PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 149, Number 9, September 2021, Pages 3819–3835
https://doi.org/10.1090/proc/15469
Article electronically published on June 4, 2021

HIGH PERTURBATIONS OF CHOQUARD EQUATIONS WITH CRITICAL REACTION AND VARIABLE GROWTH

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(Communicated by Catherine Sulem)

ABSTRACT. This paper deals with the mathematical analysis of solutions for a new class of Choquard equations. The main features of the problem studied in this paper are the following: (i) the equation is driven by a differential operator with variable exponent; (ii) the Choquard term contains a nonstandard potential with double variable growth; and (iii) the lack of compactness of the reaction, which is generated by a critical nonlinearity. The main result establishes the existence of infinitely many solutions in the case of high perturbations of the source term. The proof combines variational and analytic methods, including the Hardy-Littlewood-Sobolev inequality for variable exponents and the concentration-compactness principle for problems with variable growth.

1. INTRODUCTION AND ABSTRACT SETTING

Consider the following Choquard problem with variable exponents and critical reaction:

(1)
$$\begin{cases} -\Delta_{p(x)}u + \alpha|u|^{p(x)-2}u = \left(\int_{\mathbb{R}^N} \frac{F(y,u(y))}{|x-y|^{\lambda(x,y)}}dy\right) f(x,u(x)) + \beta(x)|u|^{p^*(x)-2}u \ \text{ in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

where $\lambda : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$, $f : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}$ and $\beta : \mathbb{R}^N \mapsto \mathbb{R}$ are continuous functions, and $p : \mathbb{R}^N \mapsto \mathbb{R}$ is a Lipschitz radially symmetric function satisfying $1 < p^- \leqslant p(x) \leqslant p^+ < N$. Let $p^*(x) = Np(x)/(N-p(x))$ denote the critical Sobolev exponent and assume that $\alpha > 0$. Here, p^+ and p^- are defined by $p^+ := \sup_{x\in\mathbb{R}^N} p(x)$ and $p^- := \inf_{x\in\mathbb{R}^N} p(x)$. We assume that $F(y,t) := \int_0^t f(y,s)ds$ and $\Delta_{p(x)}u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ denotes the $p(x)$-Laplace operator with variable exponent.

Received by the editors September 22, 2020, and, in revised form, December 6, 2020.
2020 *Mathematics Subject Classification.* Primary 35A15; Secondary 35J62, 58E50.
Key words and phrases. Choquard equation, anisotropic Sobolev space, Hardy-Littlewood-Sobolev inequality, concentration-compactness principle.
This research was partially supported by the National Natural Science Foundation of China (No. 11971485), the Fundamental Research Funds for the Central Universities of Central South University (No. 2019zzts211), and the Natural Science Foundation of Hunan Province (No. 2019JJ50146). This paper was completed while the first author was visiting University of Craiova (Romania) with the financial support of the China Scholarship Council (No. 201906370079).
The work of the first and third authors was supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI-UEFISCDI, project number PCE 137/2021, within PNCDI III.
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We recall that the Choquard equation was first introduced in the pioneering work of Fröhlich [3] and Pekar [12] for the modeling of quantum polaron:

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u \text{ in } \mathbb{R}^3. \tag{2}$$

As pointed out by Fröhlich and Pekar, this model corresponds to the study of free electrons in ionic lattices interacting with phonons associated to deformations of the lattices or with the polarisation created on the medium (interaction of an electron with its own hole). In the approximation to Hartree-Fock theory of one component plasma, Choquard used equation (2) to describe an electron trapped in its own hole.

The Choquard equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with non-relativistic Newtonian gravity. The equation can also be derived from the Einstein-Klein-Gordon and Einstein-Dirac systems. Such a model was proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter. We refer for details to Elgart and Schlein [2], Giulini and Großardt [7], Jones [8], Lions [10], and Schunck and Mielke [16]. Penrose [13, 14] proposed equation (2) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon. As pointed out by Lieb [9], Choquard used equation (2) to describe steady states of the one component plasma approximation in the Hartree-Fock theory. We refer to Mingione and Rădulescu [11] for an overview of recent results concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators.

The main features of the present paper are the following:

(i) the source term of problem (1) is driven by a differential operator with variable exponent and a power-type nonhomogeneous term (the corresponding term in problem (2) is linear);

(ii) a key role in the left-hand side of problem (1) is played by the parameter α (due to the fact that we establish the main result in the case of high values of this parameter);

(iii) the presence of the variable exponent $\lambda(x, y)$ in the Choquard nonlinearity and the contribution of a critical nonlinearity in the reaction;

(iv) since the problem contains both critical and nonlocal terms, the analysis developed in this paper uses more refined techniques than in the standard case.

We start with some basic notions on variable exponent spaces (see [15] for more details).

Set $C^+(\mathbb{R}^N) := \left\{\gamma \in C(\mathbb{R}^N) : 1 < \gamma^- \leqslant \gamma^+ < +\infty\right\}$, where $\gamma^+ := \sup_{x\in\mathbb{R}^N} \gamma(x)$ and $\gamma^- := \inf_{x\in\mathbb{R}^N} \gamma(x)$.

Let $M(\mathbb{R}^N)$ be the space of all measurable functions $u : \mathbb{R}^N \mapsto \mathbb{R}$. For $\xi \in C^+(\mathbb{R}^N)$, let $L^{\xi(x)}(\mathbb{R}^N) = \left\{u : u \in M(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |u(x)|^{\xi(x)} dx < +\infty\right\}$ denote the Lebesgue space with variable exponent $\xi(\cdot)$. This space is equipped with the "Luxemburg norm" defined by

$$\|u\|_{L^{\xi(x)}(\mathbb{R}^N)} = \inf \left\{\eta > 0 : \int_{\mathbb{R}^N} \left|\frac{u(x)}{\eta}\right|^{\xi(x)} dx \leqslant 1\right\}.$$

Let $W^{1,\xi(x)}(\mathbb{R}^N) := \left\{u \in L^{\xi(x)}(\mathbb{R}^N) : |\nabla u| \in L^{\xi(x)}(\mathbb{R}^N)\right\}$ denote the Sobolev space with variable exponent $\xi(\cdot)$. On $W^{1,\xi(x)}(\mathbb{R}^N)$ we can consider one of the following equivalent norms

$$\|u\|_{W^{1,\xi(x)}(\mathbb{R}^N)} := \|u\|_{L^{\xi(x)}(\mathbb{R}^N)} + \|\nabla u\|_{L^{\xi(x)}(\mathbb{R}^N)}$$

or

$$\|u\| := \inf\left\{\eta > 0 : \int_{\mathbb{R}^N}\left(\left|\frac{\nabla u(x)}{\eta}\right|^{\xi(x)} + \left|\frac{u(x)}{\eta}\right|^{\xi(x)}\right)dx \leqslant 1\right\},$$

that is, there exist two positive constants κ_1, κ_2 such that

$$\kappa_1\|u\|_{W^{1,\xi(x)}(\mathbb{R}^N)} \leqslant \|u\| \leqslant \kappa_2\|u\|_{W^{1,\xi(x)}(\mathbb{R}^N)} \quad \text{for all } u \in W^{1,\xi(x)}(\mathbb{R}^N). \tag{3}$$

Let $C_c(\mathbb{R}^N)$ be the subspace of functions in $C(\mathbb{R}^N)$ with compact support and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the norm $|\varphi|_\infty = \sup\left\{|\varphi(x)| : x \in \mathbb{R}^N\right\}$. A finite measure on $\mathbb{R}^N$ is a continuous linear functional on $C_0(\mathbb{R}^N)$. For any finite measure μ we define $\|\mu\| := \sup\{|(\mu,\varphi)| : \varphi \in C_0(\mathbb{R}^N), |\varphi|_\infty = 1\}$, where $(\mu,\varphi) = \int_{\mathbb{R}^N}\varphi d\mu$.

Let $\mathcal{M}(\mathbb{R}^N)$ be the space of finite non-negative Borel measures on $\mathbb{R}^N$. A sequence $\mu_n \to \mu$ weakly-$*$ in $\mathcal{M}(\mathbb{R}^N)$ if $(\mu_n,\varphi) \to (\mu,\varphi)$ for all $\varphi \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

In the sequel, if h_1, $h_2 \in C(\mathbb{R}^N)$, we say that $h_1 \ll h_2$ if $\inf\left\{h_2(x) - h_1(x) : x \in \mathbb{R}^N\right\} > 0$.

