omega) : \underline{u}(x) \leqslant u(x) \leqslant \bar{u}_\lambda(x) \text{ a.e. in } \Omega\right\}. \tag{18}$$

Then from (12), (17) and (18) it follows that

$$\begin{aligned} & -\Delta u_\lambda(x) + u_\lambda(x)^{-\gamma} = \lambda f(x, u_\lambda(x)) \quad \text{a.e. in } \Omega,\ u_\lambda|_{\partial\Omega} = 0, \\ \Rightarrow\quad & u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega) \quad \text{is a solution of } (\mathrm{P}_\lambda). \end{aligned}$$

The proof is now complete. □

We introduce the following sets

$\mathcal{L} = \{\lambda > 0 : \text{ problem } (\mathrm{P}_\lambda) \text{ admits a solution}\}$

$S(\lambda) = $ the solution set of (P_λ).

We set $\lambda_* = \inf \mathcal{L}$. From the previous analysis, we have

$\mathcal{L} \neq \emptyset$ and $\lambda_* \leqslant \lambda_1{}^*$ (see Proposition 6).

The proof of the next proposition is essentially based on the well-known principle that between an ordered pair of upper and lower solutions, we can locate a solution of our problem. The presence of the singular term can be accommodated using the ideas and facts from the proof of Proposition 6. For the convenience of the reader we include the detailed proof.

Proposition 7. *Assume that hypotheses* H *hold,* $\lambda \in \mathcal{L}$ *and* $\lambda < \mu$. *Then* $\mu \in \mathcal{L}$.

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S(\lambda) \subseteq H_0^1(\Omega) \cap L^\infty(\Omega)$. We have

$$\begin{aligned} & -\Delta u_\lambda + u_\lambda{}^{-\gamma} = \lambda f(x, u_\lambda) \leqslant \mu f(x, u_\lambda) \quad \text{in } \Omega \text{ (since } \lambda < \mu \text{ and } f \geqslant 0) \\ \Rightarrow\quad & u_\lambda \text{ is a lower solution for problem } (P_\mu) \quad \text{(see Definition 2)}. \end{aligned}$$

Let $\tilde{u}_\mu \in \operatorname{int} C_+$ be the solution of problem (Au_μ) produced in Proposition 3. We can find $t \geqslant 1$ big such that $u_\lambda \leqslant t\tilde{u}_\mu = u_\mu^* \in \operatorname{int} C_+$ (see Filippakis, Kristaly and Papageorgiou [10]). We have

$$\begin{aligned} & -\Delta u_\mu{}^* = -t\Delta\tilde{u}_\mu = \mu t f(x, \tilde{u}_\mu) \geqslant \mu f(x, t\tilde{u}_\mu) = \mu f(x, u_\mu^*) \quad \text{in } \Omega \text{ (see H(iv))}, \\ \Rightarrow\quad & -\Delta u_\mu^* + (u_\mu^*)^{-\gamma} \geqslant \mu f(x, u_\mu^*) \quad \text{in } \Omega, \\ \Rightarrow\quad & u_\mu^* \in \operatorname{int} C_+ \text{ is an upper solution for problem } (P_\mu) \quad \text{(see Definition 2)}. \end{aligned}$$

We introduce the following Carathéodory function

$$g_\mu(x, u) = \begin{cases} \mu f(x, u_\lambda(x)) - u_\lambda(x)^{-\gamma} & \text{if } u < u_\lambda(x) \\ \mu f(x, u) - x^{-\gamma} & \text{if } u_\lambda(x) \leqslant u \leqslant u_\mu^*(x) \\ \mu f(x, u_\mu^*(x)) - u_\mu^*(x)^{-\gamma} & \text{if } u_\mu^*(x) < u. \end{cases} \tag{19}$$

Note that $u_\mu^*(x)^{-\gamma} \leqslant u_\lambda(x)^{-\gamma}$ and so $u_\mu^*(\cdot)^{-\gamma} \in L^1(\Omega)$ (see Definition 1). So, we can introduce $G_\mu(x, u) = \int_0^u g_\mu(x, s)ds$ and consider the functional $\psi_\mu : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\mu(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega G_\mu(x, u(x))dx \quad \text{for all } u \in H_0^1(\Omega).$$

