PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Dave Renfro, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before October 31, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11649. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Let p be real with $p > 1$. Let $(x_0, x_1, \dots)$ be a sequence of nonnegative real numbers. Prove that

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right)^p < \infty \;\Rightarrow\; \sum_{j=0}^{\infty}\left(\frac{1}{j+1}\sum_{k=0}^{j}x_k\right)^p < \infty.$$

11650. *Proposed by Michael Becker, University of South Carolina at Sumter, Sumter, SC.* Evaluate

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-(x-y)^2}\sin^2(x^2+y^2)\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx.$$

11651. *Proposed by Marcel Celaya and Frank Ruskey, University of Victoria, Victoria, BC, Canada.* Show that the equation

$$\left\lfloor \frac{n+1}{\phi}\right\rfloor = n - \left\lfloor \frac{n}{\phi}\right\rfloor + \left\lfloor \frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor - \left\lfloor \frac{\left\lfloor \frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor + \left\lfloor \frac{\left\lfloor \frac{\left\lfloor \frac{\lfloor n/\phi\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor}{\phi}\right\rfloor - \cdots$$

holds for every nonnegative integer n if and only if $\phi = (1+\sqrt{5})/2$.

11652. *Proposed by Ajai Choudhry, Foreign Service Institute, New Delhi, India.* For $a, b, c, d \in \mathbb{R}$, and for nonnegative integers i, j, and n, let

$$t_{i,j} = \sum_{s=0}^{i}\binom{n-i}{j-s}\binom{i}{s}a^{n-i-j+s}b^{j-s}c^{i-s}d^s.$$

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Let $T(a, b, c, d, n)$ be the $(n+1)$-by-$(n+1)$ matrix with (i, j)-entry given by $t_{i,j}$, for $i, j \in \{0, \dots, n\}$. Show that $\det T(a, b, c, d, n) = (ad - bc)^{n(n+1)/2}$.

11653. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Let n be a positive integer. Determine all entire functions f that satisfy, for all complex s and t, the functional equation

$$f(s+t) = \sum_{k=0}^{n-1} f^{(n-1-k)}(s) f^{(k)}(t).$$

Here, $f^{(m)}$ denotes the mth derivative of f.

11654. *Proposed by David Borwein, University of Western Ontario, Canada, and Jonathan M. Borwein and James Wan, CARMA, University of Newcastle, Australia.* Let Cl denote the *Clausen* function, given by $\mathrm{Cl}(\theta) = \sum_{n=1}^{\infty} \sin(n\theta)/n^2$. Let ζ denote the Riemann zeta function.
(**a**) Show that

$$\int_{y=0}^{2\pi} \int_{x=0}^{2\pi} \log(3 + 2\cos x + 2\cos y + 2\cos(x-y))\, dx\, dy = 8\pi \mathrm{Cl}(\pi/3).$$

(**b**) Show that

$$\int_{y=0}^{\pi} \int_{x=0}^{\pi} \log(3 + 2\cos x + 2\cos y + 2\cos(x-y))\, dx\, dy = \frac{28}{3}\zeta(3).$$

11655. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let $ABCD$ be a convex quadrilateral, and let α, β, γ, and δ be the radian measures of angles DAB, ABC, BCD, and CDA, respectively. Suppose $\alpha + \beta > \pi$ and $\alpha + \delta > \pi$, and let $\eta = \alpha + \beta - \pi$ and $\phi = \alpha + \delta - \pi$. Let a, b, c, d, e, f be real numbers with $ac = bd = ef$. Show that if $abe > 0$, then

$$a\cos\alpha + b\cos\beta + c\cos\gamma + d\cos\delta + e\cos\eta + f\cos\phi \le \frac{be}{2a} + \frac{cf}{2b} + \frac{de}{2c} + \frac{af}{2d},$$

while for $abe < 0$ the inequality is reversed.

