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QUALITATIVE ANALYSIS OF SOLUTIONS FOR A CLASS OF ANISOTROPIC ELLIPTIC EQUATIONS WITH VARIABLE EXPONENT

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Abstract We are concerned with the degenerate anisotropic problem

$$-\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ -2}u = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

We first establish the existence of an unbounded sequence of weak solutions. We also obtain the existence of a non-trivial weak solution if the nonlinear term f has a special form. The proofs rely on the fountain theorem and Ekeland's variational principle.

Keywords: degenerate anisotropic Sobolev spaces; variable exponent; Dirichlet boundary value condition; fountain theorem; Ekeland variational principle

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) be a bounded domain with smooth boundary. The purpose of this paper is to analyse the existence of multiple weak solutions to the anisotropic problem

$$\left.\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ -2}u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

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where $b \in L^{\infty}(\bar{\Omega})$, $f\colon \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $a_i\colon \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions fulfilling some natural hypotheses.

The solution of problems of type (1.1) is strongly motivated and important research efforts have been made recently with the aim of understanding anisotropic phenomena described by non-homogeneous differential operators. We recall that equations of this type can be regarded as models for phenomena arising in the study of electrorheological fluids (see [**6**,**13**,**22**]), elasticity (see [**25**]) or image processing and restoration (see [**5**,**9**]). A survey of the history of this research field with a comprehensive bibliography is provided by Diening *et al*. [**7**].

The anisotropic differential operator $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is a $\boldsymbol{p}(\cdot)$-Laplace-type operator, where $\boldsymbol{p}(x) = (p_1(x), p_2(x), \dots, p_N(x))$ and $P_+^+ = \max_{i\in\{1,\dots,N\}}\{\sup_{x\in\Omega} p_i(x)\}$. For $i = 1, \dots, N$ we assume that p_i is a continuous function on $\bar{\Omega}$.

Let $a_i(x, \eta)$ denote the continuous derivative with respect to η of the mapping $A_i\colon \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $A_i = A_i(x, \eta)$, that is, $a_i(x, \eta) = (\partial/\partial\eta)A_i(x, \eta)$. Throughout this paper we assume that the following hypotheses are fulfilled.

(A_0) $A_i(x, 0) = 0$ for almost every (a.e.) $x \in \Omega$.

(A_1) There exists a positive constant $\bar{c}_i$ such that a_i satisfies the growth condition

$$|a_i(x, \eta)| \leqslant \bar{c}_i(1 + |\eta|^{p_i(x)-1})$$

for all $x \in \bar{\Omega}$ and $\eta \in \mathbb{R}^N$.

(A_2) The inequalities

$$|\eta|^{p_i(x)} \leqslant a_i(x, \eta)\eta \leqslant p_i(x)A_i(x, \eta)$$

hold for all $x \in \bar{\Omega}$ and $\eta \in \mathbb{R}^N$.

(A_3) There exists $k_i > 0$ such that

$$A_i\left(x, \frac{\eta + \xi}{2}\right) \leqslant \tfrac{1}{2}A_i(x, \eta) + \tfrac{1}{2}A_i(x, \xi) - k_i|\eta - \xi|^{p_i(x)}$$

for all $x \in \bar{\Omega}$ and $\eta, \xi \in \mathbb{R}^N$, with equality if and only if $\eta = \xi$.

(A_4) The mapping A_i is even with respect to its second variable, that is,

$$A_i(x, -\eta) = A_i(x, \eta)$$

for all $x \in \bar{\Omega}$ and $\eta \in \mathbb{R}^N$.

The differential operator $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is the anisotropic $\boldsymbol{p}(x)$-Laplace-type operator (where $\boldsymbol{p}(x) = (p_1(x), \dots, p_N(x))$) because when we take

$$a_i(x, \eta) = |\eta|^{p_i(x)-2}\eta$$

for all $i \in \{1, \dots, N\}$, we have $A_i(x, \eta) = (1/p_i(x))|\eta|^{p_i(x)}$ for all $i \in \{1, \dots, N\}$, that is,

$$\Delta_{\boldsymbol{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u).$$

Obviously, there are many other operators deriving from $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$. Indeed, to give another interesting example, if we take

$$a_i(x, \eta) = (1 + |\eta|^2)^{(p_i(x)-2)/2}\eta$$

for all $i \in \{1, \dots, N\}$, we have $A_i(x, \eta) = (1/p_i(x))[(1 + |\eta|^2)^{p_i(x)/2} - 1]$ for all $i \in \{1, \dots, N\}$. Thus, we obtain the anisotropic variable mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i}[(1 + |\partial_{x_i} u|^2)^{(p_i(x)-2)/2}\partial_{x_i} u].$$

