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Small perturbations of elliptic problems with variable growth

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ABSTRACT

This paper deals with the study of a nonlinear eigenvalue problem driven by a new class of non-homogeneous differential operators with variable exponent and involving a nonlinear term with variable growth. The framework in the present paper corresponds to the case of small perturbations of the nonlinear term. Combining variational arguments with energy estimates, we establish the existence of eigenvalues in a neighborhood of the origin. Our abstract setting includes several models described by nonhomogeneous differential operators, including the case of the capillarity equation.

1. Introduction and abstract setting

The qualitative analysis of nonlinear problems with one or several variable exponents was extended recently by I.H. Kim and Y.H. Kim [1] in the abstract setting of a new class of non-homogeneous differential operators. Their work is an important contribution to the refined mathematical analysis of nonlinear problems with one or more variable exponents, mainly because it allows the understanding of some classes of nonlinear problems with possible lack of uniform convexity. For these stationary problems, the associated energy density changes its ellipticity as well as its growth properties according to the point. We refer to S. Baraket, S. Chebbi, N. Chorfi, and V. Rădulescu [2] for recent advances in this new abstract setting.

Nonlinear problems with this structure are motivated by numerous models in the applied sciences that are driven by partial differential equations with one or more variable exponents. The variable exponents describe the geometry of a material which is allowed to change its hardening exponent according to the point. This leads to the analysis of variable exponents Lebesgue and Sobolev function spaces (denoted by $L^{p(x)}$ and

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$W^{1,p(x)}$), where p is a real-valued (non-constant) function. We point out here the important contributions of T.C. Halsey [3] and V.V. Zhikov [4] in strong relationship with the behavior of strongly anisotropic materials. This is mainly achieved in the framework of the homogenization and nonlinear elasticity. We refer here to the monograph by V. Rădulescu and D. Repovš [5], which includes a thorough variational and topological analysis of several classes of problems with variable exponent (see also the important contributions by P. Pucci et al. [6,7]).

We are concerned in this paper with the study of a class of nonlinear non-homogeneous eigenvalue problems with Dirichlet boundary condition. Our main result establishes the existence of nontrivial weak solutions in the case of small perturbations, namely if the right-hand side of the problem is sufficiently small, as described by a suitable positive parameter. The abstract setting of this paper includes several important applications, such as the capillarity equation or the generalized version of the mean curvature equation. Related qualitative properties concerning the qualitative analysis of anisotropic elliptic problems have been established by Repovš et al., see [8] and [9].

We assume throughout the present paper that $\Omega \subset \mathbb{R}^N$ is a smooth bounded.

Set $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}$. For all $p \in C_+(\overline{\Omega})$ we denote $p^+ = \sup_{x\in\Omega} p(x)$ and $p^- = \inf_{x\in\Omega} p(x)$.

Let $L^{p(x)}(\Omega)$ be the Lebesgue space with variable exponent, namely

$$L^{p(x)}(\Omega) = \left\{u;\ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

This vector space is a Banach space if it is endowed with the Luxemburg norm defined by

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Let $L^{p'(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Then for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder-type inequality holds:

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)}|v|_{p'(x)}. \tag{1}$$

Let $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ be the *modular* of $L^{p(x)}(\Omega)$, which is defined by $\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx$. Then

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}. \tag{2}$$

We define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

or

$$\|u\|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \le 1\right\}.$$

Let $W_0^{1,p(x)}(\Omega)$ denote the closure of the set of compactly supported $W^{1,p(x)}$-functions with respect to the norm $\|u\|_{p(x)}$. By the Poincaré inequality, the space $W_0^{1,p(x)}(\Omega)$ can be also defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{p(x)} = |\nabla u|_{p(x)}$. The vector space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. Moreover, if $0 < |\Omega| < \infty$ and $p_1,\, p_2 \in C_+(\overline{\Omega})$ such that $p_1 \le p_2$ then there exists the continuous embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega)$.

Set $\varrho_{p(x)}(u) = \int_\Omega |\nabla u(x)|^{p(x)}\, dx$. Then

$$\|u\| < 1 \;\Rightarrow\; \|u\|^{p^+} \le \varrho_{p(x)}(u) \le \|u\|^{p^-}. \tag{3}$$

Set

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

We recall that if $p, q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

2. Main result

Fix $p_1, p_2 \in C_+(\overline{\Omega})$ and consider the mappings $\phi, \psi : \Omega \times [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(H1) the functions $\phi(\cdot,\xi)$ and $\psi(\cdot,\xi)$ are measurable on Ω for all $\xi \ge 0$ and $\phi(x,\cdot)$, $\psi(x,\cdot)$ are locally absolutely continuous on $[0,\infty)$ for almost all $x \in \Omega$;