SOLUTIONS

A Triangle Inequality

11527 [2010, 742]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* Prove that in an acute triangle with sides of length a, b, c, inradius r, and circumradius R,

$$\frac{a^2}{b^2+c^2-a^2} + \frac{b^2}{c^2+a^2-b^2} + \frac{c^2}{a^2+b^2-c^2} \ge \frac{3}{2} \cdot \frac{R}{r}.$$

Solution by Thomas Smotzer, Youngstown State University, Youngstown, OH. Let $\triangle ABC$ be acute, with side lengths a, b, c, area K, and semiperimeter p. Let $S = S(a, b, c)$ be the sum on the left in the required inequality. Note that $K = \frac{1}{2}bc\sin A$, and by the law of cosines $b^2 + c^2 - a^2 = 2bc \cos A$, and similar formulas hold for angles B and C. So

$$S = \frac{a^2}{2bc \cos A} + \frac{b^2}{2ca \cos B} + \frac{c^2}{2ab \cos C}$$
$$= \frac{a^2 \tan A}{4K} + \frac{b^2 \tan B}{4K} + \frac{c^2 \tan C}{4K}.$$

It is enough to show $a^2 \tan A + b^2 \tan B + c^2 \tan C \geq 6KR/r = 6pR$, since $K = rp$. By the law of sines $a = 2R \sin A$, etc., so we must show that

$$a\, 2R\ \sin A \tan A + b\, 2R\ \sin B \tan B + c\, 2R\ \sin C \tan C \geq 3(a + b + c)R.$$

Equivalently, we must show that $a \sin A \tan A + b \sin B \tan B + c \sin C \tan C \geq \frac{3}{2}(a + b + c)$. Note that since the triangle is acute, $\sin A \tan A$, $\sin B \tan B$, and $\sin C \tan C$ occur in the same order after sorting as do the corresponding quantities a, b, and c.

Using Chebyshev's inequality, it suffices to show that $\frac{1}{3}(a + b + c)(\sin A \tan A + \sin B \tan B + \sin C \tan C) \geq \frac{3}{2}$. This simplifies to

$$\sin A \tan A + \sin B \tan B + \sin C \tan C \geq \tfrac{9}{2}.$$

Since $\sin x \tan x$ is a convex function of x on $[0, \pi/2)$, by Jensen's inequality we have $\sin A \tan A + \sin B \tan B + \sin C \tan C \geq 3 \sin \frac{1}{3}(A + B + C) \tan \frac{1}{3}(A + B + C)$. The right side of this simplifies to $3 \sin(\pi/6) \tan(\pi/6) = 9/2$.

Also solved by A. Alt, G. Apostolopoulos (Greece), R. Bagby, M. Bataille (France), D. Beckwith, M. Can, M. Caragiu, C. Curtis, P. P. Dályay (Hungary), H. Y. Far, O. Faynshteyn (Germany), M. Goldenberg & M. Kaplan, J. G. Heuver (Canada), E. Hysnelaj & E. Bojaxhiu (Australia, Germany), E. Jee & S. Kim (S. Korea), W.-D. Jiang (China), O. Kouba (Syria), K.-W. Lau (China), J. H. Lee (Korea), K. McInturff, N. Minculete (Romania), P. Nüesch (Switzerland), Á. Plaza (Spain), C. R. Pranesachar (India), E. A. Smith, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), S. Wagon, H. Wang & J. Wojydylo, J. B. Zacharias, Barclays Capital Problems Solving Group (U. K.), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

An Inequality for Three Circumradii

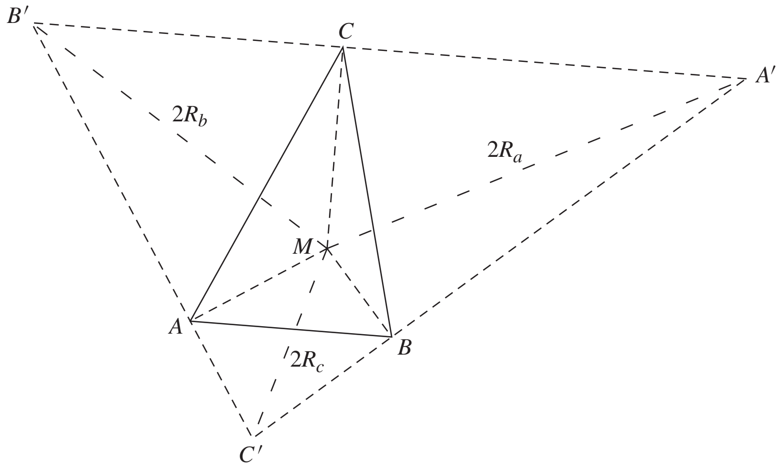
11531 [2010, 834]. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC and let λ_1, λ_2, λ_3 be positive real numbers. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Show that

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1 \lambda_2 \lambda_3 \left(\frac{|MA|}{\lambda_1} + \frac{|MB|}{\lambda_2} + \frac{|MC|}{\lambda_3} \right).$$

(Here, for $V \in \{A, B, C\}$, $|MV|$ denotes the length of the line segment MV.)

