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MULTIPLICITY OF SOLUTIONS FOR NONLINEAR NONHOMOGENEOUS ROBIN PROBLEMS

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Abstract. We consider a nonlinear nonhomogeneous Robin problem, with an indefinite concave term near the origin and a perturbation of arbitrary growth. By modifying the perturbation and using a variant of the symmetric mountain pass theorem due to Heinz (J. Diff. Equ. 66 (1987)), we show that the problem has a whole sequence of distinct nontrivial smooth solutions converging to the trivial solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous Robin problem

(1)
$$\left\{ \begin{array}{l} -\mathrm{div}\, a(Du(z)) + \xi(x)|u(z)|^{p-2}u(z) = \vartheta(z)|u(z)|^{q-2}u(z) + f(z,u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega,\ 1 < q < p < \infty\,. \end{array} \right\}.$$

In this problem, $a : \mathbb{R}^N \to \mathbb{R}^N$ is a strictly monotone and continuous map (hence maximal monotone, too) which satisfies certain other regularity and growth conditions listed in hypotheses $H(a)$ below. These conditions are general enough to incorporate in our framework many differential operators of interest such as the p-Laplacian and the (p,τ)-Laplacian (that is, the sum of a p-Laplacian and a τ-Laplacian, $1 < \tau < p < \infty$). The potential function $\xi \in L^\infty(\Omega)$ satisfies $\xi \geq 0$ and the reaction term (right-hand side of the equation) has a "concave" part (this is the $(p-1)$-sublinear function $x \mapsto \vartheta(z)|x|^{q-2}x$, recall that $1 < q < p$) and a perturbation $f(z,x)$, which is a continuous, odd in the x-variable function and can have arbitrary growth near $\pm\infty$.

As it was documented by the important works of Ambrosetti, Brezis and Cerami [1], Ambrosetti, Garcia Azorero and Peral [2] and Wang [17], the presence of a concave term of the form $\lambda|x|^{q-2}x$ with $\lambda > 0$, $x \in \mathbb{R}$, $1 < q < p$, leads to a bifurcation from the trivial pair. The aforementioned works deal with Dirichlet problems, in Ambrosetti, Brezis and Cerami [1] and Wang [17] the differential

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operator is the Laplacian and in Ambrosetti, Garcia Azorero and Peral [2] is the p-Laplacian ($1 < p < \infty$).

The novelties of this work are that the differential operator is more general and it is not homogeneous, the boundary condition is Robin (including the Neumann case when $\beta \equiv 0$), the concave term is indefinite (that is, the coefficient function $\vartheta \in L^\infty(\Omega)$ is sign-changing) and there is no global growth restriction on $f(z, \cdot)$ (as is the case also in Wang [17]). We mention also the recent work of Papageorgiou and Winkert [15], who examine Dirichlet $(p, 2)$-equations with a negative concave term and prove multiplicity results. In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the generalized normal derivative defined by extension of the map

$$u \mapsto \frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N} \text{ for all } u \in C^1(\overline{\Omega}),$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This normal derivative is dictated by the nonlinear Green's identity (see Gasiński and Papageorgiou [4, p. 210]) and was also used by Lieberman [7]. The boundary coefficient $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ satisfies $\beta(z) \geq 0$ for all $z \in \partial\Omega$. We assume that $\xi \neq 0$ or $\beta \neq 0$. So, our framework includes Neumann problems (when $\beta \equiv 0$).

Under these conditions we show that problem (1) admits a sequence of nontrivial smooth solutions $u_n \in C^1(\overline{\Omega})$ ($n \in \mathbb{N}$), converging to the trivial solution. Our method is based on an abstract multiplicity result of Heinz [5] and on a variant of a result of Wang [17], which permits the replacement of the perturbation $f(z, x)$ by a more convenient one. We mention that recently Papageorgiou and Rădulescu [12] produced a sequence of nodal solutions for a class of Robin problems using similar tools. However, their hypotheses excluded the case of an indefinite concave term.

