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Positive solutions for a class of singular Dirichlet problems

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Abstract

We consider a Dirichlet elliptic problem driven by the Laplacian with singular and superlinear nonlinearities. The singular term appears on the left-hand side while the superlinear perturbation is parametric with parameter $\lambda > 0$ and it need not satisfy the AR-condition. Having as our starting point the work of Diaz-Morel-Oswald (1987) [3], we show that there is a critical parameter value λ_* such that for all $\lambda > \lambda_*$ the problem has two positive solutions, while for $\lambda < \lambda_*$ there are no positive solutions. What happens in the critical case $\lambda = \lambda_*$ is an interesting open problem.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N (\mathbb{N} \geqslant 2)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric singular Dirichlet problem

$$-\Delta u(z) + u(z)^{-\gamma} = \lambda f(z, u(z)) \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ u > 0,\ 0 < \gamma < 1. \qquad (P_\lambda)$$

In this problem, λ is a positive parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, the mapping $x \mapsto f(z, x)$ is continuous). We assume that for almost all $z \in \Omega$, $f(z, \cdot)$ exhibits superlinear growth near $+\infty$, but it need not satisfy the usual in such cases Ambrosetti-Robinowitz condition (the AR-condition for short).

The distinguishing feature of our work is that the singular term $u^{-\gamma}$ appears on the left-hand side of the equation. This is in contrast with almost all previous works on singular elliptic equations driven by the Laplacian, where the forcing term (the right-hand side of the equation) is $u \mapsto u^{-\gamma} + \lambda f(z, u)$, so the singular term $u^{-\gamma}$ appears on the right-hand side of the equation. We mention the works of Coclite & Palmieri [2], Sun, Wu & Long [13], and Haitao [7], which also deal with equations that have the competing effects of singular and superlinear terms. A comprehensive bibliography on semilinear singular Dirichlet problems can be found in the book by Ghergu & Rădulescu [5]. The present class of singular equations was first considered by Diaz, Morel & Oswald [3], for the case when the perturbation f is independent of u. They produced a necessary and sufficient condition for the existence of positive solutions in terms of the integral $\int_\Omega f \hat{u}_1 dz$, with $\hat{u}_1$ being the positive L^2-normalized principal eigenfunction of $(-\Delta, H_0^1(\Omega))$. More recently, Papageorgiou & Rădulescu [9] considered problem (P_λ) with $f(z, \cdot)$ being sublinear.

Our aim is to study the precise dependence of the set of positive solutions of problem (P_λ) with respect to the parameter $\lambda > 0$. In this direction, we show that there exists a critical parameter value $\lambda_* > 0$ such that

- for all $\lambda > \lambda_*$, problem (P_λ) has at least two positive smooth solutions;
- for all $\lambda \in (0, \lambda_*)$, problem ($P_\lambda$) has no positive solutions.

It is an open problem what happens in the critical case $\lambda = \lambda_*$. We describe the difficulties one encounters when treating the critical case $\lambda = \lambda_*$ and why we think that $\lambda_* > 0$ is not admissible.

2. Preliminaries and hypotheses

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. We say that $\varphi(\cdot)$ satisfies the "C-condition", if the following property holds

> "Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that
> $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1 + ||u_n||_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
> admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ and it leads to the minimax theory of the critical values of φ (see, for example, Papageorgiou, Rădulescu & Repovš [12]).

The main spaces used in the analysis of problem (P_λ) are the Sobolev space $H_0^1(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. By $||\cdot||$ we denote the norm of $H_0^1(\Omega)$. On account of the Poincaré inequality we have

$$||u|| = ||Du||_2 \text{ for all } u \in H_0^1(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial\Omega} < 0 \right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

We will also use two other ordered Banach spaces, namely $C^1(\overline{\Omega})$ and

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

The order cones are

$$\hat{C}_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}$$

and

$$K_+ = \{u \in C_0(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\},$$

respectively. Both have nonempty interiors given by

$$D_+ = \{u \in \hat{C}_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\},$$
$$\mathring{K}_+ = \{u \in K_+ : c_u \hat{d} \leqslant u \text{ for some } c_u > 0\},$$

with $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$.

Concerning ordered Banach spaces, the following result is helpful (see Papageorgiou, Rădulescu & Repovš [12, Proposition 4.1.22, p. 226]).

Proposition 1. *If X is an ordered Banach space with order (positive) cone K, $\operatorname{int} K \neq \emptyset$, and $e \in \operatorname{int} K$, then for every $u \in X$, we can find $\lambda_u > 0$ such that $\lambda_u e - u \in K$.*

Next, we introduce the main notation which we will use in the sequel. Given $\varphi \in C^1(H_0^1(\Omega))$, we denote by K_φ the critical set of φ, that is,

$$K_\varphi = \{u \in H_0^1(\Omega) : \varphi'(u) = 0\}.$$

For $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Given $u \in H_0^1(\Omega)$, we set $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. We know that

$$u^{\pm} \in H_0^1(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Given $u, y \in H_0^1(\Omega)$ with $u \leqslant y$, we define

$$[u, y] = \{h \in H_0^1(\Omega) : u(z) \leqslant h(z) \leqslant y(z) \text{ for almost all } z \in \Omega\},$$
$$[u) = \{h \in H_0^1(\Omega) : u(z) \leqslant h(z) \text{ for almost all } z \in \Omega\}.$$

Also, by

$$\mathrm{int}_{C_0^1(\overline{\Omega})}[u, y]$$

we denote the interior of $[u, y] \cap C_0^1(\overline{\Omega})$ in the $C_0^1(\overline{\Omega})$-norm topology.