The papers [**4**, **18**] studied the anisotropic quasilinear elliptic problem

$$\left.\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) = f(x, u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) is a bounded domain with smooth boundary. In [**4**], Boureanu *et al*. considered problem (1.2), where f is a Carathéodory function verifying some appropriate conditions. Their arguments are based on the symmetric mountain pass theorem of Ambrosetti and Rabinowitz [**1**]. In [**18**] Mihăilescu *et al*. studied eigenvalue problems where $f(x, u) = \lambda|u|^{q(x)-2}u$ and established the multiplicity of the solution by combining the mountain pass theorem with the Ekeland variational principle [**8**]. Kone *et al*. [**14**] established the existence and uniqueness of a weak energy solution to the nonlinear problem

$$\left.\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.3}$$

In [**16**], Mihăilescu and Moroşanu considered (1.3) where $f = \lambda(x)|u|^{q(x)-2}u$. Combining the mountain pass theorem with Ekeland's variational principle, they proved that under suitable conditions (1.3) has two non-trivial weak solutions. Boureanu [**3**] proved that problem (1.3) has a sequence of weak solutions by means of the symmetric mountain pass theorem. Motivated by the above papers and [**17**], the goal of this paper is to establish the existence of a sequence of high-energy solutions of problem (1.1). In addition, we consider problem (1.1) in a case where the function f has a special form. A central role in our arguments will be played by the fountain theorem, which is due to Bartsch [**2**]. This result is nicely presented in [**23**] by using the quantitative deformation lemma. We also point out that the dual version of the fountain theorem is due to Bartsch and Willem (see [**23**]). Both the fountain theorem and its dual form are effective tools for studying the existence of infinitely many large or small energy solutions. It should be noted that the Palais–Smale condition plays an important role for these theorems and their applications.

2. Abstract framework

We recall in what follows some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponents $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. As pointed out in [**18**], anisotropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue- and Sobolev-type in which different space directions have different roles.

Set $C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega})\colon \min_{x\in\bar{\Omega}} h(x) > 1\}$. For any $h \in C_+(\bar{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

If $p \in C_+(\bar{\Omega})$, we define *the variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \Bigl\{u\colon u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\Bigr\}$$

endowed with the *Luxemburg norm* defined by

$$|u|_{p(x)} = \inf\Bigl\{\mu > 0;\ \int_\Omega \Bigl|\frac{u(x)}{\mu}\Bigr|^{p(x)}\,\mathrm{d}x \leqslant 1\Bigr\},$$

which is a separable and reflexive Banach space (see [**15**]).

If $p \in C_+(\bar{\Omega})$, the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ contains all functions $u \in L^{p(x)}(\Omega)$ such that the gradient ∇u exists almost everywhere and belongs to $[L^{p(x)}(\Omega)]^N$. Then $W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space with respect to the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

As shown by Zhikov [**26**,**27**] in relationship with the *Lavrentiev phenomenon*, the smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$. However, if $p \in C_+(\bar{\Omega})$ is *logarithmic Hölder continuous*, that is,

$$|p(x) - p(y)| \leqslant -\frac{M}{\log(|x-y|)} \quad \text{for all } x, y \in \Omega \text{ such that } |x-y| \leqslant 1/2, \tag{2.1}$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$. Let $W_0^{1,p(x)}(\Omega)$ denote the Sobolev space of functions with zero boundary values under the norm $\|\cdot\|$. Furthermore, if $p \in C_+(\bar{\Omega})$ satisfies (2.1), then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(x)}(\Omega)$ (see [**19**]). Since Ω is an open bounded set and $p \in C_+(\bar{\Omega})$ satisfies (2.1), the $p(x)$-*Poincaré inequality*

$$|u|_{p(x)} \leqslant C|\nabla u|_{p(x)}$$

holds for all $u \in W_0^{1,p(x)}(\Omega)$, where C depends on p, $|\Omega|$, $\mathrm{diam}(\Omega)$ and N [**19**, p. 13], and so

$$\|u\|_{1,p(x)} = |\nabla u|_{p(x)}$$

is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. Also, of course the norm

$$\|u\|_{p(x)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p(x)}$$

is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. Hence, $W_0^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. Note that if $s \in C_+(\bar{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \bar{\Omega}$, where $p^\star(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \geqslant N$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous.

We now introduce a natural generalization of the function space $W_0^{1,p(x)}(\Omega)$, which will play a central role in our statements. For this purpose, let us denote by $\boldsymbol{p}\colon \bar{\Omega} \to \mathbb{R}^N$ the vectorial function $\boldsymbol{p} = (p_1, \dots, p_N)$. We define $W_0^{1,\boldsymbol{p}(x)}(\Omega)$, the *anisotropic variable exponent Sobolev space*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\boldsymbol{p}(x)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}.$$

For the case in which $p_i \in C_+(\bar{\Omega})$ are constant functions for any $i \in \{1, \dots, N\}$, the resulting anisotropic Sobolev space is denoted by $W_0^{1,\boldsymbol{p}}(\Omega)$, where $\boldsymbol{p}$ is the constant vector $(p_1, \dots, p_N)$. The theory of such spaces was developed in [**12, 20, 21**]. We point out that $W_0^{1,\boldsymbol{p}}(\Omega)$ is a reflexive Banach space for any $\boldsymbol{p} \in \mathbb{R}^N$ with $p_i > 1$ for all $i \in \{1, \dots, N\}$. This result can be easily extended to $W_0^{1,\boldsymbol{p}(x)}(\Omega)$. Indeed, defining $X = L^{p_1(\cdot)}(\Omega) \times \dots \times L^{p_N(\cdot)}(\Omega)$ and considering the operator $T\colon W_0^{1,\boldsymbol{p}(x)}(\Omega) \to X$, defined by $T(u) = \nabla u$, it is clear that $W_0^{1,\boldsymbol{p}(x)}(\Omega)$ and X are isometric with respect to T, since

$$\|Tu\|_X = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)} = \|u\|_{\boldsymbol{p}(x)}.$$

Thus, $T(W_0^{1,\boldsymbol{p}(x)}(\Omega))$ is a closed subspace of X, which is a reflexive Banach space, and hence $T(W_0^{1,\boldsymbol{p}(x)}(\Omega))$ is reflexive, and consequently $W_0^{1,\boldsymbol{p}(x)}(\Omega)$ is a reflexive Banach space.

