Z. Angew. Math. Phys. (2017) 68:80

DOI 10.1007/s00033-017-0827-3

Zeitschrift für angewandte Mathematik und Physik ZAMP

On a $p(x)$-biharmonic problem with singular weights

Khaled Kefi and Vicenţiu D. Rădulescu

Abstract. In this work, sufficient conditions are given to prove the existence of at least one nontrivial weak solution for a $p(x)$-biharmonic problem involving Navier boundary conditions and singular weights.

Mathematics Subject Classification. 35J20, 35J60, 35G30, 35J35, 35P30, 46E35.

Keywords. Navier condition, $p(x)$-Biharmonic operator, Variational methods, Existence result.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $a \in C(\overline{\Omega})$ such that $\inf_{x\in\Omega} a(x) > 0$. In this paper, we investigate the existence of nontrivial solutions to the following problem:

$$(\mathbf{P}_\lambda) \begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda(V_1(x)|u|^{q(x)-2}u - V_2(x)|u|^{\alpha(x)-2}u) & \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $p \in C(\overline{\Omega})$ with $1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < \dfrac{N}{2}$, V_1 and V_2 are functions in some generalized Sobolev spaces, and $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator of fourth order.

If p is a constant function, problem $(\mathbf{P}_\lambda)$ has a solid theoretical significance and a sharp physical background. For instance, this problem describes the surface tension of the height of a thin liquid film on a solid surface in lubrication approximation (see [18]). If $p \equiv 2$, problem $(\mathbf{P}_\lambda)$ becomes the generalized Cahn–Hilliard equation. Problem $(\mathbf{P}_\lambda)$ is not a trivial generalization of related problems in the constant case. The main difficulties in the study of this problem are the lack of the maximum principle and the complicated nonlinearities. In particular, the presence of several variable exponents produces difficulties in establishing *a priori* estimates.

In the sequel, X will denote the function space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

A weak solution of $(\mathbf{P}_\lambda)$ is any $u \in X \setminus \{0\}$ such that $\Delta u = 0$ on $\partial\Omega$ and

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v + a(x)|u|^{p(x)-2}uv\mathrm{d}x = \lambda \int_\Omega (V_1(x)|u|^{q(x)-2} - V_2(x)|u|^{\alpha(x)-2})uv\mathrm{d}x,$$

for all $v \in X$.

The energy functional corresponding to problem $(\mathbf{P}_\lambda)$ is defined on X as

$$\Psi_\lambda(u) = J(u) - \lambda\Phi_1(u) + \lambda\Phi_2(u),$$

Published online: 29 June 2017

where

$$J(u)=\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)}+a(x)|u|^{p(x)})\mathrm{d}x,$$

$$\Phi_1(u)=\int_\Omega \frac{V_1(x)}{q(x)}|u|^{q(x)}\mathrm{d}x \quad \text{and} \quad \Phi_2(u)=\int_\Omega \frac{V_2(x)}{\alpha(x)}|u|^{\alpha(x)}\mathrm{d}x.$$

The study of this kind of nonlinear problems described by non-homogeneous differential operators has been an interesting topic in relationship with several relevant applications, such as electrorheological fluids (see [27]). The first major discovery in electrorheological fluids was due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance, lithium polymetacrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. We refer to [26] for more details.

We also refer to elastic mechanics (see [29]), stationary thermo-rheological viscous flows of non-Newtonian fluids, image processing (see [6]) as well as the mathematical description of the filtration process of a barotropic gas through a porous medium (see [1]). Problems of this type are characterized by the fact that the associated energy density changes its ellipticity and growth properties according to the point. These non-homogeneous problems with one or more variable exponents have been intensively studied starting with the pioneering contributions of Halsey [15] and Zhikov [29–31] in relationship with the analysis of the behavior of strongly anisotropic materials in the context of the homogenization and nonlinear elasticity.

We also point out that fourth-order elliptic equations arise in many domains like microelectromechanical systems, surface diffusion on solids, thin film theory, flow in Hele-Shaw cells and phase field models of multiphasic systems (see [13]). For recent contributions concerning this type of equations, we refer to [3,25,26].