By $\hat{\lambda}_1 > 0$ we denote the principal eigenvalue of $(-\Delta, H_0^1(\Omega))$ and by $\hat{u}_1$ the corresponding positive L^2-normalized (that is, $||\hat{u}_1||_2 = 1$) eigenfunction. Standard regularity theory and the Hopf maximum principle imply that $\hat{u}_1 \in \mathrm{int}\, C_+$.

Finally, by 2^* we denote the critical Sobolev exponent, $2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geqslant 3 \\ +\infty & \text{if } N = 2 \end{cases}$.

Now we will introduce our hypotheses on the perturbation $f(z, x)$.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $f(z, x) \leqslant a(z)(1 + x^{r-1})$ for almost all $z \in \Omega$ and all $x \geqslant 0$, with $a \in L^{\infty}(\Omega)$, $2 < r < 2^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x \to +\infty} \frac{F(z, x)}{x^2} = +\infty$ uniformly for almost all $z \in \Omega$;

(iii) there exists $\tau \in (r - 2, 2^*)$ such that

$$0 < \beta_0 \leqslant \liminf_{x \to \infty} \frac{f(z, x)x - 2F(z, x)}{x^{\tau}} \text{ uniformly for almost all } z \in \Omega;$$

(iv) for every $\rho > 0$ and every $\lambda > 0$, there exists $\hat{\xi}_{\rho}^{\lambda} > 0$ such that for almost all $z \in \Omega$, the function

$$x \mapsto \lambda f(z, x) + \hat{\xi}_{\rho}^{\lambda} x$$

is nondecreasing on $[0, \rho]$ and for every $s > 0$ we have

$$\inf\{f(z, x) : x \geqslant s\} = m_s > 0 \text{ for almost all } z \in \Omega;$$

(v) there exist $q > 2$, $\delta_0 > 0$, $\hat{c} > 0$ such that

$$\hat{c}x^{q-1} \leqslant f(z,x) \text{ for almost all } z \in \Omega \text{ and all } 0 \leqslant x \leqslant \delta_0,$$
$$\lim_{x\to 0^+} \frac{F(z,x)}{x^2} = 0 \text{ uniformly for almost all } z \in \Omega.$$

Remark 1. Since we are looking for positive solutions and all of the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume without any loss of generality that

$$f(z,x) = 0 \text{ for almost all } z \in \Omega \text{ and all } x \leqslant 0. \tag{1}$$

Hypotheses H(f)(ii), (iii) imply that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x} = +\infty \text{ uniformly for almost all } z \in \Omega.$$

So, the perturbation $f(z,\cdot)$ is superlinear. However, we do not express this superlinearity of $f(z,\cdot)$ by using the traditional (for superlinear problems) AR-condition. We recall that the AR-condition (the unilateral version due to (1)) says that there exist $\vartheta > 2$ and $M > 0$ such that

$$0 < \vartheta F(z,x) \leqslant f(z,x)x \text{ for almost all } z \in \Omega \text{ and all } x \geqslant M \tag{2a}$$
$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M). \tag{2b}$$

Integrating (2a) and using (2b), we obtain the following weaker condition

$$c_1 x^{\vartheta} \leqslant F(z,x) \text{ for almost all } z \in \Omega,\ \text{all } x \geqslant M,\ \text{and some } c_1 > 0,$$
$$\Rightarrow c_1 x^{\vartheta-1} \leqslant f(z,x) \text{ for almost all } z \in \Omega \text{ and all } x \geqslant M \text{ (see (2a))}.$$

So, the AR-condition dictates at least $(\vartheta - 1)$-polynomial growth for $f(z,\cdot)$. Here, instead of the AR-condition, we employ hypothesis $H(f)(iii)$ which is less restrictive and incorporates in our framework superlinear nonlinearities with "slower" growth near $+\infty$. Consider the following function (for the sake of simplicity we drop the z-dependence)

$$f(x) = \begin{cases} cx^{q-1} & \text{if } 0 \leqslant x \leqslant 1 \\ x\ln x + cx^{\vartheta-1} & \text{if } 1 < x \end{cases}$$

with $c > 0, q > 2 > \vartheta > 1$ (see (1)). Then $f(\cdot)$ satisfies hypotheses $H(f)$ but it fails to satisfy the AR-condition.

Finally, we mention that for $u \in H_0^1(\Omega)$ we have

$$c_u\hat{d} \leqslant u \text{ for some } c_u > 0 \text{ if and only if } \hat{c}_u\hat{u}_1 \leqslant u \text{ for some } \hat{c}_u > 0. \tag{3}$$

For $\delta > 0$ let $\Omega_\delta = \{z \in \Omega : d(z, \partial\Omega) < \delta\}$ and let $\tilde{C}^1(\Omega_\delta) = \{u \in C^1(\overline{\Omega}_\delta) : u|_{\partial\Omega} = 0\}$ with order cone

$$\tilde{C}^1(\overline{\Omega}_\delta)_+ = \{u \in \tilde{C}^1(\overline{\Omega}_\delta) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}_\delta\},$$

which has nonempty interior given by

$$\operatorname{int} \tilde{C}^1(\overline{\Omega}_\delta)_+ = \left\{ u \in \tilde{C}^1(\overline{\Omega}_\delta)_+ : u(z) > 0 \text{ for all } z \in \Omega_\delta \text{ and } \frac{\partial u}{\partial n}|_{\partial\Omega} < 0 \right\}.$$