We stress that contrary to all the previous works involving a concave contribution in the reaction, here the coefficient of this term is indefinite (that is, sign-changing). This is the source of several difficulties, which require a more careful analysis in order to be able to apply the extension of the symmetric mountain pass theorem of Heinz [5]. Also, in contrast to the recent work of Papageorgiou and Rădulescu [12], we do not assume that the reaction term has zeros of constant sign. Our hypothesis on the perturbation term $f(z, x)$ are minimal. We only impose a symmetry condition, namely that $f(z, \cdot)$ is odd (needed to apply the result of Heinz [5]) and we also assume that $f(z, \cdot)$ is $(p-1)$-sublinear in a neighborhood of the origin. So, near zero our reaction term exhibits competition phenomena between "sublinear" and "superlinear" terms (concave-convex nonlinearity). Such a situation is excluded from the hypotheses in [12]. An interesting open problem resulting from our analysis is whether we can have a sequence of nodal (that is, sign-changing) solutions converging to zero. Our result (see Theorem 3.3) provides no information about the sign of the solutions $\{u_n\}_{n \geq 1}$. Finally we point out that our differential operator is general (it includes as special cases many differential operators of interest) and also is nonhomogeneous. So, our work provides a unifying framework for a large class of nonlinear equations. The same can be said about the boundary condition, since the case $\beta \equiv 0$ is allowed, incorporating in our analysis also the usual Neumann problem. Our work illustrates that the cut-off techniques initiated by Wang [17] can be implemented to a much broader class of boundary value problems and opens the way for more results concerning the existence and multiplicity of solutions for equations with a reaction term of arbitrary growth.

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that $\varphi(\cdot)$ satisfies the "Palais-Smale condition" ("PS-condition", for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that

$\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(u_n) \to 0$ in X^*,

admits a strongly convergent subsequence".

The result of Heinz [5] which we will use, is a variant of the symmetric mountain pass theorem (see, for example, Gasinski and Papageorgiou [4, p. 688]) and says the following.

Theorem 2.1. *Assume that X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$, it satisfies the PS-condition, it is even and bounded below, $\varphi(0) = 0$ and for every $n \in \mathbb{N}$ there exist an n-dimensional subspace V_n of X and $\rho_n > 0$ such that*

$$\sup[\varphi(u) : u \in V_n,\ ||u|| = \rho_n] < 0.$$

Then there exists a sequence $\{u_n\}_{n\geq 1}$ of critical points of φ such that

$$\varphi(u_n) < 0 \text{ for all } n \in \mathbb{N} \text{ and } \varphi(u_n) \to 0 \text{ as } n \to \infty\,.$$

In what follows, given $\varphi \in C^1(X, \mathbb{R})$, we define

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}$$

(the critical set of φ).

Let $\eta \in C^1(0, \infty)$ such that $\eta(t) > 0$ for all $t > 0$ and

(2)

$$0 < \hat{c} \leq \frac{\eta'(t)t}{\eta(t)} \leq c_0 \text{ and } c_1 t^{p-1} \leq \eta(t) \leq c_2(1 + t^{p-1}) \text{ for all } t > 0, \text{ some } c_1, c_2 > 0.$$

The hypotheses on the map $a(\cdot)$ are the following:

$H(a):$ $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, +\infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $(0, +\infty)$, $a_0(t)t \to 0^+$ as $t \to 0^+$ and
$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) there exists $c_3 > 0$ such that
$$|\nabla a(y)| \leq c_3 \frac{\eta(|y|)}{|y|} \text{ for all } y \in \mathbb{R}^N \backslash \{0\};$$

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{\eta(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N \backslash \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)sds$, then $a_0(t)t^2 - qG_0(t) \geq \hat{c}_0 t^p$ for some $\hat{c}_0 > 0$, all $t > 0$ and
$$\lim_{t\to 0^+} \frac{G_0(t)}{t^q} = 0.$$

Remark 2.2. Hypotheses $H(a)(i), (ii), (iii)$ are motivated by the nonlinear regularity theory of Lieberman [7]. Hypothesis $H(a)(iv)$ serves the needs of our problem here. However, it is not restrictive and it is satisfied in all cases of interest as the examples which follow illustrate. Similar conditions on the map $a(\cdot)$ can be found in the work of Papageorgiou and Rădulescu [12].