Problem $(\mathbf{P}_\lambda)$ has been investigated by Baraket and Rădulescu [3] in the particular case when $V_1(x)=V_2(x)\equiv 1$. The main results in [3] establish the existence of a continuous spectrum of eigenvalues, respectively. the existence of a solution for λ large enough. Ayoujil and Amrouss [2] studied problem $(\mathbf{P}_\lambda)$, when $V_1(x)\equiv 1$ and $V_2(x)\equiv 0$ and $\max_{x\in\Omega} q(x)<\min_{x\in\Omega} p(x)$. The authors proved that the energy functional associated to problem $(\mathbf{P}_\lambda)$ has a nontrivial minimum for any positive λ. Our problem was also studied by Ge, Zhou and Wu [14], in the particular case when $V_2(x)\equiv 0$ and $a(x)\equiv 0$. The authors showed the existence of a continuous family of eigenvalues, more precisely they proved the existence of a positive λ^* such the problem has a solution for all $\lambda<\lambda^*$.

Finally, Kong [19] considers problem $(\mathbf{P}_\lambda)$ in the case when V_1 and V_2 are continuous functions on $\overline{\Omega}$ such that $\inf_{x\in\Omega} V_1(x)>0$ and $\inf_{x\in\Omega} V_2(x)\geq 0$. The main result in [19] establishes the existence of a nontrivial weak solution for any $\lambda>\lambda^*$. Recent contributions concerning this type of problems can be found in [17].

Inspired by the above-mentioned papers, we study problem $(\mathbf{P}_\lambda)$ under more general conditions than in [14] and [19]. In this new abstract setting, we show the existence of a weak solution for the problem $(\mathbf{P}_\lambda)$ and we impose less regularity for the potentials V_1 and V_2; moreover, V_1 may change sign in Ω. Due to the fact that we find a solution for all $\lambda>0$, our result is better than in the papers [14] and [19].

The paper is organized as follows. In Sect. 2, we recall the definition of variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$, as well as of Sobolev spaces $W^{k,p(x)}(\Omega)$. Moreover, some properties of these spaces will be also exhibited to be used later. In Sect. 3, we give the main results and their proofs. Remarks and some open problems are included in the final part of this paper.

2. Abstract setting

To study $p(x)$-biharmonic problems, we need some results on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$; see [16,23,25,26] for details, complements and proofs.

Let

$$C_+(\overline{\Omega}) := \left\{h : h \in C(\overline{\Omega}), h(x) > 1, \text{for all } \ x \in \overline{\Omega}\right\}.$$

For any $p \in C_+(\overline{\Omega})$, we set $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty$ and

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \right\}.$$

These spaces $L^{p(x)}(\Omega)$ have been introduced by Orlicz [24].

We recall the Luxemburg norm on this space, which is defined by

$$|u|_{p(x)} = \inf\left\{ \mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} \mathrm{d}x \leq 1 \right\}.$$

Clearly, when $p(x) \equiv p$, the space $L^{p(x)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$ and the norm $|u|_{p(x)}$ reduces to the standard norm $\|u\|_{L^p} = (\int_\Omega |u|^p \mathrm{d}x)^{\frac{1}{p}}$ in $L^p(\Omega)$.

For any positive integer k, let

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots . \partial^{\alpha_N} x_N}$. Then $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)}.$$

The space $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then the following Hölder-type inequality

$$\left| \int_\Omega uv \mathrm{d}x \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)}, \quad u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega). \tag{2.1}$$

holds. Moreover, if h_1, h_2 and $h_3 : \overline{\Omega} \to (1, \infty)$ are Lipschitz continuous functions such that $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$, then for any $u \in L^{h_1(x)}(\Omega), v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$ the following inequality holds (see [12, Proposition 2.5]):

$$\left| \int_\Omega uvw \, \mathrm{d}x \right| \leq \left(\frac{1}{h_1^-} + \frac{1}{h_2{}^-} + \frac{1}{h_3{}^-} \right) |u|_{h_1(x)} |v|_{h_2(x)} |w|_{h_3(x)}. \tag{2.2}$$

Inequality (2.1) and its generalized version (2.2) are due to Orlicz [24].

The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} \mathrm{d}x.$$

Proposition 2.1. (See [22]) *For all* $u, v \in L^{p(x)}(\Omega)$, *we have*

1. $|u|_{p(x)} < 1$ $(resp. = 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1$ $(resp. = 1, > 1)$.
2. $\min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \le \rho_{p(x)}(u) \le \max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+})$.
3. $\rho_{p(x)}(u - v) \to 0 \Leftrightarrow |u - v|_{p(x)} \to 0$.