We define $X := W_0^{1,\boldsymbol{p}(x)}(\Omega)$. Since X is reflexive, by [**24**] there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j \colon j = 1, 2, \dots\}}, \qquad X^* = \overline{\operatorname{span}\{e_j^* \colon j = 1, 2, \dots\}}$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between X and X^*. We define

$$X_j = \operatorname{span}\{e_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

3. Main results

An important role in what follows will be played by the vectors $\boldsymbol{P}_+, \boldsymbol{P}_- \in \mathbb{R}^N$ and by the real numbers $P_+^+, P_-^+, P_+^-, P_-^- \in \mathbb{R}^+$ defined as

$$\begin{aligned} \boldsymbol{P}_+ &= (p_1^+, p_2^+, \dots, p_N^+), & \boldsymbol{P}_- &= (p_1^-, p_2^-, \dots, p_N^-), \\ P_+^+ &= \max\{p_1^+, p_2^+, \dots, p_N^+\}, & P_-^+ &= \max\{p_1^-, p_2^-, \dots, p_N^-\}, \\ P_+^- &= \min\{p_1^+, p_2^+, \dots, p_N^+\}, & P_-^- &= \min\{p_1^-, p_2^-, \dots, p_N^-\}. \end{aligned}$$

Throughout this paper we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1. \tag{3.1}$$

This condition ensures that the anisotropic space $W_0^{1,\boldsymbol{p}(x)}(\Omega)$ is embedded into some Lebesgue space $L^r(\Omega)$. If hypothesis (3.1) is no longer fulfilled, then one has embeddings into Orlicz or Hölder spaces.

Define $P_-^* \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^{N}(1/p_i^-) - 1}, \qquad P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

For the Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$, we consider the anti-derivative $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$,

$$F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t.$$

With the previous notation, the functions b, f satisfy the following conditions.

(B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geqslant b_0$ for all $x \in \Omega$.

(F_1) There exist a positive constant c_1 and $\alpha(x) \in C_+(\bar{\Omega})$ with $\alpha(x) < P_{-,\infty}$ such that

$$|f(x,t)| \leqslant c_1(1 + |t|^{\alpha(x)-1}) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.2}$$

(F_2) There exist $M > 0$, $\theta > P_+^+$ such that for all $x \in \Omega$ and all $t \in \mathbb{R}$ with $|t| \geqslant M$,

$$0 < \theta F(x,t) \leqslant t f(x,t).$$

(F_3) The function f is odd with respect to its second variable, that is,

$$f(x,-t) = -f(x,t)$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

Definition 3.1. A function $u \in W_0^{1,\boldsymbol{p}(x)}(\Omega)$ that verifies

$$\int_\Omega \left\{ \sum_{i=1}^{N} a_i(x, \partial_{x_i})\partial_{x_i}\varphi + b(x)|u|^{P_+^+ - 2}u\varphi - f(x,u)\varphi \right\} \mathrm{d}x = 0$$

for all $\varphi \in W_0^{1,\boldsymbol{p}(x)}(\Omega)$ is called a weak solution of problem (1.1).

The following result establishes the existence of infinitely many solutions of problem (1.1), provided that the right-hand side is odd.

Theorem 3.2. *Problem (1.1) admits a sequence* $(\pm u_n)$ *of weak solutions with high energies.*

In addition, we consider the case in which $f(x,u) = \lambda|u|^{q(x)-2}u$, where the parameter λ is positive and $q(x)$ is a continuous function on $\bar{\Omega}$. Problem (1.1) then becomes

$$\left.\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+-2}u = \lambda|u|^{q(x)-2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.3}$$

Definition 3.3. A function $u \in W_0^{1,\boldsymbol{p}(x)}(\Omega)$ is said to be a weak solution of problem (3.3) if and only if

$$\int_{\Omega} \left\{ \sum_{i=1}^{N} a_i(x, \partial_{x_i})\partial_{x_i}\varphi + b(x)|u|^{P_+^+-2}u\varphi - \lambda|u|^{q(x)-2}u\varphi \right\} \mathrm{d}x = 0$$

for all $\varphi \in W_0^{1,\boldsymbol{p}(x)}(\Omega)$.

Our main result concerning problem (3.3) can be described as follows.

Theorem 3.4. *Assume that* $q^- < P_-^- \leqslant P_+^+ < P_{-,\infty}$ *for all* $x \in \bar{\Omega}$. *There then exists a positive constant* λ^* *such that for any* $\lambda \in (0, \lambda^*)$, *problem (3.3) has at least one non-trivial weak solution.*

In what follows we use c_i to denote a general non-negative or positive constant (the exact value may change from line to line).