According to Lemma 14.16 of Gilbarg & Trudinger [6, p. 355], for $\delta > 0$ small enough we have $\hat{d} \in \operatorname{int} \tilde{C}^1(\overline{\Omega}_\delta)_+$. Also, we have $\hat{d} \in D_+(\overline{\Omega} \backslash \Omega_\delta)$, with the latter being the interior of the order cone of $C^1(\overline{\Omega} \backslash \Omega_\delta)$. So, using Proposition 1 we can find $0 < \hat{c}_1 < \hat{c}_2$ such that $\hat{c}_1 \hat{d} \leqslant \hat{u}_1 \leqslant \hat{c}_2 \hat{d}$ (recall that $\hat{u}_1 \in \operatorname{int} C_+$). This implies (3).

3. Positive solutions

Let $\eta > 0$. We start by considering the following auxiliary purely singular Dirichlet problem

$$-\Delta u(z) + u(z)^{-\gamma} = \eta \hat{u}_1(z) \text{ in } \Omega, \; u|_{\partial\Omega} = 0. \tag{$(Au)_\eta$}$$

By Theorem 1 of Diaz, Morel & Oswald [3] we know that for $\eta > 0$ big Problem $(Au)_\eta$ has a solution $v_\eta \in H_0^1(\Omega) \cap C_0(\overline{\Omega})$ and $v_\eta^{-\gamma} \in L^1(\Omega)$, $c_\eta \hat{u}_1 \leqslant v_\eta$ for some $c_\eta > 0$.

Also, we consider the following Dirichlet problem

$$-\Delta u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \; u|_{\partial\Omega}, \; u > 0, \; \lambda > 0. \tag{Q_λ}$$

Proposition 2. *If hypotheses $H(f)$ hold and $\lambda > 0$, then problem (Q_λ) has a solution $\hat{u}_\lambda \in \operatorname{int} C_+$.*

Proof. Let $\Psi_\lambda : H_0^1(\Omega) \to \mathbb{R}$ be the C^1- functional defined by

$$\Psi_\lambda(u) = \frac{1}{2}||Du||_2^2 - \int_\Omega \lambda F(z, u) dz \text{ for all } u \in H_0^1(\Omega).$$

Hypotheses $H(f)(ii), (iii)$ imply that

$$\Psi_\lambda(\cdot) \text{ satisfies the C-condition} \tag{4}$$

(see Papageorgiou & Rădulescu [11, Proposition 9]).

Combining hypotheses $H(f)(i), (v)$, given $\epsilon > 0$, we can find $c_\epsilon > 0$ such that

$$F(z, x) \leqslant \frac{\epsilon}{2} x^2 + c_\epsilon x^r \text{ for almost all } z \in \Omega, \text{ all } x \geqslant 0.$$

Then we have

$$\begin{aligned} \Psi_\lambda(u) \quad & \geqslant \frac{1}{2}||Du||_2^2 - \frac{\lambda\epsilon}{2}||u||_2^2 - \lambda \hat{c}_\epsilon ||u||^r \text{ for some } \hat{c}_\epsilon > 0 \\ & \geqslant c_2 ||u||^2 - \lambda \hat{c}_\epsilon ||u||^r \text{ for some } c_2 = c_2(\lambda) > 0 \text{ (choose } \epsilon > 0 \text{ small enough)}, \\ & \Rightarrow u = 0 \text{ is a local minimizer of } \Psi_\lambda(\cdot) \text{ (recall that } r > 2). \end{aligned} \tag{5}$$

We can easily see that if $u \in K_{\Psi_\lambda}$, then $u \geqslant 0$. Hence we assume that K_{Ψ_λ} is finite. On account of (5) and Theorem 5.7.6 of Papageorgiou, Rădulescu & Repovš [12, p. 367], we can find $\rho \in (0, 1)$ so small that

$$0 = \Psi_\lambda(0) < \inf\{\Psi_\lambda(u) : ||u|| = \rho\} = m_\lambda. \tag{6}$$

Hypothesis $H(f)(ii)$ implies that

$$\Psi_\lambda(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty. \tag{7}$$

Then (4), (6), (7) permit the use of the mountain pass theorem. So, we can find $\hat{u}_\lambda \in H_0^1(\Omega)$ such that

$$\begin{aligned} &\hat{u}_\lambda \in K_{\Psi_\lambda} \text{ and } m_\lambda \leqslant \Psi_\lambda(\hat{u}_\lambda), \\ &\Rightarrow \hat{u}_\lambda \geqslant 0,\ \hat{u}_\lambda \neq 0 \text{ (see (6))}. \end{aligned}$$

We have

$$\begin{aligned} &\int_\Omega (D\hat{u}_\lambda, Dh)_{\mathbb{R}^N} dz = \lambda \int_\Omega f(z, u_\lambda) h dz \text{ for all } h \in H_0^1(\Omega), \\ &\Rightarrow -\Delta u_\lambda(z) = \lambda f(z, u_\lambda(z)) \geqslant 0 \text{ for almost all } z \in \Omega. \end{aligned}$$