Throughout the paper, C will denote a positive constant and the same C may represent different constants.

2. High perturbations of the source term

Throughout this paper we assume that the following conditions are fulfilled:

(C1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $|f(x,t)| \leqslant g_1(x)|t|^{r(x)-1} + g_2(x)|t|^{s(x)-1}$, $\forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}$, where

$$0 \leqslant g_1 \in L^\infty(\mathbb{R}^N) \cap L^{\frac{p^*(x)q^+}{p^*(x)-r(x)q^+}}(\mathbb{R}^N) \cap L^{\frac{p^*(x)q^-}{p^*(x)-r(x)q^-}}(\mathbb{R}^N),$$

$$0 \leqslant g_2 \in L^\infty(\mathbb{R}^N) \cap L^{\frac{p^*(x)q^+}{p^*(x)-s(x)q^+}}(\mathbb{R}^N) \cap L^{\frac{p^*(x)q^-}{p^*(x)-s(x)q^-}}(\mathbb{R}^N)$$

and r, $s \in \mathcal{D} := \left\{\phi \in C^+(\mathbb{R}^N) : p(x) \leqslant \phi(x)q^- \leqslant \phi(x)q^+ \leqslant p^*(x),\ \forall\, x \in \mathbb{R}^N\right\}$ verify

$$p \ll rq^- \leqslant rq^+ \ll p^*,\ p \ll sq^- \leqslant sq^+ \ll p^* \text{ and } r^+,\ s^+ > p^-/2,$$

where $q \in C^+(\mathbb{R}^N)$, $\frac{1}{q(x)} + \frac{\lambda(x,y)}{N} + \frac{1}{q(y)} = 2$, for all x, $y \in \mathbb{R}^N$, $0 < \lambda^- := \inf_{x,\,y\in\mathbb{R}^N}\lambda(x,y) \leq \lambda^+ := \sup_{x,\,y\in\mathbb{R}^N}\lambda(x,y) < N$.

(C2) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(C3) $f(x,t) = f(|x|,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(C4) $\beta \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where β is radially symmetric, that is, $\beta(|x|) = \beta(x)$ for all $x \in \mathbb{R}^N$, $\beta(x) \geqslant (\not\equiv)\ 0$ and $\beta(0) = \beta(\infty) = 0$.

(C5) There exists $p \ll \theta$ such that $0 \leqslant \theta(x)F(x,t) \leqslant 2f(x,t)t$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $F(x,t) = \int_0^t f(x,s)ds$.

Let $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ denote the subspace of $W^{1,p(x)}(\mathbb{R}^N)$ containing all functions with radial symmetry.

Definition 1. We say that $u \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ is a weak solution of problem (1) if

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + \alpha |u|^{p(x)-2}uv - \beta(x)|u|^{p^*(x)-2}uv \right) dx$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u(y))f(x,u(x))v(x)}{|x-y|^{\lambda(x,y)}} dxdy \quad \text{for all } v \in W^{1,p(x)}_{rad}(\mathbb{R}^N).$$

Our main result establishes the existence of infinitely many radial solutions in the case of high perturbations of the absorption term. More precisely, we prove the following multiplicity property.

Theorem 1. *Assume that hypotheses* (C1)–(C5) *are satisfied. Then there exists* $\alpha_0 > 0$ *such that for all* $\alpha \geqslant \alpha_0$*, problem* (1) *has infinitely many radial solutions.*

2.1. **Auxiliary properties.** The energy functional associated to problem (1) is given by

$$I_\alpha(u) = \Upsilon(u) - \Phi(u) - \int_{\mathbb{R}^N} \frac{\beta(x)}{p^*(x)}|u|^{p^*(x)}dx,$$

where

$$\Upsilon(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \alpha|u|^{p(x)} \right) dx$$

and

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\lambda(x,y)}} dxdy.$$

It follows from Alves and Tavares [1, Lemma 3.2] that $\Phi \in C^1(W^{1,p(x)}_{rad}(\mathbb{R}^N),\mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u(y))f(x,u(x))v(x)}{|x-y|^{\lambda(x,y)}} dxdy \quad \text{for all } v \in W^{1,p(x)}_{rad}(\mathbb{R}^N).$$

A straightforward argument shows that $I_\alpha \in C^1(W^{1,p(x)}_{rad}(\mathbb{R}^N),\mathbb{R})$. Thus, the critical points of the functional I_α coincide with the weak solutions of problem (1).

Lemma 1. *There exists* $\alpha_0 > 0$ *such that for* $\alpha \geqslant \alpha_0$*, any* (PS) *sequence* $\{u_n\} \subset W^{1,p(x)}_{rad}(\mathbb{R}^N)$ *of* I_α *(that is,* $I_\alpha(u_n) \to c$ *and* $I'_\alpha(u_n) \to 0$ *as* $n \to \infty$*) is bounded in* $W^{1,p(x)}_{rad}(\mathbb{R}^N)$.

Proof. Let $\ell(x) := p(x) + \min\{\inf_{x\in\mathbb{R}^N}(\theta(x)-p(x)), \inf_{x\in\mathbb{R}^N}(p^*(x)-p(x))\}$. Note that p is a Lipschitz continuous and radially symmetric function on $\mathbb{R}^N$. Combining this fact and condition (C5), we obtain that ℓ is a Lipschitz symmetric function satisfying $p \ll \ell \leqslant p^*$.

By the Young inequality, we can deduce that for any $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 0$ such that

$$\left| \frac{u_n}{\ell(x)^2} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla \ell \right| \leqslant \varepsilon |\nabla u_n|^{p(x)} + C_\varepsilon |u_n|^{p(x)}. \tag{4}$$

Set

$$\ell_0 = \inf_{x\in\mathbb{R}^N} \left\{ \frac{1}{p(x)} - \frac{1}{\ell(x)} \right\} > 0.$$

Taking $\varepsilon = \frac{\ell_0}{2}$, from relation (4) we obtain

$$\left| \frac{u_n}{\ell(x)^2} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla \ell \right| \leqslant \frac{\ell_0}{2} |\nabla u_n|^{p(x)} + C_{\ell_0/2} |u_n|^{p(x)}. \tag{5}$$

Let $\alpha \geqslant 2C_{\ell_0/2}/\ell_0 =: \alpha_0 > 0$, using relation (5) and condition (C5) we have

$$\begin{aligned}
& I_\alpha(u_n) - \left\langle I'_\alpha(u_n), \frac{u_n}{\ell(x)} \right\rangle \\
&= \int_{\mathbb{R}^N} \left(\left(\frac{1}{p(x)} - \frac{1}{\ell(x)} \right) \left(|\nabla u_n|^{p(x)} + \alpha |u_n|^{p(x)} \right) + \frac{u_n}{\ell(x)^2} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \ell \right) dx \\
&\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n(y))}{|x-y|^{\lambda(x,y)}} \left(\frac{f(x, u_n(x)) u_n(x)}{\ell(x)} - \frac{F(x, u_n(x))}{2} \right) dx dy \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{\ell(x)} - \frac{1}{p^*(x)} \right) \beta(x) |u_n|^{p^*(x)} dx \\
&\geqslant \int_{\mathbb{R}^N} \left(\ell_0 |\nabla u_n|^{p(x)} + \ell_0 \alpha |u_n|^{p(x)} - \frac{\ell_0}{2} |\nabla u_n|^{p(x)} - C_{\ell_0/2} |u_n|^{p(x)} \right) dx \\
&\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n(y))}{|x-y|^{\lambda(x,y)}} \left(\frac{f(x, u_n(x)) u_n(x)}{\theta(x)} - \frac{F(x, u_n(x))}{2} \right) dx dy \\
&\geqslant \int_{\mathbb{R}^N} \left(\frac{\ell_0}{2} |\nabla u_n|^{p(x)} + \frac{\ell_0 \alpha}{2} |u_n|^{p(x)} \right) dx.
\end{aligned}$$

It follows that $\{u_n\}$ is bounded in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$. The proof is now complete. □

Lemma 2. *Any (PS) sequence has a convergent subsequence when $\alpha \geqslant \alpha_0$, where α_0 is given in Lemma* 1.