4. Infinitely many high energy solutions

In this section we are concerned with the existence of multiple weak solutions of problem (1.1). We associate with problem (1.1) the energy functional $I\colon X \to \mathbb{R}$ defined by

$$I(u) = \int_{\Omega} \left\{ \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - F(x,u) \right\} \mathrm{d}x.$$

Due to [**14**, Lemma 3.4], by a standard calculus we deduce that I is well defined and $I \in C^1(X, \mathbb{R})$ with

$$\langle I'(u), \varphi \rangle = \int_{\Omega} \left\{ \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi + b(x)|u|^{P_+^+-2}u\varphi - f(x,u)\varphi \right\} \mathrm{d}x$$

for all $u, \varphi \in X$. Hence, any critical point $u \in X$ of I is a weak solution of problem (1.1). The idea of the proof of Theorem 3.2 is to show that all the hypotheses of the fountain theorem [**23**, Theorem 3.6] are fulfilled. To this end, we will prove three corresponding auxiliary properties.

Lemma 4.1. *For every $k \in \mathbb{N}$ there exists $r_k > 0$ such that $\inf_{u\in Z_k,\, \|u\|=r_k} I(u) \to \infty$ as $k \to \infty$.*

Proof. By (A_2) and (F_1) for any $u \in Z_k$, $\|u\| = r_k > 1$ (r_k will be specified below), we have

$$\begin{aligned} I(u) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - F(x,u) \right\} \mathrm{d}x \\ &\geqslant \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, \mathrm{d}x + \frac{b_0}{P_+^+} \int_\Omega |u|^{P_+^+}\, \mathrm{d}x - c_1 \int_\Omega (1 + |u|^{\alpha(x)})\, \mathrm{d}x \\ &\geqslant \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, \mathrm{d}x + \frac{b_0}{P_+^+} |u|^{P_+^+}_{L^{P_+^+}(\Omega)} - c_2 \max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\} - c_3. \end{aligned} \tag{4.1}$$

Using (B) we can write

$$\frac{1}{P_+^+} \int_\Omega b(x)|u|^{P_+^+}\, \mathrm{d}x \geqslant \frac{b_0}{P_+^+}|u|^{P_+^+}_{L^{P_+^+}(\Omega)} \geqslant 0. \tag{4.2}$$

For each $i \in \{1, 2, \dots, N\}$ we define

$$\alpha_i = \begin{cases} P_+^+ & \text{if } |\partial_{x_i} u|_{p_i(x)} < 1, \\ P_-^- & \text{if } |\partial_{x_i} u|_{p_i(x)} > 1. \end{cases}$$

Using [**11**, Theorem 1.3] and the Jensen inequality (applied to the convex function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, $g(t) = t^{P_-^-}$, $P_-^- > 1$), we have

$$\begin{aligned} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, \mathrm{d}x &\geqslant \sum_{i=1}^N |\partial_{x_i} u|^{\alpha_i}_{p_i(x)} \\ &\geqslant \sum_{i=1}^N |\partial_{x_i} u|^{P_-^-}_{p_i(x)} - \sum_{\{i;\alpha_i = P_+^+\}} (|\partial_{x_i} u|^{P_-^-}_{p_i(x)} - |\partial_{x_i} u|^{P_+^+}_{p_i(x)}) \\ &\geqslant N \left(\frac{\sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}}{N} \right)^{P_-^-} - N \\ &= \frac{\|u\|^{P_-^-}}{N^{P_-^- - 1}} - N. \end{aligned} \tag{4.3}$$

Taking into account relations (4.2) and (4.3), the inequality (4.1) reduces to

$$I(u) \geqslant \frac{\|u\|^{P_-^-}}{P_+^+ N^{P_-^- - 1}} - c_2 \max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\} - c_4.$$

Define

$$\alpha_k = \sup\{|u|_{L^{\alpha(x)}(\Omega)};\ \|u\| = 1,\ u \in Z_k\}.$$

By [**10**, Proposition 3.5], we know that $\lim_{k\to\infty} \alpha_k = 0$.

If $\max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\} = |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}$, we have

$$I(u) \geqslant \frac{\|u\|^{P_-^-}}{P_+^+ N^{P_-^- - 1}} - c_2 \alpha_k^{\alpha^+} \|u\|^{\alpha^+} - c_4. \tag{4.4}$$

Depending on the relation between P_-^- and α^+, we distinguish the following cases.

Case 1 ($P_-^- > \alpha^+$). For sufficiently large k we have $\alpha_k < 1/(2c_2 P_+^+ N^{P_-^- - 1})$, so relation (4.4) yields

$$I(u) \geqslant \frac{1}{2c_2 P_+^+ N^{P_-^- - 1}} \|u\|^{P_-^-}.$$

By choosing r_k such that $r_k \to \infty$ as $k \to \infty$ for $u \in Z_k$ with $\|u\| = r_k$, we have that $I(u) \to \infty$.

Case 2 ($\alpha^- > P_+^+$). Choose $r_k = (c_2 N^{P_-^- - 1} \alpha^+ \alpha_k^{\alpha^+})^{1/(P_-^- - \alpha^+)}$. We deduce that

$$I(u) \geqslant \frac{1}{N^{P_-^- - 1}} \left(\frac{1}{P_+^+} - \frac{1}{\alpha^+} \right) r_k^{P_-^-} - c_4.$$

Since $\alpha_k \to 0$ and $r_k \to \infty$ as $k \to \infty$, we obtain $I(u) \to \infty$.