Then the semilinear regularity theory (see Gilbarg & Trudinger [6]) and the Hopf maximum principle (see Gasinski & Papageorgiou [4]), imply that $\hat{u}_\lambda \in \operatorname{int} C_+$. □

Hypotheses $H(f)$ imply that we can find $c_2 > 0$ such that

$$f(z, x) \geqslant c_2 min\{x, x^{q-1}\} \text{ for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{8}$$

We have $\hat{u}_\lambda \in \operatorname{int} C_+$ and $\hat{u}_\lambda^{q-1} \in \operatorname{int} K_+$. So, we can find $c_3 > 0$ such that

$$\begin{aligned} &\eta\hat{u}_1 \leqslant c_3\hat{u}_\lambda \text{ and } \eta\hat{u}_1 \leqslant c_3\hat{u}_\lambda^{q-1}, \\ \Rightarrow\quad &\eta\hat{u}_1 \leqslant c_3 \min\{\hat{u}_\lambda, \hat{u}_\lambda^{q-1}\}. \end{aligned} \tag{9}$$

From (8) and (9) we see that we can find $\lambda_0 \geqslant 0$ big such that for $\lambda \geqslant \lambda_0$

$$\begin{aligned} &\lambda f(z, \hat{u}_\lambda(z)) \geqslant \lambda_0 c_2 \min\{\hat{u}_\lambda(z), \hat{u}_\lambda(z)^{q-1}\} \\ &\geqslant \eta\hat{u}_1(z) \text{ for almost all } z \in \Omega. \end{aligned} \tag{10}$$

Recall that $c_\eta\hat{u}_1 \leqslant v_\eta$ for some $c_\eta > 0$. Hence by (3), $\hat{c}_\eta\hat{d} \leqslant v_\eta$ for some $\hat{c}_\eta > 0$. Therefore

$$v_\eta^{-\gamma} \leqslant \frac{1}{\hat{c}_\eta^\gamma}\hat{d}^{-\gamma}.$$

For every $h \in H_0^1(\Omega)$, we have

$$\begin{aligned}\int_\Omega \frac{|h|}{v_\eta^\gamma} dz &\leqslant \frac{1}{\hat{c}_\eta^\gamma} \int_\Omega \frac{|h|}{\hat{d}^\gamma} dz \\ &= \frac{1}{\hat{c}_\eta^\gamma} \int_\Omega \frac{|h|}{\hat{d}} \hat{d}^{1-\gamma} dz \leqslant c_4 \int_\Omega \frac{|h|}{\hat{d}} dz \text{ for some } c_4 > 0.\end{aligned}$$

Invoking Hardy's inequality (see Brezis [1, p. 313]), we infer that $\frac{|h|}{\hat{d}} \in L^2(\Omega)$. Therefore

$$\begin{aligned}& c_4 \int_\Omega \frac{|h|}{\hat{d}} dz \leqslant c_5 \left(\int_\Omega \frac{|h|^2}{\hat{d}^2} dz \right)^{\frac{1}{2}} < \infty \text{ for some } c_5 > 0, \\ \Rightarrow \quad & |\int_\Omega v_\eta^{-\gamma} h dz| < \infty \text{ for all } h \in H_0^1(\Omega).\end{aligned}$$

Therefore we have

$$-\Delta v_\eta(z) + v_\eta(z)^{-\gamma} = \eta \hat{u}_1(z) \text{ for almost all } z \in \Omega. \tag{11}$$

If $\lambda \geqslant \lambda_0$, from (10) and (11) we have

$$\begin{aligned}-\Delta \hat{u}_\lambda(z) = \lambda f(z, \hat{u}_\lambda(z)) \geqslant \eta \hat{u}_1(z) = -\Delta v_\eta(z) + v_\eta(z)^{-\gamma} \geqslant -\Delta v_\eta(z) \\ \text{for almost all } z \in \Omega.\end{aligned} \tag{12}$$

Since $v_\eta|_{\partial\Omega} = \hat{u}_\lambda|_{\partial\Omega} = 0$, from (12) and the weak comparison principle (see Tolksdorf [14, Lemma 3.1]), we have

$$v_\eta \leqslant \hat{u}_\lambda \quad (\lambda \geqslant \lambda_0). \tag{13}$$

Now we introduce the following two sets

$$\begin{aligned}&\mathcal{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\}, \\ &S_\lambda = \text{ the set of positive solutions of } (P_\lambda).\end{aligned}$$

Here by a solution of (P_λ), following [9], we understand a function $u \in H_0^1(\Omega)$ such that

(a) $u \in L^\infty(\Omega)$, $u(z) > 0$ for almost all $z \in \Omega$, and $u^{-\gamma} \in L^1(\Omega)$;
(b) there exists $c_u > 0$ such that $c_u \hat{d} \leqslant u$;
(c) $\int_\Omega (Du, Dh)_{\mathbb{R}^N} dz + \int_\Omega u^{-\gamma} h dz = \lambda \int_\Omega f(z, u) h dz$ for all $h \in H_0^1(\Omega)$.

From (3) we know that (b) is equivalent to saying that $\hat{c}_u \hat{u}_1 \leqslant u$ for some $\hat{c}_u > 0$. Also the previous discussion reveals that (c) makes sense. Regularity theory will provide additional structure for the solutions of (P_λ).