Proof. Let $\{u_n\} \subset W^{1,p(x)}_{rad}(\mathbb{R}^N)$ be a (PS) sequence. By Lemma 1, we get that $\{u_n\}$ is bounded for $\alpha \geqslant \alpha_0$. Since $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ is reflexive, up to a subsequence, we may assume that there exists $u \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ such that $u_n \to u$ weakly in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, $u_n \to u$ weakly in $W^{1,p(x)}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$.

We first prove that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \to 0, \text{ as } n \to \infty.$$

Indeed, since $u_n \to u$ weakly in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, as $n \to \infty$, we obtain

$$\langle \Phi'(u), u_n - u \rangle \to 0, \text{ as } n \to \infty,$$

where $\Phi'(u) \in \left(W^{1,p(x)}_{rad}(\mathbb{R}^N) \right)^*$.

It remains to prove that

$$\langle \Phi'(u_n), u_n - u \rangle \to 0 \text{ as } n \to \infty.$$

By the Hardy-Littlewood-Sobolev inequality for variable exponents (see Alves and Tavares [1, Proposition 2.4]), we have

$$\begin{aligned}
(6) \qquad |\langle \Phi'(u_n), u_n - u \rangle| &\leqslant C \|F(\cdot, u_n)\|_{L^{q^+}(\mathbb{R}^N)} \|f(\cdot, u_n)(u_n - u)\|_{L^{q^+}(\mathbb{R}^N)} \\
&\quad + C \|F(\cdot, u_n)\|_{L^{q^-}(\mathbb{R}^N)} \|f(\cdot, u_n)(u_n - u)\|_{L^{q^-}(\mathbb{R}^N)}.
\end{aligned}$$

By condition (C1) and the boundedness of $\{u_n\}$ in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, we obtain

$$\begin{aligned}
&\|F(\cdot,u_n)\|_{L^{q^+}(\mathbb{R}^N)}\\
&\leqslant C\left(\int_{\mathbb{R}^N}\left(|u_n|^{q^+r(x)}+|u_n|^{q^+s(x)}\right)dx\right)^{\frac{1}{q^+}}\\
&\leqslant C\left(\int_{\mathbb{R}^N}|u_n|^{q^+r(x)}dx\right)^{\frac{1}{q^+}}+C\left(\int_{\mathbb{R}^N}|u_n|^{q^+s(x)}dx\right)^{\frac{1}{q^+}}\\
&\leqslant C\max\left\{\|u_n\|^{r^+}_{L^{q^+r(x)}(\mathbb{R}^N)},\ \|u_n\|^{r^-}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\}\\
&\quad+C\max\left\{\|u_n\|^{s^+}_{L^{q^+s(x)}(\mathbb{R}^N)},\ \|u_n\|^{s^-}_{L^{q^+s(x)}(\mathbb{R}^N)}\right\}\\
&\leqslant C
\end{aligned}\tag{7}$$

and

$$\begin{aligned}
\|F(\cdot,u_n)\|_{L^{q^-}(\mathbb{R}^N)}&\leqslant C\max\left\{\|u_n\|^{r^+}_{L^{q^-r(x)}(\mathbb{R}^N)},\ \|u_n\|^{r^-}_{L^{q^-r(x)}(\mathbb{R}^N)}\right\}\\
&\quad+C\max\left\{\|u_n\|^{s^+}_{L^{q^-s(x)}(\mathbb{R}^N)},\ \|u_n\|^{s^-}_{L^{q^-s(x)}(\mathbb{R}^N)}\right\}\\
&\leqslant C.
\end{aligned}\tag{8}$$

Moreover, the compact embeddings

$$\begin{aligned}
&W^{1,p(x)}_{rad}(\mathbb{R}^N)\hookrightarrow L^{q^+r(x)}(\mathbb{R}^N),\ W^{1,p(x)}_{rad}(\mathbb{R}^N)\hookrightarrow L^{q^+s(x)}(\mathbb{R}^N),\\
&W^{1,p(x)}_{rad}(\mathbb{R}^N)\hookrightarrow L^{q^-r(x)}(\mathbb{R}^N),\ W^{1,p(x)}_{rad}(\mathbb{R}^N)\hookrightarrow L^{q^-s(x)}(\mathbb{R}^N)
\end{aligned}$$

combined with condition (C1) and the boundedness of $\{u_n\}$ in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ imply that

$$\begin{aligned}
&\|f(\cdot,u_n)(u_n-u)\|^{q^+}_{L^{q^+}(\mathbb{R}^N)}\\
\leqslant&C\||u_n|^{q^+(r(\cdot)-1)}\|_{L^{\frac{r(x)}{r(x)-1}}(\mathbb{R}^N)}\||u_n-u|^{q^+}\|_{L^{r(x)}(\mathbb{R}^N)}\\
&+C\||u_n|^{q^+(s(\cdot)-1)}\|_{L^{\frac{s(x)}{s(x)-1}}(\mathbb{R}^N)}\||u_n-u|^{q^+}\|_{L^{s(x)}(\mathbb{R}^N)}\\
\leqslant&C\max\left\{\|u_n-u\|^{q^+}_{L^{q^+r(x)}(\mathbb{R}^N)},\|u_n-u\|^{\frac{q^+r^-}{r^+}}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\}\\
&+C\max\left\{\|u_n-u\|^{\frac{q^+r^+}{r^-}}_{L^{q^+r(x)}(\mathbb{R}^N)},\|u_n-u\|^{q^+}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\}\\
&+C\max\left\{\|u_n-u\|^{q^+}_{L^{q^+s(x)}(\mathbb{R}^N)},\|u_n-u\|^{\frac{q^+s^-}{s^+}}_{L^{q^+s(x)}(\mathbb{R}^N)}\right\}\\
&+C\max\left\{\|u_n-u\|^{\frac{q^+s^+}{s^-}}_{L^{q^+s(x)}(\mathbb{R}^N)},\|u_n-u\|^{q^+}_{L^{q^+s(x)}(\mathbb{R}^N)}\right\}\\
=&o_n(1),\ \text{ as } n\to\infty
\end{aligned}\tag{9}$$

and

$$\begin{aligned}
(10)\qquad &\|f(\cdot,u_n)(u_n-u)\|^{q^-}_{L^{q^-}(\mathbb{R}^N)}\\
\leqslant &C\max\left\{\|u_n-u\|^{q^-}_{L^{q^-r(x)}(\mathbb{R}^N)},\ \|u_n-u\|^{\frac{q^-r^-}{r^+}}_{L^{q^-r(x)}(\mathbb{R}^N)}\right\}\\
&+C\max\left\{\|u_n-u\|^{\frac{q^-r^+}{r^-}}_{L^{q^-r(x)}(\mathbb{R}^N)},\ \|u_n-u\|^{q^-}_{L^{q^-r(x)}(\mathbb{R}^N)}\right\}\\
&+C\max\left\{\|u_n-u\|^{q^-}_{L^{q^-s(x)}(\mathbb{R}^N)},\ \|u_n-u\|^{\frac{q^-s^-}{s^+}}_{L^{q^-s(x)}(\mathbb{R}^N)}\right\}\\
&+C\max\left\{\|u_n-u\|^{\frac{q^-s^+}{s^-}}_{L^{q^-s(x)}(\mathbb{R}^N)},\ \|u_n-u\|^{q^-}_{L^{q^-s(x)}(\mathbb{R}^N)}\right\}\\
=&o_n(1),\ \text{ as } n\to\infty.
\end{aligned}$$

By relations (6)–(10), we have $\langle \Phi'(u_n), u_n-u\rangle \to 0$ as $n\to\infty$.