Similarly, if $\max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\} = |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}$, we can deduce that for $u \in Z_k$ with $\|u\| = r_k > 1$, $I(u) \to \infty$ as $k \to +\infty$ and the proof is complete. □

Lemma 4.2. *For every $k \in \mathbb{N}$ there exists $\rho_k > r_k$ (r_k given by Lemma 4.1) such that*

$$\max_{u \in Y_k,\, \|u\| = \rho_k} I(u) \leqslant 0.$$

Proof. From (A_0) and (A_1) we have

$$A_i(x, \eta) = \int_0^1 a_i(x, t\eta)\eta \,\mathrm{d}t \leqslant c_5 \left(|\eta| + \frac{1}{p_i(x)} |\eta|^{p_i(x)} \right)$$

for all $x \in \bar{\Omega}$ and $\eta \in \mathbb{R}^N$, where $c_5 = \max_{i \in \{1,\ldots,N\}} \bar{c}_i$. Therefore,

$$\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u) \,\mathrm{d}x \leqslant c_5 \left(\sum_{i=1}^N \int_\Omega |\partial_{x_i} u| \,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} \,\mathrm{d}x \right).$$

Moreover, by rewriting condition (F_2) we can obtain the existence of a positive constant c_6 such that

$$F(x, s) \geqslant c_6 |s|^\theta$$

for all $x \in \Omega$ and $s \in \mathbb{R}$. Then, for any $u \in Y_k \backslash \{0\}$ with $\|u\| = 1$ and $1 < \rho_k = t_k$ with $t_k \to \infty$, we have

$$\begin{aligned} I(t_k u) &= \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i}(t_k u))\,\mathrm{d}x + \frac{1}{P_+^+}\int_\Omega b(x)|t_k u|^{P_+^+}\,\mathrm{d}x - \int_\Omega F(x, t_k u)\,\mathrm{d}x \\ &\leqslant c_5\bigg(\sum_{i=1}^N \int_\Omega |\partial_{x_i}(t_k u)|\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i}(t_k u)|^{p_i(x)}}{p_i(x)}\,\mathrm{d}x\bigg) + \frac{1}{P_+^+}\int_\Omega b(x)|t_k u|^{P_+^+}\,\mathrm{d}x \\ &\quad - c_6 \int_\Omega |t_k u|^\theta\,\mathrm{d}x + c_7 \\ &\leqslant c_5 t_k^{P_+^+} \sum_{i=1}^N \int_\Omega \bigg(|\partial_{x_i} u| + \frac{|\partial_{x_i} u|^{p_i(x)}}{P_-^-}\bigg)\,\mathrm{d}x + \frac{t_k^{P_+^+}}{P_+^+}\int_\Omega b(x)|u|^{P_+^+}\,\mathrm{d}x \\ &\quad - c_6 t_k^\theta \int_\Omega |u|^\theta\,\mathrm{d}x + c_7. \end{aligned}$$

Since $\dim Y_k < \infty$ and all norms are equivalent in the finite-dimensional space, it is easy to see that $I(t_k u) \to -\infty$ as $k \to \infty$ for $u \in Y_k$, due to $\theta > P_+^+$. Therefore, we deduce that for ρ_k large enough ($\rho_k > r_k$),

$$\max_{u \in Y_k,\, \|u\| = \rho_k} I(u) \leqslant 0.$$

This completes the proof. □

Lemma 4.3. *The energy functional I satisfies the Palais–Smale condition.*

Proof. Let $(u_n) \subset X$ be a sequence such that

$$|I(u_n)| < c_8 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{4.5}$$

We claim that (u_n) is bounded. Arguing by contradiction, we assume that, passing eventually to a subsequence still denoted by (u_n), $\|u_n\| \to \infty$ as $n \to \infty$.

Using (4.5), for n large enough we have

$$\begin{aligned} 1 + c_8 + \|u_n\| &\geqslant I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\ &\geqslant \sum_{i=1}^N \int_\Omega \bigg[A_i(x, \partial_{x_i} u_n) - \frac{1}{\theta} a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n\bigg]\,\mathrm{d}x \\ &\qquad + \bigg(\frac{1}{P_+^+} - \frac{1}{\theta}\bigg)\int_\Omega b(x)|u|^{P_+^+}\,\mathrm{d}x - \bigg[\int_\Omega F(x, u_n) - \frac{1}{\theta} u_n f(x, u_n)\bigg]\,\mathrm{d}x. \end{aligned} \tag{4.6}$$

From (A_2), for all $x \in \Omega$ and $i \in \{1, \ldots, N\}$ we have

$$a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n \leqslant p_i(x) A_i(x, \partial_{x_i} u_n) \leqslant P_+^+ A_i(x, \partial_{x_i} u_n),$$

which implies that

$$-\frac{1}{\theta}a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n \geqslant -\frac{P_+^+}{\theta}A_i(x,\partial_{x_i}u_n).$$

Combining the previous inequality with relation (4.6) and using (F_2) we obtain

$$1+c_8+\|u_n\| \geqslant \left(1-\frac{P_+^+}{\theta}\right)\sum_{i=1}^N \int_\Omega A_i(x,\partial_{x_i}u_n)\,\mathrm{d}x.$$

Again from (A_2), we have

$$A_i(x,\partial_{x_i}u_n) \geqslant \frac{1}{p_i(x)}|\partial_{x_i}u_n|^{p_i(x)} \geqslant \frac{1}{P_+^+}|\partial_{x_i}u_n|^{p_i(x)}$$

for all $x\in\Omega$ and $i\in\{1,\dots,N\}$.