(H2) there exist $b > 0$ and $a_2 \in L^{p_2'}(\Omega)$ such that

$$|\phi(x,|v|)v| \le b|v|^{p_1(x)-1}, \quad |\psi(x,|v|)v| \le a_2(x) + b|v|^{p_2(x)-1}$$

for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$;

(H3) there exists $c > 0$ such that for almost all $x \in \Omega$ and for all $\xi > 0$

$$\phi(x,\xi) \ge c\xi^{p_1(x)-2}, \quad \phi(x,\xi) + \xi \frac{\partial \phi}{\partial \xi}(x,\xi) \ge c\xi^{p_1(x)-2}$$

and

$$\psi(x,\xi) \ge c\xi^{p_2(x)-2}, \quad \psi(x,\xi) + \xi \frac{\partial \psi}{\partial \xi}(x,\xi) \ge c\xi^{p_2(x)-2};$$

(H4) we have $\min_{x\in\overline{\Omega}} p_2(x) < \min_{x\in\overline{\Omega}} p_1(x)$ and $\max_{x\in\overline{\Omega}}\{p_1(x), p_2(x)\} < p_1^*(x)$ for all $x \in \overline{\Omega}$.

Consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}(\phi(x,|\nabla u|)\nabla u) = \lambda\, \psi(x,|u|)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Problem (4) is driven by the non-homogeneous operator $\operatorname{div}(\phi(x,|\nabla u|)\nabla u)$. If $\phi(x,\xi) = \xi^{p(x)-2}$ then we obtain the standard $p(x)$-Laplace operator, $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. Our abstract setting includes the case

$$\phi(x,\xi) = \left(1 + \frac{\xi^{p(x)}}{\sqrt{1+\xi^{2p(x)}}}\right)\xi^{p(x)-2}, \quad x \in \Omega,\ \xi > 0,$$

which corresponds to the capillarity equation described by the differential operator

$$\operatorname{div}\left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right].$$

We say that $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ is a solution of problem (4) if

$$\int_\Omega \phi(x, |\nabla u|)\nabla u \cdot \nabla v dx = \lambda \int_\Omega \psi(x, |u|)uvdxdx \quad \text{for all } v \in W_0^{1,p_1(x)}(\Omega).$$

In this case, u is an *eigenfunction* of problem (4) and the corresponding $\lambda \in \mathbb{R}$ is an *eigenvalue* of (4).

For ϕ and ψ described in hypotheses (H1)–(H4) we set

$$A_0(x, t) := \int_0^t \phi(x, s)ds; \quad B_0(x, t) := \int_0^t \psi(x, s)ds, \quad \text{for all } t \geq 0. \tag{5}$$

Consider the functionals A, $B : W_0^{1,p_1(x)}(\Omega) \to \mathbb{R}$ defined by $A(u) := \int_\Omega A_0(x, |\nabla u|)dx$ and $B(u) := \int_\Omega B_0(x, |u|)dx$. The energy functional associated to problem (4) is $C_\lambda : W_0^{1,p_1(x)}(\Omega) \to \mathbb{R}$, which is defined by $C_\lambda(u) := A(u) - \lambda B(u)$. Thus, by [1, Lemma 3.2], C_λ is of class C^1 and weak solutions of (4) coincide with nontrivial critical points of C_λ.

Theorem 2.1. *Assume that hypotheses* (H1)–(H4) *are fulfilled. Then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem* (4) *has at least one solution.*

Proof. In our arguments we use some ideas developed by Rădulescu et al., see [10] and [5, Chapter 3].

Throughout this paper, we denote by $\|\cdot\|$ the norm in the Banach space $W_0^{1,p_1(x)}(\Omega)$. We split the proof into several steps.

Step 1. There exist positive numbers λ^*, r and a such that $C_\lambda(u) \geq a$ for all $\lambda \in (0, \lambda^*)$ and for all $u \in W_0^{1,p_1(x)}(\Omega)$ satisfying $\|u\| = r$.

By hypothesis (H2), we observe that

$$B(u) \leq \int_\Omega \left(\int_0^{|u|} (|a_2(x)| + b|s|^{p_2(x)-1})ds \right) dx \leq 2\, |a_2|_{p_2'(x)} |u|_{p_2(x)} + \frac{b}{p_2^-}\, \rho_{p_2(x)}(u). \tag{6}$$

Fix $r \in (0, 1)$ and take $u \in W_0^{1,p_1(x)}(\Omega)$ with that $\|u\| = r$. Next, using the continuous embedding of $W_0^{1,p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$ we can assume that r is chosen small enough such that $|u|_{p_2(x)} < 1$. We fix $r \in (0, 1)$ with this property. Using (2), relation (6) yields for all $u \in W_0^{1,p_1(x)}(\Omega)$ with $\|u\| = r$

$$B(u) \leq C_1|u|_{p_2(x)} + C_2|u|_{p_2(x)}^{p_2^-} \leq C_3'\|u\| + C_3''\|u\|^{p_2^-} \leq C_3(r + r^{p_2^-}). \tag{7}$$

On the other side, by (H3) we have

$$A(u) \geq c \int_\Omega \left(\int_0^{|\nabla u|} s^{p_1(x)-1} \right) dx \geq \frac{c}{p_1^+} \int_\Omega |\nabla u|^{p_1(x)} dx = \frac{c}{p_1^+}\, \varrho_{p_1(x)}(u),$$

for all $u \in W_0^{1,p_1(x)}(\Omega)$.