Proposition 2.2. (See [8]) *Let* p *and* q *be measurable functions such that* $p \in L^\infty(\Omega)$, *and* $1 \le p(x)q(x) \le \infty$, *for a.e.* $x \in \Omega$. *Let* $u \in L^{q(x)}(\Omega)$, $u \neq 0$. *Then*

$$\min\left(|u|_{p(x)q(x)}^{p^+}, |u|_{p(x)q(x)}^{p^-}\right) \le ||u|^{p(x)}|_{q(x)} \le \max\left(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+}\right).$$

Definition 2.3. Assume that spaces E, F are Banach spaces, we define the norm on the space $X := E \cap F$ as $\|u\|_X = \|u\|_E + \|u\|_F$.

In order to discuss problem $(\mathbf{P}_\lambda)$, we need some properties of the space $X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$. From Definition 2.3, we know that for any $u \in X$ we have $\|u\| = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$, thus $\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}$. In Zang and Fu [28], the equivalence of the norms was proved, and it was even proved that the norm $|\Delta u|_{p(x)}$ is equivalent to the norm $\|u\|$ (see [28, Theorem 4.4]). Note that $(X, \|.\|)$ is a separable and reflexive Banach space.

Let

$$\|u\|_a = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\Delta u}{\mu}\right|^{p(x)} + a(x)\left|\frac{u}{\mu}\right|^{p(x)}\right) \mathrm{d}x \le 1\right\} \quad \text{for} \quad u \in X.$$

In view of $a^- > 0$, it is easy to see that $\|u\|_a$ is equivalent to the norms $\|u\|$ and $|\Delta u|_{p(x)}$ in X. In our paper, we will use the norm $\|u\|_a$. The modular on X is the mapping $\rho_{p(x)} : X \to \mathbb{R}$ defined by $\rho_{p(x)}(u) = \int_\Omega |\Delta u|^{p(x)} + a(x)|u|^{p(x)}\mathrm{d}x$. This mapping satisfies the same properties as in Proposition 2.4. More precisely, we have the following result (see [10]).

Proposition 2.4. *For all* $u \in L^{p(x)}(\Omega)$, *we have*

1. $\|u\|_a < 1$ $(resp. = 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1$ $(resp. = 1, > 1)$.
2. $\min(\|u\|_a^{p^-}, \|u\|_a^{p^+}) \le \rho_{p(x)}(u) \le \max(\|u\|_a^{p^-}, \|u\|_a^{p^+})$.

We recall that the critical Sobolev exponent is defined as follows:

$$\begin{cases} p^*(x) = \dfrac{Np(x)}{N - 2p(x)}, & p(x) < \dfrac{N}{2}, \\ p^*(x) = +\infty, & p(x) \ge \dfrac{N}{2}. \end{cases}$$

Remark 2.5. Assume that $q \in C^+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$. Then, by Theorem 3.2 in [2], the function space X is continuously and compactly embedded in $L^{q(x)}(\Omega)$.

As pointed out in [26], the function spaces with variable exponent have some striking properties, such as:

(i) If $1 < p^- \le p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula

$$\int_\Omega |u(x)|^p \mathrm{d}x = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| \mathrm{d}t$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do not have the mean continuity property. More precisely, if p is continuous and nonconstant in an open ball B, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) The function spaces with variable exponent are *never* translation invariant. The use of convolution is also limited, for instance, the Young inequality

$$|f * g|_{p(x)} \leq C\,|f|_{p(x)}\,\|g\|_{L^1}$$

holds if and only if p is constant.

3. Main result

Throughout this section, the letters $c, c_i, i = 1, 2, \ldots$. denote positive constants which may change from line to line.

In the sequel, we impose the following hypothesis:

(**H**) $1 < q(x) < \alpha(x) < p(x) < \frac{N}{2} < \min\{s_1(x), s_2(x)\}$, for all $x \in \overline{\Omega}$, where $s_1,\ s_2 \in C(\overline{\Omega})$, $V_1 \in L^{s_1(x)}(\Omega)$ such that $V_1 > 0$ in $\Omega_0 \subset\subset \Omega$, with $|\Omega_0| > 0$ and $V_2 \in L^{s_2(x)}(\Omega)$ such that $V_2 \geq 0$ in Ω.