Next, we prove that

$$\int_{\mathbb{R}^N}\beta(x)\left(|u_n|^{p^*(x)-2}u_n-|u|^{p^*(x)-2}u\right)(u_n-u)dx\to 0,\ \text{ as } n\to\infty.$$

Note that $u_n\to u$ weakly in $W^{1,p(x)}(\mathbb{R}^N)$, as $n\to\infty$. Up to a subsequence, still denoted by $\{u_n\}$, we may assume that there exist μ, $\nu\in\mathcal{M}(\mathbb{R}^N)$ such that $|\nabla u_n|^{p(x)}+\alpha|u_n|^{p(x)}\to\mu$ and $|u_n|^{p^*(x)}\to\nu$ weakly-$*$ in $\mathcal{M}(\mathbb{R}^N)$. By the concentration-compactness principle for variable exponents (see Fu and Zhang [6, Theorem 2.2]), we know that

$$\mu=|\nabla u|^{p(x)}+\alpha|u|^{p(x)}+\sum_{j\in J}\mu_j\delta_{x_j}+\widetilde{\mu}$$

and

$$\nu=|u|^{p^*(x)}+\sum_{j\in J}\nu_j\delta_{x_j},$$

where J is a countable set, $\{\mu_j\}$, $\{\nu_j\}\subset[0,+\infty)$, $\{x_j\}\subset\mathbb{R}^N$, δ_{x_j} is the Dirac mass centered at x_j, $\widetilde{\mu}\in\mathcal{M}(\mathbb{R}^N)$ is a non-atomic non-negative measure. By the concentration-compactness principle for variable exponents, we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*(x)}dx=\int_{\mathbb{R}^N}d\nu+\nu_\infty=\int_{\mathbb{R}^N}|u|^{p^*(x)}dx+\sum_{j\in J}\nu_j+\nu_\infty.$$

(i) We prove that $\nu_j=0$. For any $\varepsilon>0$, we choose a radially symmetric function $\varphi\in C_0^\infty(B_{2\varepsilon}(0))$ such that $0\leqslant\varphi\leqslant 1$, $|\nabla\varphi|\leqslant 2/\varepsilon$; $\varphi=1$ on $B_\varepsilon(0)$. Since $\{u_n\varphi\}$ is bounded in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, we obtain $\langle I'_\alpha(u_n),u_n\varphi\rangle\to 0$ as $n\to\infty$. It follows that

$$\begin{aligned}
&\langle I'_\alpha(u_n),u_n\varphi\rangle\\
=&\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n\varphi)+\alpha|u_n|^{p(x)}\varphi-\beta(x)|u_n|^{p^*(x)}\varphi\right)dx\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,u_n(y))f(x,u_n(x))u_n(x)\varphi(x)}{|x-y|^{\lambda(x,y)}}dxdy\\
=&\int_{\mathbb{R}^N}\left(\left(|\nabla u_n|^{p(x)}+\alpha|u_n|^{p(x)}\right)\varphi+|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\varphi u_n-\beta(x)|u_n|^{p^*(x)}\varphi\right)dx\\
&-\langle\Phi'(u_n),u_n\varphi\rangle.
\end{aligned}$$

Next, we show that

$$\langle \Phi'(u_n), u_n\varphi\rangle \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u(y))f(x,u(x))u(x)\varphi(x)}{|x-y|^{\lambda(x,y)}}dxdy = \langle \Phi'(u), u\varphi\rangle, \text{ as } n\to\infty.$$

By condition (C1) and using again the compact embeddings

$$W^{1,p(x)}_{rad}(\mathbb{R}^N) \hookrightarrow L^{q^+r(x)}(\mathbb{R}^N),\ W^{1,p(x)}_{rad}(\mathbb{R}^N) \hookrightarrow L^{q^+s(x)}(\mathbb{R}^N),$$
$$W^{1,p(x)}_{rad}(\mathbb{R}^N) \hookrightarrow L^{q^-r(x)}(\mathbb{R}^N),\ W^{1,p(x)}_{rad}(\mathbb{R}^N) \hookrightarrow L^{q^-s(x)}(\mathbb{R}^N),$$

the Hardy-Littlewood-Sobolev inequality for variable exponents, the boundedness of $\{u_n\}$ in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, relations (7)–(8), and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&|\langle \Phi'(u_n), u_n\varphi\rangle - \langle \Phi'(u), u\varphi\rangle| \\
&\leqslant \left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u_n(y))(f(x,u_n(x))u_n(x)-f(x,u(x))u(x))}{|x-y|^{\lambda(x,y)}}dxdy\right| \\
&\quad + \left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(F(y,u_n(y))-F(y,u(y)))f(x,u(x))u(x)}{|x-y|^{\lambda(x,y)}}dxdy\right| \\
&\leqslant C\|F(\cdot,u_n)\|_{L^{q^+}(\mathbb{R}^N)}\|f(\cdot,u_n)u_n - f(\cdot,u)u\|_{L^{q^+}(\mathbb{R}^N)} \\
&\quad + C\|F(\cdot,u_n)\|_{L^{q^-}(\mathbb{R}^N)}\|f(\cdot,u_n)u_n - f(\cdot,u)u\|_{L^{q^-}(\mathbb{R}^N)} \\
&\quad + C\|F(\cdot,u_n)-F(\cdot,u)\|_{L^{q^+}(\mathbb{R}^N)}\|f(\cdot,u)u\|_{L^{q^+}(\mathbb{R}^N)} \\
&\quad + C\|F(\cdot,u_n)-F(\cdot,u)\|_{L^{q^-}(\mathbb{R}^N)}\|f(\cdot,u)u\|_{L^{q^-}(\mathbb{R}^N)} \\
&\leqslant C\|f(\cdot,u_n)u - f(\cdot,u)u\|_{L^{q^+}(\mathbb{R}^N)} + C\|f(\cdot,u_n)u - f(\cdot,u)u\|_{L^{q^-}(\mathbb{R}^N)} \\
&\quad + C_u\|F(\cdot,u_n)-F(\cdot,u)\|_{L^{q^+}(\mathbb{R}^N)} + C_u\|F(\cdot,u_n)-F(\cdot,u)\|_{L^{q^-}(\mathbb{R}^N)} \\
&= o_n(1), \text{ as } n\to\infty,
\end{aligned}$$

where C_u is a positive constant.

Thus, we get $\langle \Phi'(u_n), u_n\varphi\rangle \to \langle \Phi'(u), u\varphi\rangle$ as $n\to\infty$. Therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\varphi u_n dx = \int_{\mathbb{R}^N} -\varphi d\mu + \int_{\mathbb{R}^N}\beta(x)\varphi d\nu + \langle \Phi'(u), u\varphi\rangle.$$

Since $u_n \to u$ in $L^{p(x)}(B_{2\varepsilon}(0))$, we have $\|\nabla\varphi u_n\|_{L^{p(x)}(\mathbb{R}^N)} \to \|\nabla\varphi u\|_{L^{p(x)}(\mathbb{R}^N)}$ as $n\to\infty$. It follows that

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\varphi u_n dx\right| \\
&\quad\leqslant \limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-1}|\nabla\varphi u_n|dx \\
&\quad\leqslant \limsup_{n\to\infty} C\||\nabla u_n|^{p(x)-1}\|_{L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)}\|\nabla\varphi u_n\|_{L^{p(x)}(\mathbb{R}^N)} \\
&\quad\leqslant C\|\nabla\varphi u\|_{L^{p(x)}(\mathbb{R}^N)}.
\end{aligned}$$

Furthermore, by a straightforward computation we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\varphi u|^{p(x)}dx \\
&= \int_{B_{2\varepsilon}(0)} |\nabla\varphi u|^{p(x)}dx \leqslant C\||\nabla\varphi|^{p(x)}\|_{L^{\frac{p^*(x)}{p^*(x)-p(x)}}(B_{2\varepsilon}(0))} \||u|^{p(x)}\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\varepsilon}(0))} \\
&\leqslant C\max\left\{\left(\int_{B_{2\varepsilon}(0)} |\nabla\varphi|^N dx\right)^{\frac{p^+}{N}}, \left(\int_{B_{2\varepsilon}(0)} |\nabla\varphi|^N dx\right)^{\frac{p^-}{N}}\right\} \||u|^{p(x)}\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\varepsilon}(0))} \\
&\leqslant C\max\left\{\left(\frac{4^N w_N}{N}\right)^{\frac{p^+}{N}}, \left(\frac{4^N w_N}{N}\right)^{\frac{p^-}{N}}\right\} \||u|^{p(x)}\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\varepsilon}(0))} \\
&= o_\varepsilon(1), \text{ as } \varepsilon \to 0,
\end{aligned}$$

where w_N is the surface area of the unit sphere in $\mathbb{R}^N$. Similarly, we can also infer that