Proposition 3. *If hypotheses $H(f)$ hold, then $\mathcal{L} \neq \emptyset$ and $S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. Let $\lambda \geqslant \lambda_0$. Using (13) we can introduce the Carathéodory function $g_\lambda(z, x)$ defined by

$$g_\lambda(z,x) = \begin{cases} \lambda f(z, v_\eta(z)) - v_\eta(z)^{-\gamma} & \text{if } x < v_\eta(z) \\ \lambda f(z,x) - x^{-\gamma} & \text{if } v_\eta(z) \leqslant x \leqslant \hat{u}_\lambda(z) \\ \lambda f(z, \hat{u}_\lambda(z)) - \hat{u}_\lambda(z)^{-\gamma} & \text{if } \hat{u}_\lambda(z) < x. \end{cases} \tag{14}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the functional $\varphi_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{2}||Du||_2^2 - \int_\Omega G_\lambda(z,u)dz \text{ for all } u \in H_0^1(\Omega).$$

From Papageorgiou & Rădulescu [9] (see Claim 1 in the proof of Proposition 6), we have that $\varphi_\lambda \in C^1(H_0^1(\Omega))$. It is clear from (14) that $\varphi_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore we can find $u_\lambda \in H_0^1(\Omega)$ such that

$$\begin{aligned} & \varphi_\lambda(u_\lambda) = \inf\left\{\varphi_\lambda(u) : u \in H_0^1(\Omega)\right\}, \\ \Rightarrow\ & \varphi_\lambda'(u_\lambda) = 0, \\ \Rightarrow\ & \int_\Omega (Du_\lambda, Dh)_{\mathbb{R}^N} dz = \int_\Omega g_\lambda(z,u_\lambda) h dz \text{ for all } h \in H_0^1(\Omega). \end{aligned} \tag{15}$$

In (15) first we choose $h = (u_\lambda - \hat{u}_\lambda)^+ \in H_0^1(\Omega)$. We have

$$\begin{aligned} \int_\Omega (Du_\lambda, D(u_\lambda - \hat{u}_\lambda)^+)_{\mathbb{R}^N} dz &= \int_\Omega [\lambda f(z, \hat{u}_\lambda) - \hat{u}_\lambda^{-\gamma}](u_\lambda - \hat{u}_\lambda)^+ dz \text{ (see (14))} \\ &\leqslant \int_\Omega \lambda f(z, \hat{u}_\lambda)(u_\lambda - \hat{u}_\lambda)^+ dz \\ &= \int_\Omega (D\hat{u}_\lambda, D(u_\lambda - \hat{u}_\lambda)^+)_{\mathbb{R}^N} dz \text{ (see Proposition 2)}, \end{aligned}$$
$$\Rightarrow ||D(u_\lambda - \hat{u}_\lambda)^+||_2^2 \leqslant 0,$$
$$\Rightarrow u_\lambda \leqslant \hat{u}_\lambda.$$

Next, in (15) we choose $h = (v_\eta - u_\lambda)^+ \in H_0^1(\Omega)$. Then we have

$$\int_\Omega (Du_\lambda, D(v_\eta - u_\lambda)^+)_{\mathbb{R}^N} dz = \int_\Omega [\lambda f(z, v_\eta) - v_\eta^{-\gamma}](v_\eta - u_\lambda)^+ dz. \tag{16}$$

As we proved (10), using (8), (9), we see that by taking $\lambda \geqslant \lambda_0$ even bigger if necessary, we can have

$$\lambda f(z, v_\eta(z)) \geqslant \eta \hat{u}_1(z) \text{ for almost all } z \in \Omega. \tag{17}$$

Hence from (16) and (17) we have

$$\begin{aligned}
\int_\Omega (Du_\lambda, D(v_\eta - u_\lambda)^+)_{\mathbb{R}^N} dz &\geqslant \int_\Omega [\eta \hat{u}_1 - v_\eta^{-\gamma}](v_\eta - u_\lambda)^+ dz \\
&= \int_\Omega (Dv_\eta, D(v_\eta - u_\lambda)^+)_{\mathbb{R}^N} dz \\
\Rightarrow ||D(v_\eta - u_\lambda)^+||_2^2 \leqslant 0,& \\
\Rightarrow v_\eta \leqslant u_\lambda.&
\end{aligned}$$

So, we have proved that

$$u_\lambda \in [v_\eta, \hat{u}_\lambda]. \tag{18}$$

It follows from (14), (15) and (18) that

$$\int_\Omega (Du_\lambda, Dh)_{\mathbb{R}^N} dz + \int_\Omega u_\lambda^{-\gamma} h dz = \int_\Omega f(z, u_\lambda) h dz \text{ for all } h \in H_0^1(\Omega).$$

Recall that

$$\begin{aligned}
&c_\eta \hat{d} \leqslant v_\eta \leqslant u_\lambda \\
\text{and} \quad &u_\lambda^{-\gamma} \leqslant v_\eta^{-\gamma} \in L^1(\Omega) \text{ (see (18))}.
\end{aligned}$$

Therefore u_λ is a solution of (P_λ). We have proved that for $\lambda \geqslant \lambda_0$ big enough, we have $\lambda \in \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$.