The main result of this paper is the following.

Theorem 3.1. *Assume that hypothesis* (**H**) *holds. Then for all $\lambda > 0$, problem* ($\boldsymbol{P}_\lambda$) *has at least one nontrivial weak solution with negative energy.*

We denote by $s_1'(x)$, respectively, $s_2'(x)$ the conjugate exponents of the functions $s_1(x)$, respectively, $s_2(x)$, $r_1(x) := \frac{s_1(x)q(x)}{s_1(x)-q(x)}$ and $r_2(x) := \frac{s_2(x)\alpha(x)}{s_2(x)-\alpha(x)}$. Then the following embedding properties hold.

Remark 3.2. Under assumption (**H**), we have $\max(r_1(x), s_1'(x)q(x)) < p^*(x)$, for all $x \in \overline{\Omega}$. It follows that the embeddings $X \hookrightarrow L^{s_1'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{r_1(x)}(\Omega)$ are compact and continuous. In addition, $\max(r_2(x)s_2'(x)\alpha(x)) < p^*(x)$, for all $x \in \overline{\Omega}$. Consequently, the embeddings $X \hookrightarrow L^{s_2'(x)\alpha(x)}(\Omega)$ and $X \hookrightarrow L^{r_2(x)}(\Omega)$ are compact and continuous.

Note that under Remark 3.2, we have for all $u \in X$

$$|\Phi_1(u)| \leq \frac{1}{q^-}|V_1|_{s_1(x)}||u|^{q(x)}|_{s_1'(x)} \leq \begin{cases} \frac{1}{q^-}|V_1|_{s_1(x)}|u|^{q^-}_{s_1'(x)q(x)}, & \text{if } |u|_{s_1'(x)q(x)} \leq 1,\\ \frac{1}{q^-}|V_1|_{s_1(x)}|u|^{q^+}_{s_1'(x)q(x)}, & \text{if } |u|_{s_1'(x)q(x)} > 1.\end{cases}$$

and

$$|\Phi_2(u)| \leq \frac{1}{\alpha^-}|V_2|_{s_2(x)}||u|^{\alpha(x)}|_{s_2'(x)} \leq \begin{cases} \frac{1}{\alpha^-}|V_2|_{s(x)}|u|^{\alpha^-}_{s_2'(x)\alpha(x)}, & \text{if } |u|_{s_2'(x)\alpha(x)} \leq 1,\\ \frac{1}{\alpha^-}|V_2|_{s(x)}|u|^{\alpha^+}_{s_2'(x)\alpha(x)}, & \text{if } |u|_{s_2'(x)\alpha(x)} > 1.\end{cases}$$

Using Proposition 2.2, we deduce that Ψ_λ is well defined on X. On the other hand, in [9] it is proved the following property.

Proposition 3.3. *The energy functional $J : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous and of class C^1. Moreover, the mapping $J' : X \to X^*$ is a strictly monotone bounded homeomorphism and is of type (S_+), that is, if $u_n \rightharpoonup u$ and $\limsup\limits_{n\to+\infty} J'(u_n)(u_n - u) \leq 0$, then $u_n \to u$.*

By Proposition 3.3, we obtain that $J \in C^1(X, \mathbb{R})$. Moreover, under assumption (**H**), Proposition 2 in [5] implies that $\Phi_1,\ \Phi_2 \in C^1(X, \mathbb{R})$. Thus, $\Psi_\lambda \in C^1(X, \mathbb{R})$ and for all $u, v \in X$

$$\langle \mathrm{d}\Psi_\lambda(u), v\rangle = \int\limits_\Omega \left(|\Delta u|^{p(x)-2}\,\Delta u \Delta v + a(x)|u|^{p(x)-2}uv\right)\mathrm{d}x - \lambda\int\limits_\Omega \left(V_1(x)|u|^{q(x)-2} - V_2(x)|u|^{\alpha(x)-2}\right)uv\mathrm{d}x.$$

3.1. Proof of Theorem 3.1

By Remark 3.2, we observe that

$$|u|_{s_1'(x)q(x)} \leq c\|u\|_a \quad \text{ and } \quad |u|_{s_2'(x)\alpha(x)} \leq c_1\|u\|_a, \text{ for all } u \in X. \tag{3.1}$$

We establish in what follows some qualitative properties of the Euler–Lagrange functional Ψ_λ.