Evidently the above hypotheses imply that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing. We set

$$G(y) = G_0(|y|) \text{ for all } y \in \mathbb{R}^N.$$

Then we see that

$$G(\cdot) \text{ is convex and } G(0) = 0. \tag{3}$$

Also, we have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \text{ for all } y \in \mathbb{R}^N\backslash\{0\},\ \nabla G(0) = 0.$$

Hence $G(\cdot)$ is the primitive of $a(\cdot)$ and so from (3) it follows that

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{4}$$

The next lemma summarizes the main properties of the map $a(\cdot)$. It is a straightforward consequence of hypotheses $H(a)(i),(ii),(iii)$.

Lemma 2.3. *If hypotheses $H(a)(i),(ii),(iii)$ hold, then*

(a) *the map $y \mapsto a(y)$ is strictly monotone and continuous (thus, maximal monotone too);*
(b) $|a(y)| \leq c_4(1 + |y|^{p-1})$ *for all* $y \in \mathbb{R}^N$, *some* $c_4 > 0$;
(c) $(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

This lemma together with (4) lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.4. *If hypotheses $H(a)(i),(ii),(iii)$ hold, then* $\frac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_5(1 + |y|^p)$ *for all* $y \in \mathbb{R}^N$, *some* $c_5 > 0$.

The examples which follow illustrate the generality of our framework. They cover many differential operators of interest.

Example 2.5. The following maps $a(\cdot)$ satisfy hypotheses $H(a)$ above.

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$.
This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$.
This map corresponds to the (p, q)-Laplace differential operator defined by

$$\Delta_p u + \Delta_q u \text{ for all } u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics. Recently there have been some multiplicity results for such equations. We mention the works of Mugnai and Papageorgiou [8], Papageorgiou and Rădulescu [9, 10], Papageorgiou and Winkert [15] and Sun, Zhang and Su [16].

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y$ with $1 < p < \infty$.
This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}((1 + |Du|^2)^{\frac{p-2}{2}}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y + \frac{|y|^{p-2}y}{1+|y|^p}$ with $1 < p < \infty$.
This map corresponds to the differential operator

$$\Delta_p u + \operatorname{div}\left(\frac{|Du|^{p-2}Du}{1+|Du|^p}\right) \text{ for all } u \in W^{1,p}(\Omega).$$

This differential operator arises in problems of plasticity.

The hypotheses on the other data of problem (1) are the following:

H(ϑ) : $\vartheta \in L^\infty(\Omega)$ and there exists an open set $U \subseteq \Omega$ such that $\vartheta(z) > 0$ for almost all $z \in U$.

H(f) : $f \in C(\Omega \times (-\tau, \tau))$ for some $\tau > 0$, for all $z \in \Omega$, $f(z, \cdot)$ is odd and

$$\lim_{x \to 0} \frac{f(z,x)}{|x|^{p-2}x} = 0 \text{ uniformly for almost all } z \in \Omega\,.$$

Remark 2.6. We stress that we do not impose any global growth condition on $f(z, \cdot)$. We have only conditions concerning the behavior of $f(z, \cdot)$ near zero.

H(ξ) : $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for almost all $z \in \Omega$.
H(β) : $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$.
H_0 : $\xi \neq 0$ or $\beta \neq 0$.

In what follows, by $\sigma(\cdot)$ we denote the $(N-1)$-dimensional Hausdorff (surface) measure on $\partial\Omega$. Also, the restriction of any Sobolev function on $\partial\Omega$ is understood in the sense of traces.