Assume that $\|u\| = r$ and use (3). It follows that

$$A(u) \geq \frac{c}{p_1^+}\, \|u\|^{p_1^+} = C_4 r^{p_1^+}. \tag{8}$$

Combining relations (7), (8) with hypothesis (H4) we deduce that for all $u \in W_0^{1,p_1(x)}(\Omega)$ with $\|u\| = r$ we have

$$C(u) \geq C_4 r^{p_1^+} - \lambda C_3(r + r^{p_2^-}) = C_3 r \left(\frac{C_4}{C_3}\, r^{p_1^+ - p_2^-} - \lambda(1 + r^{p_2^- - 1}) \right) \geq a > 0,$$

provided that $\lambda \in (0, \lambda^*)$ for some λ^*. We observe that as soon as $r \in (0,1)$ with the above properties is fixed then a and λ^* can be also assumed to be fixed.

Step 2. There exist $v \in W_0^{1,p_1(x)}(\Omega)$ and $t_0 > 0$ such that $C_\lambda(t_0 v) < 0$.

In order to construct the convenient test function v, we use our hypothesis (H4). Indeed, since $p_2^- < p_1^-$, we fix $\varepsilon_0 > 0$ such that $p_2^- + \varepsilon_0 < p_1^-$ and choose a bounded subset ω of Ω such that $p_2(x) < p_2^- + \varepsilon_0$ for all $x \in \omega$. Next, we choose $v \in C_0^\infty(\Omega)$ such that $v \geq 0$ and $\operatorname{supp} v \supset \omega$. It follows that for all $t \in (0,1)$

$$\begin{aligned}
C_\lambda(tv) &\leq b \int_\Omega \int_0^{t|\nabla v|} |s|^{p_1(x)-1} ds dx - \lambda c \int_\Omega \int_0^{tv} s^{p_2(x)-1} ds dx \\
&\leq \frac{bt^{p_1^-}}{p_1^-} \int_\Omega |\nabla v|^{p_1(x)} dx - \frac{\lambda c}{p_2^+} \int_\Omega t^{p_2(x)} v^{p_2(x)} dx \\
&\leq \frac{bt^{p_1^-}}{p_1^-} \int_\Omega |\nabla v|^{p_1(x)} dx - \frac{\lambda c}{p_2^+} \int_\omega t^{p_2(x)} v^{p_2(x)} dx \\
&\leq C_5' t^{p_1^-} - C_6 t^{p_2^- + \varepsilon_0} \int_\omega v^{p_2(x)} dx \\
&= C_5' t^{p_1^-} - C_6' t^{p_2^- + \varepsilon_0}.
\end{aligned}$$

Since $p_2^- + \varepsilon_0 < p_1^-$, we can choose small $t_0 > 0$ such that $C_\lambda(t_0 v) < 0$.

Using Step 2, there exist $v \in W_0^{1,p_1(x)}(\Omega)$ and $t_0 > 0$ such that $C_\lambda(t_0 v) < 0$. We can assume that t_0 is small enough so that $\|t_0 v\| < r$, where r is given in Step 1. Set $B := \{u \in W_0^{1,p_1(x)}(\Omega);\ \|u\| \leq r\}$. Combining steps 1 and 2 we deduce that

$$\inf\{C_\lambda(u);\ u \in B\} \leq C_\lambda(t_0 v) < 0 \quad \text{and} \quad \inf\{C_\lambda(u);\ u \in \partial B\} \geq a > 0.$$

Step 3. Existence of “almost critical” points and completion of the proof.