Lemma 1. *Assume that hypothesis* (**H**) *is fulfilled. Then the functional* Ψ_λ *is coercive on* X.

Proof. By virtue of assumption (**H**), Remark 3.2 and Proposition 2.4, we have for all $u \in X$ with $\|u\|_a > 1$

$$\begin{aligned}\Psi_\lambda(u) &\geq \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)}\mathrm{d}x + a(x)|u|^{p(x)}) - \frac{\lambda}{q^-}\int_\Omega |V_1(x)||u|^{q(x)}\mathrm{d}x + \frac{\lambda}{\alpha^+}\int_\Omega |V_2(x)||u|^{\alpha(x)}\mathrm{d}x,\\ &\geq \frac{1}{p^+}\rho_{p_{(x)}}(u) - \frac{\lambda}{q^-}|V_1|_{s(x)}|u^{q(x)}|_{s_1'(x)},\\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{\lambda}{q^-}|V_1|_{s(x)}\min\left(c^{q^-}\|u\|_a^{q^-}, c_1^{q^+}\|u\|_a^{q^+}\right).\end{aligned}$$

Since $q^+ < p^-$, we infer that $\Psi_\lambda(u) \to \infty$ as $\|u\| \to \infty$; in other words, Ψ_λ is coercive on X. □

In the sequel, we set $\alpha_0^- = \inf_{x\in\overline{\Omega}_0} \alpha(x)$, $q_0^- = \inf_{x\in\overline{\Omega}_0} q(x)$ and $p_0^- = \inf_{x\in\overline{\Omega}_0} p(x)$. The following result asserts the existence of a "valley" for Ψ_λ near the origin.

Lemma 2. *Assume that hypothesis* (**H**) *is fulfilled. Then there exists* $u_0 \in X$ *such that* $u_0 \geq 0, u_0 \neq 0$ *and* $\Psi_\lambda(tu_0) < 0$, *for* $t > 0$ *small enough.*

Proof. Since $q_0^- < \alpha_0^-$, let $\epsilon_0 > 0$ be such that $q_0^- + \epsilon_0 < \alpha_0^-$. Since $q \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset\subset \Omega_0$ such that $|q(x) - q_0^-| < \epsilon_0$ for all $x \in \Omega_1$. Thus, $q(x) \leq q_0^- + \epsilon_0 < \alpha_0^-$ for all $x \in \Omega_1$.

Let $u_0 \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(u_0) \subset \Omega_1 \subset \Omega_0$, $u_0 = 1$ in a subset $\Omega_1' \subset \operatorname{supp}(u_0)$, $0 \leq u_0 \leq 1$ in Ω_1. Therefore

$$\begin{aligned}\Psi_\lambda(tu_0) &= \int_\Omega \frac{t^{p(x)}}{p(x)}(|\Delta u_0|^{p(x)} + a(x)|u_0|^{p(x)}\mathrm{d}x) - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}V_1(x)|u_0|^{q(x)}\mathrm{d}x\\ &\quad + \lambda\int_\Omega \frac{t^{\alpha(x)}}{\alpha(x)}V_2(x)|u_0|^{\alpha(x)}\mathrm{d}x\\ &= \int_{\Omega_0} \frac{t^{p(x)}}{p(x)}(|\Delta u_0|^{p(x)} + a(x)|u_0|^{p(x)}\mathrm{d}x) - \lambda\int_{\Omega_1} \frac{t^{q(x)}}{q(x)}V_1(x)|u_0|^{q(x)}\mathrm{d}x\\ &\quad + \lambda\int_{\Omega_0} \frac{t^{\alpha(x)}}{\alpha(x)}V_2(x)|u_0|^{\alpha(x)}\mathrm{d}x\\ &\leq \frac{t^{p_0^-}}{p_0^-}\int_{\Omega_0}(|\Delta u_0|^{p(x)} + a(x)|u_0|^{p(x)}\mathrm{d}x) - \frac{\lambda t^{q_0^- + \epsilon_0}}{q_0^-}\int_{\Omega_1} V_1(x)|u_0|^{q(x)}\\ &\quad + \lambda\frac{t^{\alpha_0^-}}{\alpha_0^-}\int_{\Omega_0} V_2(x)|u_0|^{\alpha(x)}\mathrm{d}x\end{aligned}$$