Now let $u \in S_\lambda$. Then by definition we have

$$\begin{aligned}
&\hat{c}_u \hat{u}_1 \leqslant u \text{ for some } \hat{c}_u > 0, \\
\Rightarrow \quad &u \in \operatorname{int} L^\infty(\Omega)_+.
\end{aligned} \tag{19}$$

Let $s > N$. Since $\hat{u}_1^{1/s} \in K_+$, we can find $c_6 > 0$ such that

$$\begin{aligned}
&\hat{u}_1^{1/s} \leqslant c_6 u \text{ (see Proposition 1)}, \\
\Rightarrow \quad &u^{-\gamma} \leqslant c_7 \hat{u}_1^{-\frac{\gamma}{s}} \text{ for some } c_7 > 0.
\end{aligned}$$

However, by Lemma in Lazer & McKenna [8], we have that $\hat{u}_1^{-\frac{\gamma}{s}} \in L^s(\Omega)$ (recall that $0 < \gamma < 1$). So, it follows that $u^{-\gamma} \in L^s(\Omega)$. Then Theorem 9.15 of Gilbarg & Trudinger [6, p. 241] implies that $u \in W^{2,s}(\Omega)$. Since $s > N$, from the Sobolev embedding theorem, we have $u \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{s}$. We conclude that $u \in \operatorname{int} C_+$ (see (19)) and so $S_\lambda \subseteq \operatorname{int} C_+$. □

Next, we prove a structural property for the set $\mathcal{L}$ and a kind of monotonicity property for the set S_λ with respect to $\lambda \in \mathcal{L}$.

Proposition 4. *If hypotheses $H(f)$ hold, $\lambda \in \mathcal{L}$, $\mu > \lambda$, and $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$.*

Proof. Let $\rho = ||u_\lambda||_\infty$. Hypotheses $H(f)$ imply that we can find $c_\rho > 0$ such that

$$0 \leqslant f(z, x) \leqslant c_\rho x \text{ for almost all } z \in \Omega \text{ and all } 0 \leqslant x \leqslant \rho. \tag{20}$$

Also from (8) we know that

$$f(z, x) \geqslant c_2 \min\{x, x^{q-1}\} \text{ for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{21}$$

Recall that for $\vartheta \geqslant \lambda_0$ we have $\hat{u}_\theta \geqslant v_\eta$ (see (13)) and $v_\eta \in \operatorname{int} K_+$. So, for $\vartheta \geqslant \lambda_0$ big enough we have

$$\vartheta c_2 \min\{\hat{u}_\theta, \hat{u}_\theta^{q-1}\} \geqslant \lambda c_\rho u_\lambda. \tag{22}$$

It follows that

$$\begin{aligned} -\Delta \hat{u}_\theta = \vartheta f(z, \hat{u}_\theta) \quad &\geqslant \vartheta c_2 \min\{\hat{u}_\theta, \hat{u}_\theta^{q-1}\} \text{ (see (21))} \\ &\geqslant \lambda c_\rho u_\lambda \text{ (see (22))} \\ &\geqslant \lambda f(z, u_\lambda) \text{ (see (20))} \\ &= -\Delta u_\lambda + u_\lambda^{-\gamma} \text{ (since } u_\lambda \in S_\lambda) \\ &\geqslant -\Delta u_\lambda \text{ for almost all } z \in \Omega, \end{aligned}$$
$$\Rightarrow \hat{u}_\theta \geqslant u_\lambda \text{ (by the weak comparison principle, see Tolksdorf [14]).}$$

Therefore we can introduce the Carathéodory function $k_\mu(z, x)$ defined by

$$k_\mu(z, x) = \begin{cases} \mu f(z, u_\lambda(z)) - u_\lambda(z)^{-\gamma} & \text{if } x < u_\lambda(z) \\ \mu f(z, x) - x^{-\gamma} & \text{if } u_\lambda(z) \leqslant x \leqslant \hat{u}_\theta(z) \\ \mu f(z, \hat{u}_\theta(z)) - \hat{u}_\theta(z)^{-\gamma} & \text{if } \hat{u}_\theta(z) < x. \end{cases} \tag{23}$$

We set $K_\mu(z, x) = \int_0^x k_\mu(z, s) ds$ and consider the functional $\sigma_\mu : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\mu(u) = \frac{1}{2}||Du||_2^2 - \int_\Omega k_\mu(z, u) dz \text{ for all } u \in H_0^1(\Omega).$$

Again we have $\sigma_\mu \in C^1(H_0^1(\Omega))$ (see Papageorgiou & Rădulescu [9]). From (23) it is clear that $\sigma_\mu(\cdot)$ is coercive. Also, by the Sobolev embedding theorem we see that $\sigma_\mu(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\mu \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&\sigma_\mu(u_\mu) = \inf\left\{\sigma_\mu(u) : u \in H_0^1(\Omega)\right\},\\
\Rightarrow\ &\sigma'_\mu(u_\mu) = 0,\\
\Rightarrow\ &\int_\Omega (Du_\lambda, Dh)_{\mathbb{R}^N} dz = \int_\Omega k_\mu(z, u_\lambda) h dz \text{ for all } h \in H_0^1(\Omega).
\end{aligned}$$

Choosing first $h = (u_\mu - \hat{u}_\theta)^+ \in H_0^1(\Omega)$ and then $h = (u_\lambda - u_\mu)^+ \in H_0^1(\Omega)$ as in the proof of Proposition 3, we can show that

$$\begin{aligned}
&u_\mu \in [u_\lambda, \hat{u}_\theta],\\
\Rightarrow\ &u_\mu \in S_\mu \subseteq \operatorname{int} C_+ \text{ (see (23))}.
\end{aligned} \tag{24}$$