$$\begin{aligned}
&\left|\langle \Phi'(u), u\varphi\rangle\right| \\
&\quad\leqslant C\|F(\cdot,u)\|_{L^{q^+}(\mathbb{R}^N)}\|f(\cdot,u)u\varphi\|_{L^{q^+}(\mathbb{R}^N)} \\
&\qquad\qquad + C\|F(\cdot,u)\|_{L^{q^-}(\mathbb{R}^N)}\|f(\cdot,u)u\varphi\|_{L^{q^-}(\mathbb{R}^N)} \\
&\quad\leqslant C\|f(\cdot,u)u\varphi\|_{L^{q^+}(\mathbb{R}^N)} + C\|f(\cdot,u)u\varphi\|_{L^{q^-}(\mathbb{R}^N)} \\
&\quad\leqslant C\left(\int_{B_{2\varepsilon}(0)} g_1^{q^+}|u|^{r(x)q^+}dx\right)^{\frac{1}{q^+}} + C\left(\int_{B_{2\varepsilon}(0)} g_2^{q^+}|u|^{s(x)q^+}dx\right)^{\frac{1}{q^+}} \\
&\qquad + C\left(\int_{B_{2\varepsilon}(0)} g_1^{q^-}|u|^{r(x)q^-}dx\right)^{\frac{1}{q^-}} + C\left(\int_{B_{2\varepsilon}(0)} g_2^{q^-}|u|^{s(x)q^-}dx\right)^{\frac{1}{q^-}} \\
&\quad\leqslant C\left(\int_{B_{2\varepsilon}(0)} |u|^{r(x)q^+}dx\right)^{\frac{1}{q^+}} + C\left(\int_{B_{2\varepsilon}(0)} |u|^{s(x)q^+}dx\right)^{\frac{1}{q^+}} \\
&\qquad + C\left(\int_{B_{2\varepsilon}(0)} |u|^{r(x)q^-}dx\right)^{\frac{1}{q^-}} + C\left(\int_{B_{2\varepsilon}(0)} |u|^{s(x)q^-}dx\right)^{\frac{1}{q^-}} = o_\varepsilon(1), \text{ as } \varepsilon \to 0.
\end{aligned}$$

Therefore, $\mu(\{0\}) = \beta(0)\nu(\{0\}) = 0$ (since $\beta(0) = 0$), hence 0 is not an atom of μ.

Now, we prove that for any $j \in J$, $\nu_j = 0$. From the above information, we may assume that there exists $x_{j_0} \neq 0$ ($j_0 \in J$) such that $\nu_{j_0} = \nu_{j_0}(\{x_{j_0}\}) > 0$. Due to $u_n \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$, the measure ν is $O(N)$-invariant, where $O(N)$ is the group of orthogonal linear transformations in $\mathbb{R}^N$. For any $g \in O(N)$, $\nu_{j_0}(\{gx_{j_0}\}) = \nu_{j_0}(\{x_{j_0}\}) > 0$. We know that

$$|O(N)| = \inf_{x\in\mathbb{R}^N,\, x\neq 0} |O(N)_x| = +\infty,$$

where $|O(N)_x|$ denotes the cardinality of $\{gx : g \in O(N)\}$. Then, $\nu_{j_0}(\{gx_{j_0} : g \in O(N)\}) = +\infty$. But the measure ν is finite, hence we get a contradiction. So, we obtain $\nu_j = 0$ for any $j \in J$.

(ii) We show that $\nu_\infty = 0$. For any $R > 0$, we take a radially symmetric function $w_R \in C^\infty(\mathbb{R}^N)$ such that $0 \leqslant w_R \leqslant 1$, $|\nabla w_R| < 2/R$; $w_R = 1$ in $\mathbb{R}^N \setminus B_{2R}(0)$, $w_R = 0$ in $B_R(0)$. Clearly, $\{u_n w_R\}$ is bounded in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$. So, we can easily obtain $\langle I'_\alpha(u_n), u_n w_R\rangle \to 0$, as $n \to \infty$. Hence, we have

$$\text{(11)}\quad \langle I'_\alpha(u_n), u_n w_R\rangle = \int_{\mathbb{R}^N} \Big(\Big(|\nabla u_n|^{p(x)} + \alpha |u_n|^{p(x)} \Big) w_R + |\nabla u_n|^{p(x)-2} \nabla u_n \nabla w_R u_n - \beta(x)|u_n|^{p^*(x)} w_R \Big)\, dx - \langle \Phi'(u_n), u_n w_R\rangle.$$

Due to $\beta(\infty) = 0$, we have

$$\text{(12)}\quad \lim_{R\to+\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \beta(x)|u_n|^{p^*(x)} dx = 0.$$

Since $1 < p^- \leqslant p(x) \leqslant p^+ < N$, by the definition of w_R we get

$$\lim_{R\to+\infty} \int_{\mathbb{R}^N} |\nabla w_R u|^{p(x)} dx = 0.$$

Thanks to $u_n \to u$ strongly in $L^{p(x)}(B_{2R}(0) \setminus B_R(0))$, we can easily observe that

$$\lim_{n\to\infty} \|\nabla w_R u_n\|_{L^{p(x)}(\mathbb{R}^N)} = \|\nabla w_R u\|_{L^{p(x)}(\mathbb{R}^N)}.$$

So, by Hölder's inequality and the above inequalities we obtain

$$\text{(13)}\quad \begin{aligned} &\lim_{R\to+\infty} \limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla w_R u_n dx \right| \\ &\leqslant C \lim_{R\to+\infty} \limsup_{n\to\infty} \|\nabla w_R u_n\|_{L^{p(x)}(\mathbb{R}^N)} \\ &\leqslant C \lim_{R\to+\infty} \|\nabla w_R u\|_{L^{p(x)}(\mathbb{R}^N)} = 0. \end{aligned}$$

Since

$$0 \leqslant g_1 \in L^{\frac{p^*(x)q^+}{p^*(x)-r(x)q^+}}(\mathbb{R}^N) \bigcap L^{\frac{p^*(x)q^-}{p^*(x)-r(x)q^-}}(\mathbb{R}^N)$$

and

$$0 \leqslant g_2 \in L^{\frac{p^*(x)q^+}{p^*(x)-s(x)q^+}}(\mathbb{R}^N) \bigcap L^{\frac{p^*(x)q^-}{p^*(x)-s(x)q^-}}(\mathbb{R}^N),$$

we can deduce that

$$\lim_{R\to+\infty} \int_{\mathbb{R}^N\setminus B_R(0)} g_1^{\frac{p^*(x)q^+}{p^*(x)-r(x)q^+}} dx = 0, \quad \lim_{R\to+\infty} \int_{\mathbb{R}^N\setminus B_R(0)} g_2^{\frac{p^*(x)q^+}{p^*(x)-s(x)q^+}} dx = 0,$$

$$\lim_{R\to+\infty} \int_{\mathbb{R}^N\setminus B_R(0)} g_1^{\frac{p^*(x)q^-}{p^*(x)-r(x)q^-}} dx = 0, \quad \lim_{R\to+\infty} \int_{\mathbb{R}^N\setminus B_R(0)} g_2^{\frac{p^*(x)q^-}{p^*(x)-s(x)q^-}} dx = 0.$$

By the above four relations, condition (C1) and the boundedness of $\{u_n\}$ in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, we get

$$\begin{aligned} &\lim_{R\to+\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} |f(x,u_n)u_n|^{q^+} dx \\ &\leqslant C \lim_{R\to+\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} {g_1}^{q^+} |u_n|^{r(x)q^+} dx \\ &\qquad + C \lim_{R\to+\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} {g_2}^{q^+} |u_n|^{s(x)q^+} dx \\ &\leqslant C \lim_{R\to+\infty} \|{g_1}^{q^+}\|_{L^{\frac{p^*(x)}{p^*(x)-r(x)q^+}}(\mathbb{R}^N\setminus B_R(0))} \\ &\qquad + C \lim_{R\to+\infty} \|{g_2}^{q^+}\|_{L^{\frac{p^*(x)}{p^*(x)-s(x)q^+}}(\mathbb{R}^N\setminus B_R(0))} = 0 \end{aligned}$$

and

$$\begin{aligned}\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}|f(x,u_n)u_n|^{q^-}dx\\ \leqslant C\lim_{R\to+\infty}\|g_1{}^{q^-}\|_{L^{\frac{p^*(x)}{p^*(x)-r(x)q^-}}(\mathbb{R}^N\setminus B_R(0))}\\ +C\lim_{R\to+\infty}\|g_2{}^{q^-}\|_{L^{\frac{p^*(x)}{p^*(x)-s(x)q^-}}(\mathbb{R}^N\setminus B_R(0))}=0.\end{aligned}$$