$$\leq \frac{t^{\alpha_0^-}}{\alpha_0^-}\left[\int\limits_{\Omega_0}\left(|\Delta u_0|^{p(x)} + a(x)|u_0|^{p(x)}\mathrm{d}x)\right)\mathrm{d}x + \lambda\int\limits_{\Omega_0} V_2(x)|u_0|^{\alpha(x)}\mathrm{d}x\right]$$
$$- \frac{\lambda t^{q_0^- + \epsilon_0}}{q_0^+}\int\limits_{\Omega_1} V_1(x)|u_0|^{q(x)}\mathrm{d}x$$

It follows that

$$\Psi_\lambda(tu_0) < 0,$$

for $t < \delta^{1/(\alpha_0^- - q_0^- - \epsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda\alpha_0^- \int\limits_{\Omega_1} V_1(x)|u_0|^{q(x)}}{q_0^+\left[\int\limits_{\Omega_0}\left((|\Delta u_0|^{p(x)} + a(x)|u_0|^{p(x)}\mathrm{d}x)\right)\mathrm{d}x + \lambda\int\limits_{\Omega_0} V_2(x)|u_0|^{\alpha(x)}\mathrm{d}x\right]}\right\}.$$

Finally, we point out that

$$\int\limits_{\Omega_0}\left(|\Delta u_0|^{p(x)} + a(x)|u_0|^{p(x)}\mathrm{d}x\right)\mathrm{d}x + \lambda\int\limits_{\Omega_0} V_2(x)|u_0|^{\alpha(x)}\mathrm{d}x > 0.$$

In fact, if

$$\int\limits_{\Omega_0}\left(|\Delta u_0|^{p(x)} + a(x)|u_0|^{p(x)}\mathrm{d}x\right)\mathrm{d}x + \lambda\int\limits_{\Omega} V_2(x)|u_0|^{\alpha(x)}\mathrm{d}x = 0,$$

then

$$\int\limits_{\Omega_0}(|\Delta u_0|^{p(x)} + a(x)|u_0|^{p(x)}\mathrm{d}x) = 0.$$

It follows that $\|u_0\|_a = 0$, hence $u_0 = 0$ in Ω which is a contradiction. The proof of lemma is now complete. $\square$

Proof of Theorem 3.1 completed. Since Ψ_λ is coercive and weakly lower semi-continuous, it admits a global minimizer u, which is a critical point of Ψ_λ. By Lemma 2, we have $u \neq 0$.

In order to show that u is a solution of problem $(\mathbf{P}_\lambda)$, it remains to show that $\Delta u = 0$ on $\partial\Omega$. Since $u \in X \setminus \{0\}$ is a critical point of Ψ_λ, it follows that

$$\int\limits_{\Omega}|\Delta u|^{p(x)-2}\Delta u\Delta v\mathrm{d}x = \int\limits_{\Omega} m(x)v\mathrm{d}x \quad \text{for all } v \in X, \tag{3.2}$$

where

$$m(x) = \lambda(V_1(x)|u|^{q(x)-2}u - V_2(x)|u|^{\alpha(x)-2}u) - a(x)|u|^{p(x)-2}u.$$

Relation (3.2) implies that

$$\int\limits_{\Omega}|\Delta u|^{p(x)-2}\Delta u\Delta v\mathrm{d}x = \int\limits_{\Omega} m(x)v\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega). \tag{3.3}$$

Let ζ be the unique solution of the problem

$$\begin{cases}\Delta\zeta = m(x) & \text{in } \Omega\\ \zeta = 0 & \text{on } \partial\Omega.\end{cases}$$

Relation (3.3) yields

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v\mathrm{d}x = \int_\Omega (\Delta\zeta)v\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega).$$

Using the Green formula, we have

$$\int_\Omega (\Delta\zeta)v\mathrm{d}x = \int_\Omega \zeta\Delta v\mathrm{d}x.$$

Therefore

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v\mathrm{d}x = \int_\Omega \zeta\Delta v\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega). \tag{3.4}$$