When $\beta \neq 0$, with $\hat{\beta} = \frac{p-1}{c_1}\beta$ we define

$$\tilde{\lambda}_1 = \inf\left[\frac{||Du||_p^p + \int_{\partial\Omega} \hat{\beta}(z)|u|^p d\sigma}{||u||_p^p} : u \in W^{1,p}(\Omega), u \neq 0\right].$$

When $\xi \neq 0$, with $\hat{\xi} = \frac{p-1}{c_1}\xi$, we define

$$\tilde{\lambda}_1 = \inf\left[\frac{||Du||_p^p + \int_{\Omega} \hat{\xi}(z)|u|^p dz}{||u||_p^p} : u \in W^{1,p}(\Omega), u \neq 0\right].$$

From Papageorgiou and Rădulescu [11] (case $\beta \neq 0$) and from Cardinali, Papageorgiou and Rubbioni [3] (Lemma 3.1, case $\xi \neq 0$), we have

$$\tilde{\lambda}_1 > 0. \tag{5}$$

We consider the C^1-functional $\gamma : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma(u) = ||Du||_p^p + \int_\Omega \hat{\xi}(z)|u|^p dz + \int_{\partial\Omega} \hat{\beta}(z)|u|^p d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

We set

$$\hat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{||u||_p^p} : u \in W^{1,p}(\Omega), u \neq 0\right].$$

Then

$$\hat{\lambda}_1 \geq \tilde{\lambda}_1 > 0 \text{ (see (5))}. \tag{6'}$$

3. Multiple solutions

We start with a lemma, which is inspired by Lemma 2.3 of Wang [17]. In our case we have to accommodate the fact that the concave term is indefinite.

Lemma 3.1. *If hypotheses $H(f)$ hold and $\mu \in (0, \hat{\lambda}_1)$, then there exist $b \in \left(0, \frac{\tau}{2}\right)$ and $\hat{f} \in C(\Omega \times \mathbb{R})$ such that*

- *$\hat{f}(z, \cdot)$ is odd for all $z \in \Omega$;*
- *$\hat{f}(z, x) = f(z, x)$ for all $z \in \Omega$, all $|x| \leq \dfrac{b}{2}$;*
- *if $\hat{F}(z, x) = \displaystyle\int_0^x \hat{f}(z, s)ds$,*
  then $|\hat{F}(z, x)| \leq \dfrac{\mu}{p}|x|^p$ for all $(z, x) \in \Omega \times \mathbb{R}$;
- *$\hat{f}(z, x)x - q\hat{F}(z, x) \leq \mu c_6 |x|^p$ for all $(z, x) \in \Omega \times \mathbb{R}$ and with*
  $c_6 = \min\left\{\hat{c}_0, \dfrac{c_1(p-q)}{p(p-1)}\right\}$.

Proof. Hypotheses $H(f)$ imply that given $\epsilon > 0$, we can find $b = b(\epsilon) \in (0, \tau)$ such that

$$|f(z, x)| \leq \epsilon |x|^p \text{ and } |F(z, x)| \leq \epsilon |x|^p \text{ for almost all } z \in \Omega, \text{ all } |x| \leq b. \tag{6}$$

We choose a cut-off function $\chi \in C^1(\mathbb{R})$ such that

$$\chi(\cdot) \text{ is even}, \ \chi(t) = 1 \text{ if } |t| \leq \frac{b}{2}, \ \chi(t) = 0 \text{ if } |t| \geq b$$
$$|\chi'(t)| \leq \frac{4}{b} \text{ and } \chi'(t)t \leq 0 \text{ for all } t \geq 0. \tag{7}$$

Given $k > 0$, we define

$$\hat{F}(z, x) = \chi(x)F(z, x) + (1 - \chi(x))k|x|^p. \tag{8}$$

We set $\hat{f}(z, x) = \hat{F}'_x(z, x)$. We have

$$\hat{f}(z, x) = \chi'(x)F(z, x) + \chi(x)f(z, x) - \chi'(x)k|x|^p + pk(1 - \chi(x))|x|^{p-2}x,$$
$$\Rightarrow \quad \hat{f}(z, \cdot) \text{ is odd and } \hat{f}(z, x) = f(z, x) \text{ for all } z \in \Omega, \text{ all } |x| \leq \frac{b}{2} \text{ (see (7))}.$$