By (7), (8) and the Hardy-Littlewood-Sobolev inequality for variable exponents, we have

$$\begin{aligned}&\lim_{R\to+\infty}\limsup_{n\to\infty}|\langle\Phi'(u_n),u_nw_R\rangle|\\ &\leqslant C\lim_{R\to+\infty}\limsup_{n\to\infty}\left(\|f(\cdot,u_n)u_nw_R\|_{L^{q^+}(\mathbb{R}^N)}+\|f(\cdot,u_n)u_nw_R\|_{L^{q^-}(\mathbb{R}^N)}\right)\\ &\leqslant C\lim_{R\to+\infty}\limsup_{n\to\infty}\left(\|f(\cdot,u_n)u_n\|_{L^{q^+}(\mathbb{R}^N\setminus B_R(0))}+\|f(\cdot,u_n)u_n\|_{L^{q^-}(\mathbb{R}^N\setminus B_R(0))}\right)\\ &=0.\end{aligned}\tag{14}$$

By relations (11), (12), (13) and (14), we obtain

$$\mu_\infty=\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla u_n|^{p(x)}+\alpha|u_n|^{p(x)})w_Rdx=0.$$

Furthermore, we can conclude that

$$\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla(u_nw_R)|^{p(x)}+\alpha|u_nw_R|^{p(x)})dx=0.$$

It follows that

$$\nu_\infty=\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|w_Ru_n|^{p^*(x)}dx=0.$$

Using (i) and (ii), we obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*(x)}dx=\int_{\mathbb{R}^N}|u|^{p^*(x)}dx.$$

Next, by the Brezis-Lieb-type lemma (see [5, Lemma 2.1]) we find

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^{p^*(x)}dx=0,$$

that is, $\|u_n-u\|_{L^{p^*(x)}(\mathbb{R}^N)}\to 0$, as $n\to\infty$. Combing this fact and $\beta\in L^\infty(\mathbb{R}^N)$, we can deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\beta(x)\left(|u_n|^{p^*(x)-2}u_n-|u|^{p^*(x)-2}u\right)(u_n-u)dx=0.$$

Furthermore, from the above information, we have

$$\lim_{n\to\infty}\langle I'_\alpha(u_n)-I'_\alpha(u),u_n-u\rangle+\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle=0.$$

It follows that

$$\begin{aligned}&\lim_{n\to\infty}\langle\Upsilon'(u_n)-\Upsilon'(u),u_n-u\rangle\\ &=\lim_{n\to\infty}\Big(\langle I'_\alpha(u_n)-I'_\alpha(u),u_n-u\rangle+\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle\\ &\qquad+\int_{\mathbb{R}^N}\beta(x)\left(|u_n|^{p^*(x)-2}u_n-|u|^{p^*(x)-2}u\right)(u_n-u)dx\Big)=0.\end{aligned}$$

Finally, by taking similar steps as Fu [4, Theorem 3.1], we can derive that

$$\lim_{n\to\infty} \|u_n - u\|_{W^{1,p(x)}(\mathbb{R}^N)} = 0.$$

The proof is now complete. □

Since $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ is a separable and reflexive Banach space, we can find $\{e_n\}_{n=1}^{\infty} \subset W^{1,p(x)}_{rad}(\mathbb{R}^N)$ and $\{\psi_m\}_{m=1}^{\infty} \subset \left(W^{1,p(x)}_{rad}(\mathbb{R}^N)\right)^*$ such that $\psi_m(e_n) = \delta_{nm}$ ($\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ if $n \neq m$), $W^{1,p(x)}_{rad}(\mathbb{R}^N) = \overline{\operatorname{span}\{e_n\}_{n=1}^{\infty}}$ and $\left(W^{1,p(x)}_{rad}(\mathbb{R}^N)\right)^* = \overline{\operatorname{span}\{\psi_m\}_{m=1}^{\infty}}$.

In the sequel, we use V_k^+ to denote $\overline{\operatorname{span}\{e_i : i = k, \ldots\}}$ ($k = 1, 2, \ldots$). Then we have the following auxiliary property.

Lemma 3. *For any large enough $k \in \mathbb{N}$, there exist $\tau_k > 0$ and $\rho_k > 0$ such that $I_\alpha(u) \geqslant \tau_k$ for any $u \in V_k^+$ with $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \rho_k$.*

Proof. For any $u \in V_k^+$ with $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \geqslant \max\left\{\frac{1}{\kappa_1}, 1\right\}$ (κ_1 is given in (3)), combining condition (C1), the growth of F and the Hardy-Littlewood-Sobolev inequality for variable exponents we have

$$\begin{aligned} I_\alpha(u) \geqslant & \int_{\mathbb{R}^N} \frac{\min\{1,\alpha_0\}}{p_+} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx - \int_{\mathbb{R}^N} \frac{\beta(x)}{p^*(x)} |u|^{p^*(x)} dx \\ & - C\|F(\cdot,u)\|^2_{L^{q^+}(\mathbb{R}^N)} - C\|F(\cdot,u)\|^2_{L^{q^-}(\mathbb{R}^N)} \\ \geqslant & \int_{\mathbb{R}^N} \frac{\min\{1,\alpha_0\}}{p_+} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx - \int_{\mathbb{R}^N} \frac{\beta(x)}{p^*(x)} |u|^{p^*(x)} dx \\ & - C\left(\int_{\mathbb{R}^N} |u|^{r(x)q^+} dx\right)^{\frac{2}{q^+}} - C\left(\int_{\mathbb{R}^N} |u|^{s(x)q^+} dx\right)^{\frac{2}{q^+}} \\ & - C\left(\int_{\mathbb{R}^N} |u|^{r(x)q^-} dx\right)^{\frac{2}{q^-}} - C\left(\int_{\mathbb{R}^N} |u|^{s(x)q^-} dx\right)^{\frac{2}{q^-}}. \end{aligned}$$

Set

$$\begin{aligned} \sigma_k^{r+} &= \sup\left\{\int_{\mathbb{R}^N} |u|^{r(x)q^+} dx : u \in V_k^+,\ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = 1\right\}, \\ \sigma_k^{r-} &= \sup\left\{\int_{\mathbb{R}^N} |u|^{r(x)q^-} dx : u \in V_k^+,\ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = 1\right\}, \\ \sigma_k^{s+} &= \sup\left\{\int_{\mathbb{R}^N} |u|^{s(x)q^+} dx : u \in V_k^+,\ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = 1\right\}, \\ \sigma_k^{s-} &= \sup\left\{\int_{\mathbb{R}^N} |u|^{s(x)q^-} dx : u \in V_k^+,\ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = 1\right\}. \end{aligned}$$

We first show that $\sigma_k^{r+} \to 0$, as $k \to \infty$. We observe that $\sigma_k^{r+} \geqslant \sigma_{k+1}^{r+} \geqslant 0$, hence $\sigma_k^{r+} \to \sigma^{r^+} \geqslant 0$, as $k \to \infty$. Choose $u_k \in V_k^+$ with $\|u_k\|_{W^{1,p(x)}(\mathbb{R}^N)} = 1$ such that

$$0 \leqslant \sigma_k^{r+} - \int_{\mathbb{R}^N} |u_k|^{r(x)q^+} dx < \frac{1}{k},$$

for each $k \in \mathbb{N}^+$. Since $W^{1,p(x)}_{rad}(\mathbb{R}^N)$ is reflexive, $\{u_k\}$ admits weakly convergent subsequence, up to a subsequence, still denoted by $\{u_k\}$. Then there exists $u \in W^{1,p(x)}_{rad}(\mathbb{R}^N)$ such that $u_k \to u$ weakly in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, as $k \to \infty$. Now we assert that $u = 0$. Indeed, for any $\psi_m \in \{\psi_n : n = 1, 2, \dots, m, \dots\}$, $\psi_m(u_k) = 0$ for any $k > m$. So, $\psi_m(u_k) \to 0$, as $k \to \infty$. This implies that $\psi_m(u) = 0$ for any $\psi_m \in \{\psi_n : n = 1, 2, \dots, m, \dots\}$. Due to the denseness of $\{\psi_n : n = 1, 2, \dots, m, \dots\}$ in $\left(W^{1,p(x)}_{rad}(\mathbb{R}^N)\right)^*$, we obtain $u = 0$. By condition (C1) and the compact embedding $W^{1,p(x)}_{rad}(\mathbb{R}^N) \hookrightarrow L^{r(x)q^+}$, we have

$$\int_{\mathbb{R}^N} |u_k|^{r(x)q^+} dx \to 0, \text{ as } k \to \infty.$$

Thus, we conclude that $\sigma_k^{r+} \to 0$ (as $k \to \infty$) holds true.