Taking into consideration relation (4.3), we obtain

$$1+c_8+\|u_n\| \geqslant \left(\frac{1}{P_+^+}-\frac{1}{\theta}\right)\left(\frac{\|u_n\|^{P_-^-}}{N^{p_-^- -1}}-N\right).$$

Dividing the above inequality by $\|u_n\|^{P_-^-}$ and passing to the limit as $n\to\infty$, we obtain a contradiction.

It follows that (u_n) is bounded in X. This information, combined with the fact that X is reflexive, implies that there exist a subsequence, still denoted by (u_n), and $u_0\in X$ such that (u_n) converges weakly to u_0 in X.

Due to [**18**, Theorem 1], the embeddings $X\hookrightarrow L^{\alpha(x)}(\Omega)$ and $X\hookrightarrow L^{P_+^+}(\Omega)$ are compact. Thus, (u_n) converges strongly to u_0 in $L^{\alpha(x)}(\Omega)$ and also in $L^{P_+^+}(\Omega)$.

Using the Hölder-type inequality and (F_1), we deduce that

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u_0)\,\mathrm{d}x = 0, \tag{4.7}$$

$$\lim_{n\to\infty}\int_\Omega b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0)\,\mathrm{d}x = 0. \tag{4.8}$$

Using (4.5) we infer that

$$\lim_{n\to\infty}\langle I'(u_n),u_n-u_0\rangle = 0;$$

more precisely,

$$\lim_{n\to\infty}\int_\Omega \left[\sum_{i=1}^N a_i(x,\partial_{x_i}u_n)(\partial_{x_i}u_n-\partial_{x_i}u_0)+b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0) - f(x,u_n)(u_n-u_0)\right]\mathrm{d}x = 0.$$

Combining the above relation with (4.7) and (4.8) we have

$$\lim_{n\to\infty}\int_\Omega \sum_{i=1}^N a_i(x,\partial_{x_i}u_n)(\partial_{x_i}u_n-\partial_{x_i}u_0)\,\mathrm{d}x=0.$$

Using [**16**, Lemma 1], we deduce that (u_n) converges strongly to u_0 in X; in other words, I satisfies the Palais–Smale condition. □

4.1. Proof of Theorem 3.2 completed

The fact that the mapping A_i is even and f is odd with respect to their second variables implies that I is even. The proof follows immediately from Lemmas 4.1–4.3 and the fountain theorem.

5. The case of small positive parameters

This section is devoted to the proof of Theorem 3.4, which is essentially based on the Ekeland variational principle [**8**]. Let us define the functional $I_\lambda\colon X\to\mathbb{R}$ by

$$I_\lambda(u)=\int_\Omega\left\{\sum_{i=1}^N A_i(x,\partial_{x_i}u)+\frac{b(x)}{P_+^+}|u|^{P_+^+}-\frac{\lambda}{q(x)}|u|^{q(x)}\right\}\mathrm{d}x.$$

Then the functional I_λ associated with problem (3.3) is well defined and of C^1 class on X. Moreover, we have

$$\langle I_\lambda'(u),\varphi\rangle=\int_\Omega\left\{\sum_{i=1}^N a_i(x,\partial_{x_i}u)\partial_{x_i}\varphi+b(x)|u|^{P_+^+-2}u\varphi-\lambda|u|^{q(x)-2}u\varphi\right\}\mathrm{d}x$$

for all $u,\varphi\in X$. Thus, weak solutions of (3.3) are exactly the critical points of the functional I_λ. Due to [**14**, Lemma 3.5], we can show that I_λ is weakly lower semi-continuous in X.

We establish the following two auxiliary properties.

Lemma 5.1. *There exists $\lambda^*>0$ such that for any $\lambda\in(0,\lambda^*)$ there exist $\rho,a>0$ such that $I_\lambda(u)\geqslant a>0$ for any $u\in X$ with $\|u\|=\rho$.*

Proof. Under the conditions of Theorem 3.4, X is continuously embedded in $L^{q(x)}(\Omega)$. Thus, there exists a positive constant c_9 such that

$$|u|_{q(x)}\leqslant c_9\|u\|\quad\text{for all } u\in X. \tag{5.1}$$

Now, let us assume that $\|u\|\leqslant\min\{1,1/c_9\}$, where c_9 is the positive constant from above. Then we have $|u|_{q(x)}<1$. Using [**11**, Theorem 1.3] we find

$$\int_\Omega |u|^{q(x)}\,\mathrm{d}x\leqslant|u|_{q(x)}^{q^-}\quad\text{for all } u\in X \text{ with } \|u\|=\rho\in(0,1). \tag{5.2}$$

Relations (5.1) and (5.2) yield

$$\int_\Omega |u|^{q(x)}\,\mathrm{d}x \leqslant c_9^{q^-}\|u\|^{q^-} \quad \text{for all } u \in X \text{ with } \|u\| = \rho. \tag{5.3}$$