Also for all $(z, x) \in \Omega \times \mathbb{R}$, we have

$$\begin{aligned} \hat{f}(z, x)x - q\hat{F}(z, x) = & \chi'(x)xF(z, x) \\ & + \chi(x)f(z, x)x - \chi'(x)xk|x|^p + pk(1 - \chi(x))|x|^p \\ & - q\chi(x)F(z, x) - q(1 - \chi(x))k|x|^p \text{ (see (8))}. \end{aligned} \tag{9}$$

From (6), (7), (8) we infer that

$$|\hat{F}(z, x)| \leq (\epsilon + k)|x|^p \text{ for all } (z, x) \in \Omega \times \mathbb{R}.$$

We choose $\epsilon, k > 0$ such that $\epsilon + k \leq \frac{\mu}{p}$. Then

$$|\hat{F}(z, x)| \leq \frac{\mu}{p}|x|^p \text{ for all } (z, x) \in \Omega \times \mathbb{R}.$$

Also, we have

$$\chi(x)\,[f(z,x)x - qF(z,x)] + pk(1-\chi(x))|x|^p \le (\epsilon(1+q)+pk-q)|x|^p, \tag{10}$$

$$\chi'(x)x\,[F(z,x)x - k|x|^p] \ge 0 \text{ for all } (z,x) \in \Omega\times\mathbb{R}, \tag{11}$$

(see (6), (7) and choose $\epsilon \in (0,k]$).

Returning to (9) and using (10) and (11), we obtain

$$\hat{f}(z,x)x - q\hat{F}(z,x) \le (\epsilon(1+q)+pk-q)|x|^p \text{ for all } (z,x)\in\Omega\times\mathbb{R}. \tag{12}$$

Choosing $\epsilon \in (0,k]$ even smaller if necessary, we can have

$$\epsilon(1+q)+pk-q \le \mu c_6.$$

Then from (12) it follows that

$$\hat{f}(z,x)x - q\hat{F}(z,x) \le \mu c_6 |x|^p \text{ for all } (z,x)\in\Omega\times\mathbb{R}\,.$$

□

Using this lemma, we introduce the C^1-functional $\hat{\varphi} : W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\begin{aligned}\hat{\varphi}(u) = \int_\Omega G(Du)dz \quad &+ \quad \frac{1}{p}\int_\Omega \xi(z)|u|^p dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p d\sigma \\ &- \quad \frac{1}{q}\int_\Omega \vartheta(z)|u|^q dz - \int_\Omega \hat{F}(z,u)dz \text{ for all } u\in W^{1,p}(\Omega).\end{aligned}$$

Using Corollary 2.4 and Lemma 3.1, we have

$$\hat{\varphi}(u) \ge \frac{c_1}{p(p-1)}\left[1-\frac{\mu}{\hat{\lambda}_1}\right]\gamma(u) - c_7||u||^q \text{ for some } c_7>0.$$

Since $\mu < \hat{\lambda}$ and $q<p$, it follows that $\hat{\varphi}$ is coercive. Therefore

- $\hat{\varphi}$ is even (see Lemma 3.1);
- $\hat{\varphi}$ is bounded below;
- $\hat{\varphi}$ satisfies the PS-condition (see Papageorgiou and Winkert [14]);
- $\hat{\varphi}(0)=0$.

Proposition 3.2. *If hypotheses $H(a)$, $H(\vartheta)$, $H(f)$, $H(\xi)$, $H(\beta)$, H_0 hold and*

$$\hat{\varphi}(u)=0,\ \hat{\varphi}'(u)=0,$$

then $u=0$.