On the other hand, for all $w \in C_0^\infty(\Omega)$ there exists a unique $v \in C_0^\infty(\Omega)$ such that $\Delta v = w$ in Ω. Thus, relation (3.4) can be rewritten as

$$\int_\Omega (|\Delta u|^{p(x)-2}\Delta u - \zeta)w\mathrm{d}x = 0 \quad \text{for all } w \in C_0^\infty(\Omega).$$

Applying the fundamental lemma of the calculus of variations, we deduce that

$$|\Delta u|^{p(x)-2}\Delta u - \zeta = 0 \quad \text{in } \Omega.$$

Since $\zeta = 0$ on $\partial\Omega$, we conclude that $\Delta u = 0$ on $\partial\Omega$. The proof is now complete. □

3.2. An alternative proof for small perturbations

We state in what follows a weaker version of Theorem 3.1, which establishes the existence of solutions to problem $(\mathbf{P}_\lambda)$ in the case of *small perturbations*.

Theorem 3.4. *Assume that hypothesis* $(\mathbf{H})$ *is fulfilled. Then there exists* $\lambda_* > 0$ *such that for all* $\lambda \in (0, \lambda_*)$ *problem* $(\boldsymbol{P}_\lambda)$ *has a solution.*

The key argument in the proof is related to Ekeland's variational principle [11]. This approach was introduced for the first time in [23] in the framework of problems with variable exponent.

We start with some preliminary results. The following property shows the existence of a *mountain* for Ψ_λ near the origin.

Lemma 3. *Assume that hypothesis* $(\mathbf{H})$ *is fulfilled. Then for all* $\rho \in (0,1)$ *there exist* $\lambda_* > 0$ *and* $b > 0$ *such that for all* $u \in X$ *with* $\|u\| = \rho$ *we have* $\Psi_\lambda(u) \geq b > 0$*, for any* $\lambda \in (0, \lambda_*)$.

Proof. Let us assume that $\|u\|_a < \min(1, 1/c)$, where c is the positive constant given in (3.1). It follows that $|u|_{s_1'(x)q(x)} < 1$.

Using assumption (**H**), Hölder inequality (2.1), Proposition (2.2) and relation (3.1), we deduce that for any $u \in X$ with $\|u\|_a = \rho$ the following inequalities hold true:

$$\begin{aligned}
\Psi_{\lambda,\mu}(u) &\geq \frac{1}{p^+}\int_\Omega (|\Delta u|^{p(x)} + a(x)|u|^{p(x)})\mathrm{d}x - \frac{\lambda}{q^-}\int_\Omega V_1(x)|u|^{q(x)}\mathrm{d}x + \frac{\lambda}{\alpha^+}\int_\Omega V_2(x)|u|^{\alpha(x)}\mathrm{d}x \\
&\geq \frac{1}{p^+}\rho_{p(x)}(u) - \frac{\lambda}{q^-}\int_\Omega V_1(x)|u|^{q(x)}\mathrm{d}x \\
&\geq \frac{1}{p^+}\|u\|_a^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}||u|^{q(x)}|_{s_1'(x)} \\
&\geq \frac{1}{p^+}\,\|u\|_a^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}|u|^{q^-}_{s_1'(x)q(x)} \\
&\geq \frac{1}{p^+}\,\|u\|_a^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}c^{q^-}\,\|u\|_a^{q^-} \\
&\geq \frac{1}{p^+}\rho^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}c^{q^-}\,\rho^{q^-} = \rho^{q^-}\left(\frac{1}{p^+}\rho^{p^+-q^-} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}c^{q^-}\right).
\end{aligned}$$

This inequality shows that if we define

$$\lambda_* = \frac{q^-}{2p^+|V_1|_{s_1(x)}c^{q^-}}\rho^{p^+-q^-}, \tag{3.5}$$

then for all $\lambda \in (0, \lambda_*)$ and for all $u \in X$ with $\|u\|_a = \rho$, there exists $b > 0$ such that

$$\Psi_\lambda(u) \geq b > 0.$$

The proof of Lemma 3 is complete. □

Let λ_* be defined as in (3.5) and $\lambda \in (0, \lambda_*)$. By Lemma 3, it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} \Psi_\lambda > 0. \tag{3.6}$$

On the other hand, Lemma 2 yields the existence of $\phi \in X$ such that $\Psi_\lambda(t\phi) < 0$ for all $t > 0$ small enough. This shows the existence of a *valley* for the Euler–Lagrange functional Ψ_λ, hence problem $(\mathbf{P}_\lambda)$ does not have a mountain pass geometry.