Solution by George Apostolopoulis, Massolonghi, Greece. With points named as in the figure, we have that $B'C'$ is perpendicular to MA, $C'A'$ is perpendicular to MB, and $A'B'$ is perpendicular to MC. Using Pappus's Theorem, we have the inequality

$$|B'C'||MA'| \geq |C'A'||MC| + |A'B'||MB|. \tag{1}$$



However, $|MA'|$ is the diameter of the circumcircle for $\triangle BAC$, that is, $|MA'| = 2R_a$. Therefore, using this in (1) gives

$$2R_a \geq \frac{|C'A'|}{|B'C'|}|MC| + \frac{|A'B'|}{|B'C'|}|MB|. \tag{2}$$

Similarly,

$$2R_b \geq \frac{|A'B'|}{|C'A'|}|MA| + \frac{|B'C'|}{|C'A'|}|MC|, \tag{3}$$

$$2R_c \geq \frac{|B'C'|}{|A'B'|}|MB| + \frac{|C'A'|}{|A'B'|}|MA|. \tag{4}$$

Multiplying (2) by λ_1^2, (3) by λ_2^2, and (4) by λ_3^2, then adding, we obtain

$$\begin{aligned} 2\lambda_1^2 R_a + 2\lambda_2^2 R_b + 2\lambda_3^2 R_c \geq & \left(\frac{\lambda_2^2|A'B'|}{|C'A'|} + \frac{\lambda_3^2|C'A'|}{|A'B'|}\right)|MA| \\ & + \left(\frac{\lambda_1^2|A'B'|}{|B'C'|} + \frac{\lambda_3^2|B'C'|}{|A'B'|}\right)|MB| + \left(\frac{\lambda_1^2|C'A'|}{|B'C'|} + \frac{\lambda_2^2|B'C'|}{|C'A'|}\right)|MC|. \end{aligned} \tag{5}$$

For any real x and y, $x^2 + y^2 \geq 2xy$. Applying this to the coefficient of $|MA|$ on the right of (5), we get

$$\frac{\lambda_2^2|A'B'|^2 + \lambda_3^2|C'A'|^2}{|A'B'||C'A'|} \geq 2\lambda_2\lambda_3.$$

Similar inequalities follow for the other two terms in (5), and so (5) implies

$$2\lambda_1^2 R_a + 2\lambda_2^2 R_b + 2\lambda_3^2 R_c \geq 2\lambda_2\lambda_3|MA| + 2\lambda_1\lambda_3|MB| + 2\lambda_1\lambda_2|MC|.$$

This is the required inequality, namely

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1\lambda_2\lambda_3\left(\frac{|MA|}{\lambda_1} + \frac{|MB|}{\lambda_2} + \frac{|MC|}{\lambda_3}\right).$$

Equality holds when $\triangle ABC$ is equilateral, M is the circumcenter, and $\lambda_1 = \lambda_2 = \lambda_3$.

Also solved by M. Bataille (France), P. P. Dályay (Hungary), O. Faynshteyn (Germany), O. Geupel (Germany), O. Kouba (Syria), J. H. Smith, T. Smotzer, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), J. B. Zacharias & K. T. Greeson, and the proposer.

Zero–Nonzero Matrices

11534 [2010, 835]. *Proposed by Christopher Hillar, Mathematical Sciences Research Institute, Berkeley, CA.* Let k and n be positive integers with $k < n$. Characterize the $n \times n$ real matrices M with the property that for all $v \in \mathbb{R}^n$ with at most k nonzero entries, Mv also has at most k nonzero entries.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We show that either M has at most k nonzero rows or M has at most 1 nonzero entry in every column. Such a matrix has the desired property: in the first case Mv has at most k nonzero entries for any v, and in the second case Mv has at most as many nonzero entries as v.