Let $\rho = ||\hat{u}_\theta||_\infty$ and let $\hat{\xi}_0 = \max\{\hat{\xi}_\rho^\lambda, \hat{\xi}_\rho^\mu\}$ (see hypothesis $H(f)(iv)$). We have

$$\begin{aligned}
-\Delta u_\lambda + \hat{\xi}_0 u_\lambda &= \lambda f(z, u_\lambda) + \hat{\xi}_0 u_\lambda - u_\lambda^{-\gamma}\\
&\leqslant \mu f(z, u_\mu) + \hat{\xi}_0 u_\mu - u_\mu^{-\gamma}\\
&\quad \text{(see hypothesis } H(f)(iv) \text{ and (24))}\\
&= -\Delta u_\mu + \hat{\xi}_0 u_\mu \text{ (since } u_\mu \in S_\mu),
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \Delta(u_\mu - u_\lambda) \leqslant \hat{\xi}_0 (u_\mu - u_\lambda),\\
&\Rightarrow u_\mu - u_\lambda \in \operatorname{int} C_+ \text{ (by Hopf's maximum principle)}.
\end{aligned}$$

The proof is now complete. □

This proposition implies that $\mathcal{L}$ is a half-line. More precisely, let $\lambda_* = inf \mathcal{L}$. We have

$$(\lambda_*, +\infty) \subseteq \mathcal{L} \subseteq [\lambda_*, +\infty). \tag{25}$$

Proposition 5. *If hypotheses $H(f)$ hold, then $\lambda_* > 0$.*

Proof. Arguing by contradiction, suppose that $\lambda_* = 0$. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L}$ such that $\lambda_n \downarrow 0$ and let $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ for all $n \in \mathbb{N}$. We know that

$$\begin{aligned}
&0 \leqslant u_n \leqslant \hat{u}_\theta \text{ for } \vartheta \geqslant \lambda_0 \text{ big enough, for all } n \in \mathbb{N}\\
&\text{(see the proof of Proposition 4)},
\end{aligned} \tag{26}$$

$$-\Delta u_n + u_n^{-\gamma} = \lambda_n f(z, u_n) \text{ for almost all } z \in \Omega \text{ and all } n \in \mathbb{N}. \tag{27}$$

Let $\eta > 0$. With $\rho = ||\hat{u}_\theta||_\infty$ (see (26)), we have

$$\begin{aligned} -\Delta u_n + u_n^{-\gamma} &= \lambda_n f(z, u_n) \\ &\leqslant \lambda_n c_\rho u_n \text{ (see (20))} \\ &\leqslant \lambda_n c_\rho \hat{u}_\theta \text{ (see (26))} \\ &\leqslant \eta \hat{u}_1 \text{ for all } n \geqslant n_0 \text{ (recall that } \hat{u}_1 \in \operatorname{int} C_+). \end{aligned} \tag{28}$$

By (28) and Theorem 1(i) of Diaz, Morel & Oswald [3] it follows that Problem $(Au)_\eta$ has a positive solution. Since $\eta > 0$ is arbitrary, we contradict Theorem 1(ii) of Diaz, Morel & Oswald [3]. This proves that $\lambda_* > 0$. □

Proposition 6. *If hypotheses $H(f)$ hold and $\lambda_* < \lambda$, then problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \neq \hat{u}$.*

Proof. Let $\lambda_* < \sigma < \lambda < \mu$. On account of Proposition 4, we can find $u_\sigma \in S_\sigma \subseteq \operatorname{int} C_+$, $u_0 \in S_\lambda \subseteq \operatorname{int} C_+$ and $u_\mu \in S_\lambda \subseteq \operatorname{int} C_+$ such that

$$\begin{aligned} & u_0 - u_\sigma \in \operatorname{int} C_+ \text{ and } u_\mu - u_0 \in \operatorname{int} C_+, \\ \Rightarrow \quad & u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[u_\sigma, u_\mu]. \end{aligned} \tag{29}$$

We introduce the Carathéodory functions $e_\lambda(z, x)$ and $\hat{e}_\lambda(z, x)$ defined by

$$e_\lambda(z, x) = \begin{cases} \lambda f(z, u_\sigma(z)) - u_\sigma(z)^{-\gamma} & \text{if } x \leqslant u_\sigma(z) \\ \lambda f(z, x) - x^{-\gamma} & \text{if } u_\sigma(z) < x \end{cases} \tag{30}$$

$$\text{and } \hat{e}_\lambda(z, x) = \begin{cases} e_\lambda(z, x) & \text{if } x \leqslant u_\mu(z) \\ e_\lambda(z, u_\mu(z)) & \text{if } u_\mu(z) < x. \end{cases} \tag{31}$$

We set $E_\lambda(z, x) = \int_0^x e_\lambda(z, s) ds$ and $\hat{E}_\lambda(z, x) = \int_0^x \hat{e}_\lambda(z, s) ds$ and consider the C^1-functionals $\beta_\lambda, \hat{\beta}_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\beta_\lambda(u) = \frac{1}{2}||Du||_2^2 - \int_\Omega E_\lambda(z, u) dz,$$

$$\hat{\beta}_\lambda(u) = \frac{1}{2}||Du||_2^2 - \int_\Omega \hat{E}_\lambda(z, u) dz \text{ for all } u \in H_0^1(\Omega).$$

Using (30), (31), as before (see the proof of Proposition 3), we can check that

$$K_{\beta_\lambda} \subseteq [u_\sigma) \cap \operatorname{int} C_+ \text{ and } K_{\hat{\beta}_\lambda} \subseteq [u_\sigma, u_\mu] \cap \operatorname{int} C_+. \tag{32}$$

Using (32), (30) and (29), we see that we may assume that

$$K_{\beta_\lambda} \text{ is finite and } K_{\beta_\lambda} \cap [u_\sigma, u_\mu] = \{u_0\}. \tag{33}$$

Otherwise, we already have additional positive solutions and so we are done.