Proof. We have

$$\begin{aligned}-\int_\Omega qG(Du)dz - \frac{q}{p}\int_\Omega \xi(z)|u|^p dz - \frac{q}{p}\int_{\partial\Omega}\beta(z)|u|^p d\sigma + \int_\Omega \vartheta(z)|u|^q dz \\ + \int_\Omega q\hat{F}(z,u)dz = 0\end{aligned} \tag{13}$$

and

$$\begin{aligned}\langle A(u),h\rangle + \int_\Omega \xi(z)|u|^{p-2}uhdz + \int_{\partial\Omega}\beta(z)|u|^{p-2}uhd\sigma - \int_\Omega \vartheta(z)|u|^{q-2}uhdz \\ - \int_\Omega \hat{f}(z,u)hdz = 0\end{aligned} \tag{14}$$

for all $h \in W^{1,p}(\Omega)$, where $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is the nonlinear map defined by

$$\langle A(v), h\rangle = \int_\Omega (a(Dv), Dh)_{\mathbb{R}^N} dz \text{ for all } v, h \in W^{1,p}(\Omega).$$

In (14) we choose $h = u \in W^{1,p}(\Omega)$ and obtain

$$\int_\Omega (a(Du), Du)_{\mathbb{R}^N} dz + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \vartheta(z)|u|^q dz - \int_\Omega \hat{f}(z, u)u dz = 0. \tag{15}$$

We add (13) and (15) and have

$$\begin{aligned}
&\int_\Omega [(a(Du), Du)_{\mathbb{R}^N} - qG(Du)]\, dz + \left(1 - \frac{q}{p}\right) \left[\int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma\right] \\
&= \int_\Omega \left[\hat{f}(z, u)u - q\hat{F}(z, u)\right] dz \\
\Rightarrow\ & \hat{c}_0||Du||_p^p + \frac{p-q}{p} \left[\int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma\right] \le \mu c_6||u||_p^p \\
& \text{(see hypothesis } H(a)(iv) \text{ and Lemma 3.1)} \\
\Rightarrow\ & c_6 \left[||Du||_p^p + \int_\Omega \hat{\xi}(z)|u|^p dz + \int_{\partial\Omega} \hat{\beta}(z)|u|^p d\sigma\right] \le \mu c_6||u||_p^p \\
& \text{(recall the choice of } c_6 > 0) \\
\Rightarrow\ & \gamma(u) \le \mu||u||_p^p,
\end{aligned}$$

a contradiction (since $\mu < \hat{\lambda}_1$), unless $u = 0$. □

Let $\overline{B}_\tau^\infty = \{u \in L^\infty(\Omega) : ||u||_\infty \le \tau\}$ and let $\varphi : W^{1,p}(\Omega) \cap \overline{B}_\tau^\infty \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\begin{aligned}
\varphi(u) = &\int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p dz \\
&+ \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \frac{1}{q}\int_\Omega \vartheta(z)|u|^q dz - \int_\Omega F(z, u)dz
\end{aligned}$$

for all $u \in W^{1,p}(\Omega) \cap \overline{B}_\tau^\infty$.

We have the following multiplicity result.

Theorem 3.3. *If hypotheses $H(a), H(\vartheta), H(f), H(\xi), H(\beta), H_0$ hold, then problem (1) admits a sequence $\{u_n\}_{n\ge1} \subseteq C^1(\overline{\Omega})$ of solutions such that*

$$||u_n||_{C^1(\overline{\Omega})} \to 0,\ \varphi(u_n) \to 0 \textit{ and } \varphi(u_n) < 0 \textit{ for all } n \in \mathbb{N}.$$

Proof. For every $m \in \mathbb{N}$, consider a family $\{v_k\}_{k=1}^m \subseteq C_c^1(U)$ (with $U \subseteq \Omega$ open as in hypothesis $H(\vartheta)$) of linearly independent functions. Set

$$V_m = \operatorname{span}\{v_k\}_{k=1}^m.$$

Hypothesis $H(a)(iv)$ and Corollary 2.4, imply that given $\epsilon > 0$, we can find $c_8 = c_8(\epsilon) > 0$ such that

$$G(y) \le \epsilon|y|^q + c_8|y|^p \text{ for all } y \in \mathbb{R}^N. \tag{16}$$

Let $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$u_n \in V_m \text{ for all } n \in \mathbb{N},\ ||u_n|| \to 0. \tag{17}$$