Fix $\lambda \in (0, \lambda^*)$, where λ^* is given by Step 1. Let n be a positive integer such that $na > 1$. Next, we apply the Ekeland variational principle. Thus, there exists $u_n \in W_0^{1,p_1(x)}(\Omega)$ such that

$$C_\lambda(u_n) < \inf\{C_\lambda(u);\ u \in B\} + \frac{1}{n} \tag{9}$$

$$C_\lambda(u_n) < C_\lambda(u) + \frac{1}{n}\|u - u_n\|, \quad \text{for all } u \in W_0^{1,p_1(x)}(\Omega) \setminus \{u_n\}. \tag{10}$$

We first observe that $u_n \notin \partial B$. This follows from

$$C_\lambda(u_n) < \inf\{C_\lambda(u);\ u \in B\} + \frac{1}{n} < \frac{1}{n} < a \leq \inf\{C_\lambda(u);\ u \in \partial B\}.$$

Next, a standard argument shows that relation (10) implies that $\|C_\lambda'(u_n)\| \leq 1/n$. Thus, $(u_n) \subset B$ is a sequence of “almost critical” points of C_λ, namely

$$\lim_{n\to\infty} C_\lambda(u_n) = \inf\{C_\lambda(u);\ u \in B\} \text{ and } \lim_{n\to\infty} \|C_\lambda'(u_n)\| = 0.$$

Since (u_n) is a bounded sequence, we can assume (up to a subsequence) that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p_1(x)}(\Omega) \tag{11}$$

$$u_n \to u \quad \text{in } L^{p_2(x)}(\Omega). \tag{12}$$

Combining hypothesis (H1) with the Hölder inequality we find

$$\begin{aligned}
\left| \int_\Omega \psi(x, u_n) u_n (u_n - u) dx \right| &\leq \int_\Omega (a_2(x) + |u_n|^{p_2(x)-1}) |u_n - u| dx \\
&\leq |u_n - u|_{p_2(x)} |a_2 + |u_n|^{p_2(x)-1}|_{p_2'(x)} \to 0.
\end{aligned} \tag{13}$$

But

$$\lim_{n\to\infty} C'_\lambda(u_n)(u_n-u)=0.$$

So, by relations (13) and (2), we obtain

$$\int_\Omega \phi(x,|\nabla u_n|)\nabla u_n\cdot\nabla(u_n-u)dx\to 0 \quad \text{as } n\to\infty.$$

Using now (11) and applying the same argument as in F. Gazzola and Rădulescu [11, p. 59] (see also X. Fan and Q.H. Zhang [12, Theorem 3.1]) we conclude that $u_n\to u$ in $W_0^{1,p_1(x)}(\Omega)$. Thus, u is a critical point of C_λ, hence a weak solution of problem (4). $\square$

2.1. Perspectives and open problems

Function spaces $L^{p(x)}$ and $W^{1,p(x)}$ with variable exponent have several unusual properties, such as: (i) they are not translation invariant; (ii) the use of convolution is limited and the Young inequality holds if and only if p is constant; (iii) these spaces do not fulfill the mean continuity property; (iv) the co-area formula has no variable exponent analogue, etc. We refer to Rădulescu and Repovš [5, pp. 8, 9] for more details.

(a) The methods used in the present paper can be extended to *perturbed* problems, namely if the left-hand side of (4) is replaced with the operator $-\mathrm{div}\,(\phi(x,|\nabla u|)\nabla u)+\alpha\,\phi(x,u)u$, provided that there exists $C>0$ such that

$$\int_\Omega A_0(x,|\nabla u|)dx+\alpha\int_\Omega A_0(x,|u|)dx\ge C\,\varrho_{p_1(x)}(u)\quad \text{for every } u\in W_0^{1,p_1(x)}(\Omega).$$

(b) We do not have any information on the qualitative behavior for *big* values of the parameter λ in problem (4). A more difficult question is to develop an exhaustive analysis for all $\lambda>0$.

(c) We appreciate that a valuable research direction is to generalize the abstract approach developed in this paper to the framework studied by G. Mingione et al. [13,14], namely to *double-phase* integral functionals of the type

$$u\mapsto\int_\Omega[|\nabla u|^{p(x)}+a(x)|\nabla u|^{q(x)}]dx \tag{14}$$

or

$$u\mapsto\int_\Omega[|\nabla u|^{p(x)}+a(x)|\nabla u|^{q(x)}\log(e+|x|)]dx, \tag{15}$$

where $p(x)\le q(x)$, $p\ne q$, and $a(x)\ge 0$. In the case of two different materials that involve power hardening exponents $p(x)$ and $q(x)$, the coefficient $a(x)$ described the geometry of a composite of these two materials. When $a(\cdot)>0$ then the $q(\cdot)$-material is present. In the contrary case, the $p(\cdot)$-material is the only one describing the composite. The integral energy functional defined in (15) has a degenerate behavior on the zero set of the gradient. At the same time, if $|\nabla u|$ is small, then there exists an unbalance between the two terms of the integrand in the double-phase energy defined in (15).

(d) We suggest the reader to generalize the main abstract result in this paper to the very general setting of Musielak–Orlicz spaces (see the monograph by Rădulescu and Repovš [5, Chapter 4] for several examples of stationary problems with one or more variable exponents).

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