Now suppose that some matrix M with the desired property is not of the form stated. Thus M has at least $k + 1$ rows with nonzero entries and at least one column with at least two nonzero entries. Build a list of columns of M as follows: say a list of columns *represents* a row if and only if at least one of the columns in the set has a nonzero entry in that row. Start with a column w_1 with at least two nonzero entries. If $w_1, \ldots, w_r$ have been chosen, and together they represent fewer than $k + 1$ rows, then choose w_{r+1} to be any column that represents a new row and append it to the list. Stop when $w_1, \ldots, w_r$ represent at least $k + 1$ rows. Now we started with a column representing two rows, and each time we added a new column we got at least one new row. Hence $r \le k$. Thus any linear combination $\sum_{j=1}^{r} a_j w_j$ is of the form Mv, where v has at most k nonzero entries. Fix $k + 1$ rows represented by the w_j. For the ith such row, let V_i be the set of all r-tuples $(a_1, \ldots, a_r)$ such that $\sum_{j=1}^{r} a_j w_j$ has a nonzero entry in that ith row. Since $w_1, \ldots, w_r$ do represent this row, V_i is the nullspace of a nontrivial linear equation on r-tuples and therefore is a codimension-1 subspace of $\mathbb{R}^r$. However, the required property of M says that for any r-tuple $(a_1, \ldots, a_r)$, the linear combination $\sum_{j=1}^{r} a_j v_j$ has at most k nonzero entries; thus $(a_1, \ldots, a_r)$ must lie in one of these $k + 1$ subspaces. But of course $\mathbb{R}^r$ cannot be covered by finitely many codimension-1 subspaces. This contradiction shows that such an M cannot exist.

Editorial comment. Several solvers noted that the same result holds for any field of characteristic 0. John Smith (Needham, MA) noted that the result holds for rectangular matrices.

Also solved by P. Budney, N. Caro (Brazil), P. P. Dályay (Hungary), E. A. Herman, Y. J. Ionin, J. H. Lindsey II, O. P. Lossers (Netherlands), R. E. Prather, J. Simons (U. K.), J. H. Smith, M. Tetiva (Romania), E. I. Verriest, Barclays Capital Problems Solving Group (U. K.), NSA Problems Group, and the proposer.

How Closely Does This Sum Approximate the Integral?

11535 [2010, 835]. *Proposed by Marian Tetiva, Bîrlad, Romania.* Let f be a continuously differentiable function on $[0, 1]$. Let $A = f(1)$ and let $B = \int_0^1 x^{-1/2} f(x)\,dx$. Evaluate

$$\lim_{n\to\infty} n\left(\int_0^1 f(x)\,dx - \sum_{k=1}^{n}\left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right) f\left(\frac{(k-1)^2}{n^2}\right)\right)$$

in terms of A and B.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA. Answer: $A - B/2$.
Proposition. *If h is continuously differentiable on* $[0, 1]$, *then*

$$\lim_{n\to\infty} n\left(\int_0^1 h(x)\,dx - \frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k-1}{n}\right)\right) = \frac{h(1) - h(0)}{2}.$$

Proof. Let H denote an antiderivative of h. By Taylor's theorem,

$$\int_0^1 h(y)\,dy - \frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k-1}{n}\right) = \sum_{k=1}^{n}\left(\int_{(k-1)/n}^{k/n} h(y)\,dy - \frac{1}{n}h\left(\frac{k-1}{n}\right)\right)$$

$$= \sum_{k=1}^{n}\left(H\left(\frac{k}{n}\right) - H\left(\frac{k-1}{n}\right) - \frac{1}{n}H'\left(\frac{k-1}{n}\right)\right) = \frac{1}{2n^2}\sum_{k=1}^{n} H''(y_k),$$

for some list $(y_1, \ldots, y_n)$ with $y_k \in ((k-1)/n, k/n)$ for $1 \le k \le n$. Therefore,

$$\lim_{n\to\infty} n\left(\int_0^1 h(y)\,dy - \sum_{k=1}^{n} h\left(\frac{k-1}{n}\right)\frac{1}{n}\right) = \frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} H''(y_k)$$

$$= \frac{1}{2}\int_0^1 h'(y)\,dy = \frac{h(1)-h(0)}{2}. \qquad \blacksquare$$

For this problem, let $h(y) = 2yf(y^2)$. Now $(h(1)-h(0))/2 = (2f(1))/2 = A$. Using the substitution $y = \sqrt{x}$, we have

$$\int_0^1 f(x)\,dx = \int_0^1 h(y)dy, \;\; B = \int_0^1 x^{-1/2} f(x)\,dx = 2\int_0^1 f(y^2)\,dy.$$