Using the hypotheses (A_2), (B) and (5.2), we deduce that for any $u \in X$ with $\|u\| = \rho$, the following holds:

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \biggl\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \frac{\lambda}{q(x)}|u|^{q(x)} \biggr\}\,\mathrm{d}x \\ &\geqslant \frac{1}{P_+^+}\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,\mathrm{d}x + \frac{b_0}{P_+^+}|u|^{P_+^+}_{L^{P_+^+}(\Omega)} - \frac{\lambda}{q^-}c_9^{q^-}\|u\|^{q^-}. \end{aligned} \tag{5.4}$$

Here, we let $\|u\| < 1$, so $|\partial_{x_i} u|_{p_i(x)} < 1$, $i \in \{1, \dots, N\}$. For such an element u, by [**11**, Theorem 1.3], we have

$$\begin{aligned} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,\mathrm{d}x \geqslant \sum_{i=1}^N |\partial_{x_i} u|^{p_i^+}_{p_i(x)} &\geqslant \sum_{i=1}^N |\partial_{x_i} u|^{P_+^+}_{p_i(x)} \\ &\geqslant N\biggl(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}}{N}\biggr)^{P_+^+} \\ &= \frac{\|u\|^{P_+^+}}{N^{P_+^+-1}}. \end{aligned} \tag{5.5}$$

Taking into account relations (4.2) and (5.5), the inequality (5.4) reduces to

$$\begin{aligned} I_\lambda(u) \geqslant \frac{\|u\|^{P_+^+}}{P_+^+ N^{P_+^+-1}} - \frac{\lambda}{q^-}c_9^{q^-}\|u\|^{q^-} &= \frac{\rho^{P_+^+}}{P_+^+ N^{P_+^+-1}} - \frac{\lambda}{q^-}c_9^{q^-}\rho^{q^-} \\ &= \rho^{q^-}\biggl(\frac{1}{P_+^+ N^{P_+^+-1}}\rho^{P_+^+-q^-} - \frac{\lambda}{q^-}c_9^{q^-}\biggr). \end{aligned}$$

If we define

$$\lambda^* = \frac{q^-}{2P_+^+ N^{P_+^+-1} c_9^{q^-}}\rho^{P_+^+-q^-}, \tag{5.6}$$

then, for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\| = \rho$, there exists $a = \rho^{P_+^+}/(2P_+^+ N^{P_+^+-1})$ such that $I_\lambda(u) \geqslant a > 0$. □

Lemma 5.2. *For any $\lambda \in (0, \lambda^*)$, where λ^* is given by (5.6), there exist $\psi \in X$ such that $\psi \geqslant 0$, ψ is not equivalent to zero and $I_\lambda(t\psi) < 0$ for all $t > 0$ small enough.*

Proof. From (A_0) and (A_1) we have

$$A_i(x, \eta) = \int_0^1 a_i(x, t\eta)\eta\,\mathrm{d}t \leqslant c_{10}\biggl(|\eta| + \frac{1}{p_i(x)}|\eta|^{p_i(x)}\biggr)$$

for all $x \in \bar{\Omega}$ and $\eta \in \mathbb{R}^N$, where $c_{10} = \max_{i\in\{1,\dots,N\}} \bar{c}_i$. Therefore,

$$\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\,\mathrm{d}x \leqslant c_{10} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i} u| + \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} \right) \mathrm{d}x.$$

By the conditions of Theorem 3.4, $q^- < P_-^-$. Let $\epsilon_0 > 0$ such that $q^- + \epsilon_0 < P_-^-$. Since $q \in C(\bar{\Omega})$, there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. It follows that $q(x) < q^- + \epsilon_0 < P_-^-$ for all $x \in \Omega_0$.

Let $\psi \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(\psi) \supset \overline{\Omega_0}$, $\psi(x) = 1$ for all $x \in \overline{\Omega_0}$, and $0 \leqslant \psi \leqslant 1$ in Ω. Then, for any $t \in (0, 1)$, we have

$$\begin{aligned}
I_\lambda(t\psi) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(t\psi)) + \frac{b(x)}{P_+^+} |t\psi|^{P_+^+} - \frac{\lambda}{q(x)} |t\psi|^{q(x)} \right\} \mathrm{d}x \\
&\leqslant c_{10} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}(t\psi)| + \frac{|\partial_{x_i}(t\psi)|^{p_i(x)}}{p_i(x)} \right) \mathrm{d}x + \frac{1}{P_+^+} \int_\Omega b(x) |t\psi|^{P_+^+}\,\mathrm{d}x \\
&\quad - \lambda \int_\Omega \frac{1}{q(x)} |t\psi|^{q(x)}\,\mathrm{d}x \\
&\leqslant c_{10} t^{P_-^-} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\psi| + \frac{1}{P_-^-} |\partial_{x_i}\psi|^{p_i(x)} \right) \mathrm{d}x + \frac{t^{P_+^+}}{P_+^+} \int_\Omega b(x) |\psi|^{P_+^+}\,\mathrm{d}x \\
&\quad - \frac{\lambda}{q^+} \int_{\Omega_0} t^{q(x)} |\psi|^{q(x)}\,\mathrm{d}x \\
&\leqslant c_{10} t^{P_-^-} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}\psi| + \frac{1}{P_-^-} |\partial_{x_i}\psi|^{p_i(x)} \right) \mathrm{d}x + \frac{t^{P_+^+}}{P_+^+} \int_\Omega b(x) |\psi|^{P_+^+}\,\mathrm{d}x \\
&\quad - \frac{\lambda t^{q^- + \epsilon_0}}{q^+} \int_{\Omega_0} |\psi|^{q(x)}\,\mathrm{d}x.
\end{aligned}$$