Let $y_n = \frac{u_n}{||u_n||}$ $(n \in \mathbb{N})$. Then $||y_n|| = 1,\ y_n \in V_m$ for all $n \in \mathbb{N}$. Since V_m is finite dimensional, we may assume that

$$y_n \to y \text{ in } W^{1,p}(\Omega),\ ||y|| = 1,\ y \in V_m. \tag{18}$$

For every $n \in \mathbb{N}$, we have

$$\hat{\varphi}(u_n) \leq \epsilon ||Du_n||_q^q - \frac{1}{q}\int_U \vartheta(z)|u_n|^q dz + c_9\, ||u_n||^p \text{ for some } c_9 > 0$$

(see (16) and hypotheses $H(\xi), H(\beta)$),

$$\Rightarrow \frac{\hat{\varphi}(u_n)}{||u_n||^q} \leq \epsilon ||Dy_n||_q^q - \frac{1}{q}\int_U \vartheta(z)|y_n|^q dz + c_9 ||u_n||^{p-q}||y_n||^q \text{ for all } n \in \mathbb{N}\,. \tag{19}$$

We set

$$\hat{\eta}_n = \epsilon ||Du_n||_q^q - \frac{1}{q}\int_U \vartheta(z)|y_n|^q dz + c_9 ||u_n||^{p-q}||y_n||^q.$$

From (17) and (18) it follows that

$$\hat{\eta}_n \to \hat{\eta} = \epsilon ||Dy||_q^q - \frac{1}{q}\int_u \vartheta(z)|y|^q dz \text{ as } n \to \infty\,.$$

Note that $\int_U \vartheta(z)|y|^q dz > 0$ (see hypothesis $H(\vartheta)$ and (18)). So, for $\epsilon > 0$ small, we have $\hat{\eta} < 0$. Then (19) implies that

$$\limsup_{n\to\infty} \frac{\hat{\varphi}(u_n)}{||u_n||^q} \leq \hat{\eta} < 0.$$

Hence we can find $\rho_m \in (0,1)$ small such that

$$\sup\,[\hat{\varphi}(u) : u \in V_m,\ ||u|| = \rho_m] < 0.$$

Then we can apply Theorem 2.1 and produce $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$\{u_n\}_{n\geq 1} \subseteq K_{\hat{\varphi}},\ \hat{\varphi}(u_n) \to 0 \text{ and } \hat{\varphi}(u_n) < 0 \text{ for all } n \in \mathbb{N}. \tag{20}$$

We know that we can find $c_{10} > 0$ such that

$$u_n \in L^\infty(\Omega) \text{ and } ||u_n||_\infty \leq c_{10} \text{ for all } n \in \mathbb{N}$$

(see Hu and Papageorgiou [6] and Papageorgiou and Rădulescu [13] (critical case)). Then from Lieberman [7, p. 320], we can find $\alpha \in (0,1)$ and $c_{11} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}),\ ||u_n||_{C^{1,\alpha}(\overline{\Omega})} \leq c_{11} \text{ for all } n \in \mathbb{N}. \tag{21}$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and by passing to a subsequence if necessary, we have

$$\begin{aligned} & u_n \to u \text{ in } C^1(\overline{\Omega}) \text{ (see (21))}, \\ \Rightarrow\ & \hat{\varphi}(u) = 0 \text{ and } \hat{\varphi}'(u) = 0, \\ \Rightarrow\ & u = 0 \text{ (see Proposition 3.2).} \end{aligned}$$

So, we have

$$u_n \to 0 \text{ in } C^1(\overline{\Omega}).$$

Therefore we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & |u_n(z)| \leq b/2 \text{ for all } n \geq n_0 \\ \Rightarrow\ & u_n \in C^1(\overline{\Omega}) \text{ is a solution of (1) for all } n \geq n_0. \end{aligned}$$

Hence we conclude that $\{u_n\}_{n\geq n_0} \subseteq C^1(\overline{\Omega})$ is the desired sequence of solutions (see (20)). □

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