Therefore, by the proposition,

$$\lim_{n\to\infty} n\left(\int_0^1 f(x)\,dx - \sum_{k=1}^{n}\left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right) f\left(\frac{(k-1)^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty} n\left(\int_0^1 h(y)\,dy - \sum_{k=1}^{n}\left(\frac{2(k-1)}{n^2} + \frac{1}{n^2}\right) f\left(\frac{(k-1)^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty}\left[n\left(\int_0^1 h(y)\,dy - \frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k-1}{n}\right)\right) - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{(k-1)^2}{n^2}\right)\right]$$

$$= \frac{h(1)-h(0)}{2} - \int_0^1 f(y^2)\,dy = A - \frac{B}{2}.$$

Also solved by P. Bracken, N. Caro (Brazil), H. Chen, D. Constales (Belgium), P. P. Dályay (Hungary), Y. Dumont (France), P. J. Fitzsimmons, D. Fleischman, J.-P. Grivaux (France), F. Holland (Ireland), S. Kaczkowski, P. Khalili, O. Kouba (Syria), W. C. Lang, J. H. Lindsey II, R. Nandan, M. Omarjee (France), K. Schilling, J. Schlosberg, J. Simons (U. K.), N. C. Singer, Z. Song & L. Yin (China), A. Stenger, R. Stong, T. Tam, J. A. Van Casteren (Belgium), E. I. Verriest, P. Xi (China), J. B. Zacharias & K. T. Greeson, Barclays Capital Problems Solving Group (U. K.), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Use Hayashi's Inequality

11536 [2010, 835]. *Proposed by Mihaly Bencze, Brasov, Romania.* Let K, L, and M denote the respective midpoints of sides AB, BC, and CA in triangle ABC, and let P be a point in the plane of ABC other than K, L, or M. Show that

$$\frac{|AB|}{|PK|} + \frac{|BC|}{|PL|} + \frac{|CA|}{|PM|} \ge \frac{|AB|\cdot|BC|\cdot|CA|}{4|PK|\cdot|PL|\cdot|PM|},$$

where $|UV|$ denotes the length of segment UV.

Solution by D. Marinescu, Colegiul Naţional "Iancu de Hunedoara", Hunedoara, Romania, and M. Monea, Colegiul Naţional"Decebral", Deva, Romania. In 1913, T. Hayashi proved **Hayashi's Inequality**: *For any triangle ABC with opposite sides of lengths a, b, c respectively, and for and an arbitrary point M in its plane,*

$$a|MB| \cdot |MC| + b|MC| \cdot |MA| + c|MA| \cdot |MB| \geq abc.$$

See D. M. Mitronović, J. E. Peĉarić, V. Volenec, *Recent Advances in Geometric Inequalities,* (Kluwer, 1989), p. 297. We now apply Hayashi's Inequality with triangle KLM and point P to get

$$|KL| \cdot |PK| \cdot |PL| + |KM| \cdot |PK| \cdot |PM| + |ML| \cdot |PM| \cdot |PL| \geq |KL| \cdot |ML| \cdot |MK|.$$

Since $|KL| = |AC|/2$, $|KM| = |BC|/2$, and$|ML| = |AC|/2$,

$$|CA| \cdot |PK| \cdot |PL| + |BC| \cdot |PK| \cdot |PM| + |AB| \cdot |PM| \cdot |PL| \geq \frac{1}{4}|CA| \cdot |BC| \cdot |AB|,$$

which is equivalent to the inequality to be proved.

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), D. Beckwith, M. Caragiu, P. P. Dályay (Hungary), O. Geupel (Germany), O. Kouba (Syria), N. Minculete (Romania), B. Mulansky (Germany), C. R. Pranesachar (India), J. Schlosberg, T. Smotzer, M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), Northwestern University Math Problem Solving Group, and the proposer.

A Circumradius Inequality

11541 [2010, 929]. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Let $|MA|$, $|MB|$, and $|MC|$ be the distances from M to A, B, and C. Show that

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \leq \frac{3}{2}.$$

Solution by Oleh Faynshteyn, Leipzig, Germany. Let $\varphi_1 = \angle CAM$, $\varphi_2 = \angle MAB$, $\varphi_3 = \angle ABM$, $\varphi_4 = \angle MBC$, $\varphi_5 = \angle BCM$, and $\varphi_6 = \angle BCM$. Observe that $\sum_{i=1}^{6} \varphi_i = \pi$. From triangles AMC and ABM, it follows that

$$|MA| = 2R_b \sin \varphi_6 = 2R_c \sin \varphi_3,$$

hence

$$\frac{|MA|}{R_b + R_c} = \frac{2}{\csc \varphi_3 + \csc \varphi_6} \leq \frac{1}{2} (\sin \varphi_3 + \sin \varphi_6),$$

where the inequality is a consequence of the arithmetic-harmonic mean inequality. Similarly we get

$$\frac{|MB|}{R_c + R_a} \leq \frac{1}{2} (\sin \varphi_2 + \sin \varphi_5), \qquad \frac{|MC|}{R_a + R_b} \leq \frac{1}{2} (\sin \varphi_1 + \sin \varphi_4).$$