Since $u_\lambda \geqslant \hat{c}d$ (recall $u_\lambda \in S(\lambda)$ and see Definition 1), reasoning as in the proof of Proposition 6 (see Claims 1 and 2), we show that $\psi_\mu \in C^1(H_0^1(\Omega))$ and that it is sequentially weakly lower semicontinuous. Moreover, from (19) it is clear that ψ_μ is coercive. So, by the Weierstrass theorem, we can find $u_\mu \in H_0^1(\Omega)$ such that

$$\begin{aligned} & \psi_\mu(u_\mu) = \inf\{\psi_\mu(u) :\ u \in H_0^1(\Omega)\}, \\ \Rightarrow\quad & \psi_\mu'(u_\mu) = 0, \\ \Rightarrow\quad & A(u_\mu) = N_{g_\mu}(u_\mu). \end{aligned} \tag{20}$$

Acting on (20) first with $(u_\mu - u_\mu^*)^+ \in H_0^1(\Omega)$ and then $(u_\lambda - u_\mu)^+ \in H_0^1(\Omega)$, as in the proof of the Proposition 6, we show that

$$\begin{aligned} & u_\mu \in [u_\lambda, u_\mu^*] = \left\{u \in H_0^1(\Omega) :\ u_\lambda(x) \leqslant u(x) \leqslant u_\mu^*(x) \text{ a.e in } \Omega\right\}, \\ \Rightarrow\quad & u_\mu \in S(\mu) \subseteq H_0^1(\Omega) \cap L^\infty(\Omega) \quad \text{and so } \ \mu \in \mathcal{L}. \end{aligned}$$

This completes the proof. $\square$

Proposition 8. *If hypotheses* H *hold, then* $\lambda_* > 0$.

Proof. We argue by contradiction. So, suppose that $\lambda_* = 0$. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L}$ with $\lambda_n \downarrow 0$ and $u_n \in S(\lambda_n) \subseteq H_0^1(\Omega) \cap L^\infty(\Omega)$. From the proof of Proposition 7, it is clear that we can choose $\{u_n\}_{n\geqslant 1}$ to be decreasing. We have

$$A(u_n) + u_n{}^{-\gamma} = \lambda_n N_f(u_n), \tag{21}$$

$$\Rightarrow\quad \|Du_n\|_2^2 \leqslant \lambda_n \int_\Omega f(x, u_n)u_n dx \leqslant \lambda_1 \int_\Omega f(x, u_n)u_n dx \quad \text{for all } n \geqslant 1. \tag{22}$$

Since $u_n \leqslant u_1$ for all $n \geqslant 1$, from (22) and hypothesis H(i) it follows that $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{23}$$

Suppose that $u = 0$. From (21) we have

$$\int_\Omega u_n{}^{-\gamma}|y|dx \leqslant M_1\|y\| \quad \text{for all } y \in C_c^1(\Omega),\ \text{all } n \geqslant 1 \text{ and some } M_1 > 0.$$

On the other hand by Fatou's lemma and since we have assumed that $u = 0$, we have

$$\int_\Omega u_n{}^{-\gamma}|y|dx \to +\infty, \quad \text{a contradiction.}$$

Therefore $u \neq 0$. From (21) we see that for every $y \in H_0^1(\Omega) \cap L^\infty(\Omega),\ y \geqslant 0$, we have

$$\int_\Omega (Du_n, Dy)_{\mathbb{R}^N} dx + \int_\Omega u_n{}^{-\gamma} y\, dx = \lambda_n \int_\Omega f(x, u_n) y\, dx \quad \text{for all } n \geqslant 1. \tag{24}$$

Note that

$$\int_\Omega (Du_n, Dy)_{\mathbb{R}^N} dx \to \int_\Omega (Du, Dy)_{\mathbb{R}^N} dx \quad (\text{see } (23)),$$

and $\lambda_n \int_\Omega f(x, u_n)y\, dx \to 0$ (recall that $\lambda_n \downarrow 0,\ u_n \leqslant u_1$ for all $n \geqslant 1$ and see H(i)).