Evidently $\hat{\beta}_\lambda(\cdot)$ is coercive (see (30)). Also, it is sequentially weakly lower semicontinuous. Thus we can find $\hat{u}_0 \in H_0^1(\Omega)$ such that

$$\begin{aligned} &\hat{\beta}_\lambda(\hat{u}_0) = \inf\left\{\hat{\beta}_\lambda(u) : u \in H_0^1(\Omega)\right\}, \\ \Rightarrow\quad &\hat{u}_0 \in K_{\hat{\beta}_\lambda} \subseteq [u_\sigma, u_\mu] \cap \operatorname{int} C_+ \text{ (see (32))}. \end{aligned} \tag{34}$$

From (30) and (31) we see that (see [10])

$$\begin{aligned} &\beta'_\lambda|_{[u_\sigma,u_\mu]} = \hat{\beta}'_\lambda|_{[u_\sigma,u_\mu]}, \\ \Rightarrow\quad &\hat{u}_0 \in K_{\beta_\lambda} \cap [u_\sigma, u_\mu] \text{ (see (34))} \\ \Rightarrow\quad &\hat{u}_0 = u_0 \text{ (see (33))}, \\ \Rightarrow\quad &u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \beta_\lambda(\cdot), \\ \Rightarrow\quad &u_0 \text{ is a local } H_0^1(\Omega)\text{-minimizer of } \beta_\lambda(\cdot) \text{ (see [10])}. \end{aligned}$$

Then from (33) and Theorem 5.7.6 of Papageorgiou, Rădulescu & Repovš [12, p. 367], we know that we can find $\rho \in (0, 1)$ so small that

$$\beta_\lambda(u_0) < \inf\{\beta_\lambda(u) : ||u - u_0|| = \rho\} = m_\lambda. \tag{35}$$

Hypothesis $H(f)(ii)$ implies that

$$\beta_\lambda(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty. \tag{36}$$

Finally, recall that hypothesis $H(f)(iii)$ implies that

$$\beta_\lambda(\cdot) \text{ satisfies the C-condition} \tag{37}$$

(see Papageorgiou & Rădulescu [11]).

Then (35), (36), (37) permit the use of the mountain pass theorem. So, we can find $\hat{u} \in H_0^1(\Omega)$ such that

$$\begin{aligned} &\hat{u} \in K_{\beta_\lambda} \text{ and } m_\lambda \leqslant \beta_\lambda(\hat{u}), \\ \Rightarrow\quad &\hat{u} \in S_\lambda \subseteq \operatorname{int} C_+, \hat{u} \neq u_0 \text{ (see (32), (31) and (35))}. \end{aligned}$$

The proof is now complete. □

Summarizing, we can state the following theorem for the set of positive solutions of problem (P_λ).

Theorem 7. *If hypotheses $H(f)$ hold, then there exists $\lambda_* > 0$ such that*

(a) *for all* $\lambda > \lambda_*$ *problem* (P_λ) *has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, u_0 \neq \hat{u};$$

(b) *for all* $\lambda \in (0, \lambda_*)$ *problem* (P_λ) *has no positive solutions.*

Remark 2. From the above Theorem is missing what happens at the critical case $\lambda = \lambda_*$. We were unable to resolve this case.

If $\lambda_n \downarrow \lambda_*$, then we can show that there exist $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ $(n \in \mathbb{N})$ such that

$$u_n \xrightarrow{w} u_* \text{ in } H_0^1(\Omega), u_* \neq 0.$$

As before (see the proof of Proposition 3), we have

$$u_n^{-\gamma} \in L^s(\Omega) \ (s > N) \text{ and } u_n^{-\gamma} \to u_*^{-\gamma} \text{ for almost all } z \in \Omega.$$

However, we can not show that $\{u_n^{-\gamma}\}_{n\geqslant 1} \subseteq L^s(\Omega)$ is bounded and therefore have that

$$\int_\Omega u_n^{-\gamma} h dz \to \int_\Omega u_*^{-\gamma} h dz \text{ for all } h \in H_0^1(\Omega)$$

(Vitali's theorem, see Gasinski & Papageorgiou [4, p. 901]).

In addition, we can not show that there exists $c_* > 0$ such that

$$u_* \geqslant c_* \hat{d}.$$

It seems that $\lambda_* > 0$ is not admissible (that is, $\lambda_* \notin \mathcal{L}$, hence $\mathcal{L} = (\lambda_*, +\infty)$, see (25)), but this needs a proof.

Another open problem is the possibility of extending this work to equations driven by the p-Laplacian. This extension requires a corresponding generalization of the work of Diaz, Morel & Oswald [3] to the case of the p-Laplacian. However, the tools of [3] are particular for the Laplacian. So, it is not clear how this generalization can be achieved. Hence new techniques are needed.

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