Similarly, we can deduce that $\sigma_k^{r-} \to 0$, $\sigma_k^{s+} \to 0$ and $\sigma_k^{s-} \to 0$, as $k \to \infty$.

Denote

$$\vartheta_k = \sup\left\{\int_{\mathbb{R}^N} \frac{\beta(x)}{p^*(x)}|u|^{p^*(x)}dx : u \in V_k^+,\ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = 1\right\}.$$

Next, with the same ideas as in the proof of Lemma 3.5 of Fu and Zhang [6], we get $\vartheta_k \to 0$, as $k \to \infty$.

From the above information, we have

$$\begin{aligned} I_\alpha(u) \geqslant{}& \frac{\kappa_1^{p^-}\min\{1,\alpha_0\}}{p^+}\|u\|^{p^-}_{W^{1,p(x)}(\mathbb{R}^N)} - \vartheta_k\|u\|^{p^{*+}}_{W^{1,p(x)}(\mathbb{R}^N)} \\ &- C(\sigma_k^{r+})^{\frac{2}{q^+}}\|u\|^{2r^+}_{W^{1,p(x)}(\mathbb{R}^N)} - C(\sigma_k^{s+})^{\frac{2}{q^+}}\|u\|^{2s^+}_{W^{1,p(x)}(\mathbb{R}^N)} \\ &- C(\sigma_k^{r-})^{\frac{2}{q^-}}\|u\|^{2r^+}_{W^{1,p(x)}(\mathbb{R}^N)} - C(\sigma_k^{s-})^{\frac{2}{q^-}}\|u\|^{2s^+}_{W^{1,p(x)}(\mathbb{R}^N)}. \end{aligned}$$

Thanks to p^{*+}, r^+, $s^+ > p^-/2$, we can take

$$\rho_k = \max\left\{1, \frac{1}{\kappa_1}, C_k{}^{1/\left(\max\{2r^+,\ 2s^+,\ p^{*+}\}-p^-\right)}\right\},$$

where

$$C_k = \frac{\kappa_1^{p^-}\min\{1,\alpha_0\}p^-}{p^+p^{*+}\left(C(\sigma_k^{r+})^{\frac{2}{q^+}} + C(\sigma_k^{r-})^{\frac{2}{q^-}} + C(\sigma_k^{s+})^{\frac{2}{q^+}} + C(\sigma_k^{s-})^{\frac{2}{q^-}} + \vartheta_k\right)}.$$

Note that $\rho_k = C_k{}^{1/\left(\max\{2r^+,\ 2s^+,\ p^{*+}\}-p^-\right)}$ for sufficiently large k. So, for any $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \rho_k$, we have

$$\begin{aligned} I_\alpha(u) \geqslant{}& \frac{\kappa_1^{p^-}\min\{1,\alpha_0\}}{p^+}\rho_k^{p^-} - \vartheta_k\rho_k^{p^{*+}} - \left(C(\sigma_k^{r+})^{\frac{2}{q^+}} + C(\sigma_k^{r-})^{\frac{2}{q^-}}\right)\rho_k^{2r^+} \\ &- \left(C(\sigma_k^{s+})^{\frac{2}{q^+}} + C(\sigma_k^{s-})^{\frac{2}{q^-}}\right)\rho_k^{2s^+} \\ \geqslant{}& \frac{\kappa_1^{p^-}\min\{1,\alpha_0\}}{p^+}\rho_k^{p^-} \\ &- \left(C(\sigma_k^{r+})^{\frac{2}{q^+}} + C(\sigma_k^{r-})^{\frac{2}{q^-}} + C(\sigma_k^{s+})^{\frac{2}{q^+}} + C(\sigma_k^{s-})^{\frac{2}{q^-}} + \vartheta_k\right)\rho_k^{\max\{2r^+,\ 2s^+,\ p^{*+}\}} \\ \geqslant{}& \rho_k^{p^-}\frac{\kappa_1^{p^-}\min\{1,\alpha_0\}(p^{*+}-p^-)}{p^+p^{*+}} =: \tau_k > 0. \end{aligned}$$

It is easy to see that $\tau_k \to +\infty$, as $k \to \infty$. The proof is now complete. □

Using condition (C4), we can find $x_0 \in \mathbb{R}^N$ such that $\beta(x_0) > 0$. Thus, there exist positive constants $\varrho_1 < \varrho_2$ such that $\varrho_1 < |x_0| < \varrho_2$, $p_{x_0} = \sup_{\varrho_1<|x|<\varrho_2} p(x) < p^*_{x_0} = \inf_{\varrho_1<|x|<\varrho_2} p^*(x)$, and $\beta(x) \geqslant \beta(x_0)/2$ for all $|x| \in (\varrho_1, \varrho_2)$. Then we can choose radially symmetric functions $\varphi_i \in C_0^\infty(B_{\varrho_2}(0)\setminus\overline{B_{\varrho_1}(0)})$ $(i = 1, 2, \dots, k)$ such that $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset$ for $i \neq j$ $(i, j = 1, 2, \dots, k)$.

Set $V_k^- = \{\varphi_i : i = 1, 2, \dots, k\} \subset W^{1,p(x)}_{rad}(\mathbb{R}^N)$. Then for any $k \in \mathbb{N}$ we have $\operatorname{codim} V_k^+ + 1 = \dim V_k^-$.

Lemma 4. *For every $k \in \mathbb{N}$, there exists $R_k > 0$ such that $I_\alpha(u) \leqslant 0$ for any $u \in V_k^-$ and $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \geqslant R_k$.*

Proof. For any $u \in V_k^-$ and $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \geqslant 1/\kappa_1$ (κ_1 is given in (3)). Using condition (C1), the growth of F and the Hardy-Littlewood-Sobolev inequality for variable exponents, we have

$$\begin{aligned}
I_\alpha(u) &= \int_{\varrho_1<|x|<\varrho_2} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \alpha|u|^{p(x)}\right)dx - \int_{\varrho_1<|x|<\varrho_2} \frac{\beta(x)}{p^*(x)}|u|^{p^*(x)}dx \\
&\quad - \frac{1}{2}\int_{\varrho_1<|x|<\varrho_2}\int_{\varrho_1<|y|<\varrho_2} \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\lambda(x,y)}}dxdy \\
&\leqslant \frac{\max\{1,\alpha\}}{p^-}\int_{\varrho_1<|x|<\varrho_2}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - \int_{\varrho_1<|x|<\varrho_2}\frac{\beta(x_0)}{2p^*(x)}|u|^{p^*(x)}dx \\
&\quad + C\|F(\cdot,u(\cdot))\|^2_{L^{q^+}(\mathbb{R}^N)} + C\|F(\cdot,u(\cdot))\|^2_{L^{q^-}(\mathbb{R}^N)} \\
&\leqslant \frac{\max\{1,\alpha\}}{p^-}\int_{\varrho_1<|x|<\varrho_2}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - \int_{\varrho_1<|x|<\varrho_2}\frac{\beta(x_0)}{2p^*(x)}|u|^{p^*(x)}dx \\
&\quad + C\left(\int_{\varrho_1<|x|<\varrho_2} g_1^{q^+}|u|^{q^+r(x)}dx\right)^{\frac{1}{q^+}} + C\left(\int_{\varrho_1<|x|<\varrho_2} g_2^{q^+}|u|^{q^+s(x)}dx\right)^{\frac{1}{q^+}} \\
&\quad + C\left(\int_{\varrho_1<|x|<\varrho_2} g_1^{q^-}|u|^{q^-r(x)}dx\right)^{\frac{1}{q^-}} + C\left(\int_{\varrho_1<|x|<\varrho_2} g_2^{q^-}|u|^{q^-s(x)}dx\right)^{\frac{1}{q^-}}.
\end{aligned}$$