So, $I_\lambda(t\psi) < 0$ for $t < \delta^{1/(P_-^- - q^- - \epsilon_0)}$, with

$$\begin{aligned}
0 &< \delta \\
&< \min\left\{ 1, \frac{\lambda}{q^+} \frac{\int_{\Omega_0} |\psi|^{q(x)}\,\mathrm{d}x}{c_{10} \sum_{i=1}^N \int_\Omega (|\partial_{x_i}\psi| + (1/P_-^-)|\partial_{x_i}\psi|^{p_i(x)})\,\mathrm{d}x + (1/P_+^+) \int_\Omega b(x) |\psi|^{P_+^+}\,\mathrm{d}x} \right\}.
\end{aligned}$$

This completes the proof. □

5.1. Proof of Theorem 3.4 completed

Let λ^* be defined as in (5.6) and let $\lambda \in (0, \lambda^*)$. By Lemma 5.1, it follows that on the boundary of the ball centred at the origin and of radius ρ in X, we have

$$\inf_{\partial B_\rho(0)} I_\lambda(u) > 0.$$

On the other hand, by Lemma 5.2, there exists $\psi \in X$ such that

$$I_\lambda(t\psi) < 0 \quad \text{for } t > 0 \text{ small enough.}$$

Moreover, for $u \in B_\rho(0)$,

$$I_\lambda(u) \geqslant \frac{\|u\|^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} - \frac{\lambda}{q^-} c_9^{q^-} \|u\|^{q^-}.$$

It follows that

$$-\infty < c_{11} = \inf_{\overline{B_\rho(0)}} I_\lambda(u) < 0.$$

We now let $0 < \varepsilon < \inf_{\partial B_\rho(0)} I_\lambda - \inf_{B_\rho(0)} I_\lambda$. Applying the Ekeland variational principle [**8**] to the functional $I_\lambda \colon \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned} I_\lambda(u_\varepsilon) &< \inf_{\overline{B_\rho(0)}} I_\lambda + \varepsilon \\ I_\lambda(u_\varepsilon) &< I_\lambda(u) + \varepsilon \|u - u_\varepsilon\|, \quad u \neq u_\varepsilon. \end{aligned}$$

Since

$$I_\lambda(u_\varepsilon) \leqslant \inf_{\overline{B_\rho(0)}} I_\lambda + \varepsilon \leqslant \inf_{B_\rho(0)} I_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} I_\lambda,$$

we deduce that $u_\varepsilon \in B_\rho(0)$. Now, we define $K_\lambda \colon \overline{B_\rho(0)} \to \mathbb{R}$ by $K_\lambda(u) = I_\lambda(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of K_λ, and thus

$$\frac{K_\lambda(u_\varepsilon + tv) - K_\lambda(u_\varepsilon)}{t} \geqslant 0$$

for small $t > 0$ and $v \in B_\rho(0)$. The above relation yields

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geqslant 0.$$

Letting $t \to 0$, it follows that $\langle I'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\| > 0$ and we infer that $\|I'_\lambda(u_\varepsilon)\| \leqslant \varepsilon$. We deduce that there exists a sequence $(v_n) \subset B_1(0)$ such that

$$I_\lambda(v_n) \to c_{11} \quad \text{and} \quad I'_\lambda(v_n) \to 0. \tag{5.7}$$

It is clear that (v_n) is bounded in X. Thus, there exists $u_1 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_1 in X. Theorem 1 in [**18**] yields that the embeddings $X \hookrightarrow L^{q(x)}(\Omega)$ and $X \hookrightarrow L^{P_+^+}(\Omega)$ are continuous and compact. Then (v_n) converges strongly to u_1 in $L^{q(x)}(\Omega)$ and in $L^{P_+^+}(\Omega)$.

Using the Hölder-type inequality, we can easily obtain that

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega |v_n|^{q(x)-2} v_n (v_n - u_1)\,\mathrm{d}x &= 0, \\ \lim_{n\to\infty} \int_\Omega b(x)|v_n|^{P_+^+ - 2} v_n (v_n - u_1)\,\mathrm{d}x &= 0. \end{aligned}$$

On the other hand, relation (5.7) yields

$$\lim_{n\to\infty}\langle I'(v_n), v_n - u_1\rangle = 0.$$

Using the above information, we find that

$$\lim_{n\to\infty}\int_\Omega \sum_{i=1}^N a_i(x,\partial_{x_i}v_n)(\partial_{x_i}v_n - \partial_{x_i}u_1)\,\mathrm{d}x = 0.$$

Using [**16**, Lemma 1], we deduce that (v_n) converges strongly to u_1 in X. So, by (5.7),

$$I_\lambda(u_1) = c_{11} < 0 \quad \text{and} \quad I'_\lambda(u_1) = 0,$$

that is, u_1 is a non-trivial weak solution for the problem (3.3). This completes the proof.

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