Also, from Fatou's lemma we have

$$\liminf_{n\to\infty} \int_\Omega u_n{}^{-\gamma} y\, dx \geqslant \int_\Omega u^{-\gamma} y\, dx \quad (\text{see } (23)).$$

So, if in (24) we pass to the limit as $n \to \infty$, then

$$\int_\Omega (Du, Dy)_{\mathbb{R}^N} dx + \int_\Omega u^{-\gamma} y\, dx \leqslant 0 \quad \text{for all } y \in H_0^1(\Omega) \cap L^\infty(\Omega),\ y \geqslant 0.$$

Let $y = u \in H_0^1(\Omega) \cap L^\infty(\Omega),\ u \geqslant 0$. Then

$$\begin{aligned} & \|Du\|_2^2 + \int_\Omega u^{1-\gamma} dx \leqslant 0, \\ \Rightarrow\quad & u = 0, \quad \text{a contradiction.} \end{aligned}$$

This proves that $\lambda_* > 0$. $\square$

So, summarizing the situation for problem (P_λ), we can formulate the following theorem describing the existence and nonexistence of solutions for problem (P_λ) as $\lambda > 0$ varies.

Theorem 9. *If hypotheses* H *hold and* $\gamma \in (0, 1)$, *then there exists* $\lambda_* > 0$ *such that for all* $\lambda > \lambda_*$ *problem* (P_λ) *admits a solution* $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *with* $u_\lambda^{-\gamma} \in L^1(\Omega)$ *and* $u_\lambda \geqslant \hat{c}d$ *for some* $\hat{c} > 0$; *moreover for* $\lambda \in (0, \lambda_*)$ *there is no positive solution.*

Remark 2. It is open what happens in the critical case $\lambda = \lambda_*$. It will be interesting to study what happens to the positive solution as $\lambda \downarrow \lambda_*$. Such a study will provide information on what happens in the critical case $\lambda = \lambda_*$.

4. Case $\gamma \geqslant 1$

It is natural to ask what happens when $\gamma \geqslant 1$. The next theorem settles this case.

Theorem 10. *Assume that hypotheses* H *hold,* $\lambda > 0$ *and* $\gamma \geqslant 1$. *Then problem* (P_λ) *has no solution* $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Arguing indirectly, suppose that for some $\lambda > 0$, we can find a solution $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for problem (P_λ). From Lazer and McKenna [13] we know that

$$\int_\Omega u_\lambda^{-\gamma} dx = +\infty \quad (\text{recall } \gamma \geqslant 1). \tag{25}$$

For every integer $k \geqslant 1$, we consider the open set

$$\Omega_k = \left\{ x \in \Omega : d(x, \partial\Omega) > \frac{1}{k} \right\}.$$

We have

$$-\Delta u_\lambda = \lambda f(x, u_\lambda) - u_\lambda^{-\gamma} \quad \text{in } \bar{\Omega}_k \quad \text{and} \quad \lambda f(\cdot, u_\lambda(\cdot)) - u_\lambda(\cdot)^{-\gamma} \in L^\infty(\Omega),$$

$$\Rightarrow \int_{\Omega_k} (-\Delta u_\lambda) dx + \int_{\Omega_k} u_\lambda^{-\gamma} dx = \lambda \int_{\Omega_k} f(x, u_\lambda) dx \leqslant \lambda \int_\Omega f(x, u_\lambda) dx \quad (\text{since } f \geqslant 0, \text{ see H(ii)}). \tag{26}$$

By Green's identity (see Gasinski and Papageorgiou [7, p. 209]), we have

$$\int_{\Omega_k} \Delta u_\lambda dx = \int_{\partial\Omega_k} \frac{\partial u_\lambda}{\partial n} d\sigma. \tag{27}$$

Using (27) in (26), we obtain

$$-\int_{\partial\Omega_k} \frac{\partial u_\lambda}{\partial n} d\sigma + \int_{\Omega_k} u_\lambda^{-\gamma} dx \leqslant M_2 \quad \text{for some } M_2 > 0 \text{ (see H(i))}.$$

Let $k \to \infty$. We obtain

$$-\int_{\partial\Omega} \frac{\partial u_\lambda}{\partial n} d\sigma + \int_\Omega u_\lambda^{-\gamma} dx \leqslant M_2,$$

which contradicts (25). This proves the theorem. □

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