By the Young inequality, for any $\varepsilon > 0$, there exist $C_1(\varepsilon)$, $C_2(\varepsilon)$, $C_3(\varepsilon)$, $C_4(\varepsilon) > 0$ such that

$$\begin{aligned}
g_1(x)^{q^+}|u|^{q^+r(x)} &\leqslant \varepsilon|u|^{p^*(x)} + C_1(\varepsilon)g_1(x)^{q^+p^*(x)/(p^*(x)-q^+r(x))}, \\
g_2(x)^{q^+}|u|^{q^+s(x)} &\leqslant \varepsilon|u|^{p^*(x)} + C_2(\varepsilon)g_2(x)^{q^+p^*(x)/(p^*(x)-q^+s(x))}, \\
g_1(x)^{q^-}|u|^{q^-r(x)} &\leqslant \varepsilon|u|^{p^*(x)} + C_3(\varepsilon)g_1(x)^{q^-p^*(x)/(p^*(x)-q^-r(x))}, \\
g_2(x)^{q^-}|u|^{q^-s(x)} &\leqslant \varepsilon|u|^{p^*(x)} + C_4(\varepsilon)g_2(x)^{q^-p^*(x)/(p^*(x)-q^-s(x))}.
\end{aligned}$$

Using the above inequalities and condition (C1) we obtain

$$\begin{aligned}
I_\alpha(u) &\leqslant \frac{\max\{1,\alpha\}}{p^-}\int_{\varrho_1<|x|<\varrho_2}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx - \int_{\varrho_1<|x|<\varrho_2}\frac{\beta(x_0)}{2p^*(x)}|u|^{p^*(x)}dx \\
&\quad + C\varepsilon^{\frac{1}{q^+}}\left(\int_{\varrho_1<|x|<\varrho_2}|u|^{p^*(x)}dx\right)^{\frac{1}{q^+}} + C\varepsilon^{\frac{1}{q^-}}\left(\int_{\varrho_1<|x|<\varrho_2}|u|^{p^*(x)}dx\right)^{\frac{1}{q^-}} \\
&\quad + C\left(C_1(\varepsilon)^{\frac{1}{q^+}} + C_2(\varepsilon)^{\frac{1}{q^+}} + C_3(\varepsilon)^{\frac{1}{q^-}} + C_4(\varepsilon)^{\frac{1}{q^-}}\right).
\end{aligned}$$

It is obvious that $\|\cdot\|_{L^{p^*}(\mathbb{R}^N)}$ is also norm of V_k^-. On the other hand, V_k^- is a finite-dimensional space, hence the norms $\|\cdot\|_{L^{p^*(x)}(\mathbb{R}^N)}$ and $\|\cdot\|_{W^{1,p(x)}(\mathbb{R}^N)}$ are equivalent. Thus, we can find a constant $C_{V_k^-} > 1/\kappa_1$ such that $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \leqslant C_{V_k^-}\|u\|_{L^{p^*(x)}(\mathbb{R}^N)}$ for any $u \in V_k^-$. Hence, for $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \geqslant C_{V_k^-}$ and $0<\varepsilon<1$ we obtain

$$\begin{aligned}
I_\alpha(u) \leqslant\ & \frac{\kappa_2^{p_{x_0}}\max\{1,\alpha\}}{p^-}\|u\|^{p_{x_0}}_{W^{1,p(x)}(\mathbb{R}^N)} - \int_{\varrho_1<|x|<\varrho_2}\frac{\beta(x_0)}{2p^*(x)}|u|^{p^*(x)}dx \\
& + C\left(\varepsilon^{\frac{1}{q^+}}+\varepsilon^{\frac{1}{q^-}}\right)\int_{\varrho_1<|x|<\varrho_2}|u|^{p^*(x)}dx \\
& + C\left(C_1(\varepsilon)^{\frac{1}{q^+}}+C_2(\varepsilon)^{\frac{1}{q^+}}+C_3(\varepsilon)^{\frac{1}{q^-}}+C_4(\varepsilon)^{\frac{1}{q^-}}\right) \\
\leqslant\ & \frac{\kappa_2^{p_{x_0}}\max\{1,\alpha\}}{p^-}\|u\|^{p_{x_0}}_{W^{1,p(x)}(\mathbb{R}^N)} - \int_{\varrho_1<|x|<\varrho_2}\frac{\beta(x_0)}{2p^{*+}}|u|^{p^*(x)}dx \\
& + C\varepsilon^{\frac{1}{q^+}}\int_{\varrho_1<|x|<\varrho_2}|u|^{p^*(x)}dx + C\left(C_1(\varepsilon)^{\frac{1}{q^+}}+C_2(\varepsilon)^{\frac{1}{q^+}}+C_3(\varepsilon)^{\frac{1}{q^-}}+C_4(\varepsilon)^{\frac{1}{q^-}}\right),
\end{aligned}$$

where κ_2 is given in (3).

Setting $\varepsilon = \min\left\{1, \left(\frac{\beta(x_0)}{4Cp^{*+}}\right)^{q^+}\right\}$, we have

$$\begin{aligned}
I_\alpha(u) \leqslant\ & \frac{\kappa_2^{p_{x_0}}\max\{1,\alpha\}}{p^-}\|u\|^{p_{x_0}}_{W^{1,p(x)}(\mathbb{R}^N)} - \int_{\varrho_1<|x|<\varrho_2}\frac{\beta(x_0)}{4p^{*+}}|u|^{p^*(x)}dx \\
& + C\left(C_1(\varepsilon)^{\frac{1}{q^+}}+C_2(\varepsilon)^{\frac{1}{q^+}}+C_3(\varepsilon)^{\frac{1}{q^-}}+C_4(\varepsilon)^{\frac{1}{q^-}}\right) \\
\leqslant\ & \frac{\kappa_2^{p_{x_0}}\max\{1,\alpha\}}{p^-}\|u\|^{p_{x_0}}_{W^{1,p(x)}(\mathbb{R}^N)} - \frac{\beta(x_0)}{4p^{*+}}\|u\|^{p^*_{x_0}}_{L^{p^*}(\mathbb{R}^N)} + C(\varepsilon) \\
\leqslant\ & \frac{\kappa_2^{p_{x_0}}\max\{1,\alpha\}}{p^-}\|u\|^{p_{x_0}}_{W^{1,p(x)}(\mathbb{R}^N)} - \frac{\beta(x_0)}{4p^{*+}}\left(\frac{1}{C_{V_k^-}}\|u\|_{W^{1,p(x)}(\mathbb{R}^N)}\right)^{p^*_{x_0}} + C(\varepsilon),
\end{aligned}$$

where $C(\varepsilon) = C\left(C_1(\varepsilon)^{\frac{1}{q^+}}+C_2(\varepsilon)^{\frac{1}{q^+}}+C_3(\varepsilon)^{\frac{1}{q^-}}+C_4(\varepsilon)^{\frac{1}{q^-}}\right)$. Thanks to $p_{x_0} < p^*_{x_0}$, we can deduce that there is $R_k>0$ such that $I_\alpha(u)\leqslant 0$ for any $u\in V_k^-$ and $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \geqslant R_k$.

The proof is now complete. □

2.2. **Proof of Theorem 1 completed.** Firstly, using condition (C2) we know that I_α is an even functional on $W^{1,p(x)}_{rad}(\mathbb{R}^N)$. Next, combining Lemmas 1–4 with Theorem 6.3 of Struwe [17], we deduce that for all $\alpha \geqslant \alpha_0$ and large enough $k\in\mathbb{N}$,

$$\zeta_k = \inf_{h\in\Gamma_k}\sup_{u\in V_k^-} I_\alpha(h(u))$$

is a critical value of I_α, and $\zeta_k \geqslant \tau_k$, where

$$\Gamma_k = \left\{h \in C\left(W^{1,p(x)}_{rad}(\mathbb{R}^N), W^{1,p(x)}_{rad}(\mathbb{R}^N)\right) : \begin{array}{l} h \text{ is odd},\ h(u)=u,\ \text{if } u\in V_k^- \\ \text{and } \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \geqslant R_k \end{array}\right\}$$

and $\alpha_0>0$ is given in Lemma 1. Finally, by Lemma 3, we have $\zeta_k\to+\infty$, if $\tau_k\to+\infty$, as $k\to\infty$. So, we infer that the functional I_α admits a sequence of critical points $\{u_k\}\subset W^{1,p(x)}_{rad}(\mathbb{R}^N)$ such that $I_\alpha(u_k)=\zeta_k\to+\infty$, as $k\to\infty$.

The proof of Theorem 1 is now complete. □

Acknowledgments

The first author would like to thank the China Scholarship Council and the Embassy of the People's Republic of China in Romania.

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