Adding these three inequalities, we obtain

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \leq \frac{1}{2} \sum_{i=1}^{6} \sin \varphi_i.$$

Since the sine function is concave down on $(0, \pi)$, Jensen's inequality gives

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \leq 3 \sin \frac{\pi}{6} = \frac{3}{2}.$$

Equality holds if and only if all the φ_i are $\pi/6$, that is, if and only if ABC is equilateral and M is its center.

Editorial comment. Pál Péter Dályay and Marian Dincă (independently) remarked that the problem and solution generalize as follows. Let M be a point in the interior of the convex n-gon$A_1 \cdots A_n$ (with all indices interpreted mod n). With R_k denoting the circumradius of triangle MA_kA_{k+1}, we have

$$\sum_{k=1}^{n} \frac{|MA_k|}{R_{k-1} + R_k} \leq n \cos \frac{\pi}{n},$$

with equality if and only if the n-gon is regular and M is its center.

Also solved by G. Apostolopoulos (Greece), M. Bataille (France), M. Can, R. Chapman (U. K.), P. P. Dályay (Hungary), M. Dincă (Romania), W. Jiang (China), O. Kouba (Syria), C. R. Pranesachar (India), J. Schlosberg, R. A. Simon (Chile), J. Simons (U. K.), R. Smith, T. Smotzer, R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), J. B. Zacharias & K. T. Greeson, GCHQ Problem Solving Group (U. K.), and the proposer.

Gamma and Beta Inequalities

11542 [2010, 929]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania.* Show that for $x, y, z > 1$, and for positive α, β, γ,

$$(2x^2 + yz)\Gamma(x) + (2y^2 + zx)\Gamma(y) + (2z^2 + xy)\Gamma(z)$$
$$\geq (x + y + z)(x\Gamma(x) + y\Gamma(y) + z\Gamma(z)),$$

and

$$B(x, \alpha)^{x^2+2yz} B(y, \beta)^{y^2+2zx} B(z, \gamma)^{z^2+2xy}$$
$$\geq (B(x, \alpha)B(y, \beta)B(z, \gamma))^{xy+yz+zx}.$$

Here, $B(x, \alpha)$ is Euler's beta function, defined by $B(x, \alpha) = \int_0^1 t^{x-1}(1 - t)^{\alpha-1} dt$.

Solution by M. A. Prasad, India. The first inequality is equivalent to

$$(x - y)(x - z)\Gamma(x) + (y - z)(y - x)\Gamma(x) + (z - x)(z - y)\Gamma(z) \geq 0.$$

It is symmetric in x, y, z, so we may assume $x \geq y \geq z$. The first and third terms are nonnegative, and the middle term is nonpositive. Note also that $\Gamma(x)$ is a convex function for $x > 0$, since $(d^2/dx^2)\Gamma(x) = (\log x)^2 \int_0^\infty e^{-t} t^{x-1} \, dx \geq 0$. Therefore, $\Gamma(y) \leq \max\{\Gamma(x), \Gamma(z)\}$. Since $|(y - z)(y - x)| \leq \min\{(x - y)(x - z), (z - x)(z - y)\}$, one of the nonnegative terms is at least as large as the nonpositive term in absolute value. This completes the proof for the first inequality. The second inequality is incorrect. For a counterexample, consider $x > y > z$, and α, γ very large, and $\beta = 1$. The inequality is equivalent to

$$B(x, \alpha)^{(x-y)(x-z)} B(y, \beta)^{(y-z)(y-x)} B(z, \gamma)^{(z-x)(z-y)} \geq 1.$$

Now as $\alpha, \gamma \to \infty$, we have $B(x, \alpha), B(z, \gamma) \to 0$, so the left side is less than 1.

Also solved by G. Apostopoulos (Greece), R. Bagby, R. Chapman (U. K.), P. P. Dályay (Hungary), R. Stong, J. V. Tejedor (Spain), and GCHQ Problem Solving Group (U. K.)