Using assumption (**H**), the Hölder inequality (2.1), Proposition 2.2 and relation (3.1), we deduce that for any $u \in X$ with $\|u\|_a = \rho$ the following inequality holds true:

$$\Psi_\lambda(u) \geq \frac{1}{p^+}\,\|u\|_a^{p^+} - \frac{\lambda}{q^-}|V_1|_{s_1(x)}c^{q^-}\,\|u\|_a^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} \Psi_\lambda < 0.$$

Fix $0 < \epsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda - \inf_{B_\rho(0)} \Psi_\lambda$. Using the above information, the functional $\Psi_{\lambda,\mu} : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$, is bounded from below on $\overline{B_\rho(0)}$ and $\Psi_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Thus, by the Ekeland variational principle, there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq \Psi_{\lambda,\mu}(u_\epsilon) \leq \underline{c} + \epsilon, \\ 0 < \Psi_\lambda(u) - \Psi_\lambda(u_\epsilon) + \epsilon\cdot \parallel u - u_\epsilon \parallel_a, \quad u \neq u_\epsilon. \end{cases}$$

Since

$$\Psi_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} \Psi_\lambda + \epsilon \leq \inf_{B_\rho(0)} \Psi_\lambda + \epsilon < \inf_{\partial B_\rho(0)} \Psi_{\lambda,\mu},$$

we deduce that $u_\epsilon \in B_\rho(0)$.

Now, we define $I_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $I_\lambda(u) = \Psi_\lambda(u) + \epsilon \cdot \| u - u_\epsilon \|_a$. Then u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{\Psi_\lambda(u_\epsilon + t \cdot v) - \Psi_\lambda(u_\epsilon)}{t} + \epsilon \cdot \| v \|_a \geq 0.$$

Letting $t \to 0$, it follows that $\langle \mathrm{d}\Psi_\lambda(u_\epsilon), v\rangle + \epsilon \ \| v \|_a \geq 0$ and we infer that $\| \mathrm{d}\Psi_\lambda(u_\epsilon) \|_a \leq \epsilon$.

We deduce that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$\Psi_\lambda(u_n) \longrightarrow \underline{c} \quad \text{and} \quad \mathrm{d}\Psi_\lambda(u_n) \longrightarrow 0_{X^*}. \tag{3.7}$$

The sequence $\{u_n\}$ is bounded in X. Thus, there exists a subsequence again denoted by $\{u_n\}$, and u in X such that, $\{u_n\}$ converges weakly to u in X. Since $r_1(x), r_2(x) < p^*(x)$, for all $x \in \overline{\Omega}$, then X is compactly embedded in $L^{r_1(x)}(\Omega)$ and $L^{r_2(x)}(\Omega)$. It follows that $\{u_n\}$ converges strongly to u in $L^{r_1(x)}(\Omega)$ and $L^{r_2}(\Omega)$, respectively.

In order to establish the strong convergence of $\{u_n\}$ on X, we need the following auxiliary property.

Proposition 3.5. *We have*

(i) $\lim_{n\to\infty} \int_\Omega V_1(x)|u_n|^{q(x)-2}u_n(u_n - u)\mathrm{d}x = 0.$

(ii) $\lim_{n\to\infty} \int_\Omega V_2(x)|u_n|^{\alpha(x)-2}u_n(u_n - u)\mathrm{d}x = 0.$

Proof. Using the generalized Hölder inequality (2.2), we have

$$\int_\Omega V_1(x)|u_n|^{q(x)-2}u_n(u_n - u)\mathrm{d}x \leq |V_1|_{s_1(x)}||u_n|^{q(x)-2}u_n|_{\frac{q(x)}{q(x)-1}}|u_n - u|_{r_1(x)},$$

Next, by Proposition 2.2, if

$$||u_n|^{q(x)-2}u_n|_{\frac{q(x)}{q(x)-1}} > 1$$

then

$$||u_n|^{q(x)-2}u_n|_{\frac{q(x)}{q(x)-1}} \leq |u_n|^{q^+}_{q(x)}.$$

Using the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, we conclude the proof of (i).

Using a similar arguments on the functions V_2, α and r_2 we get (ii). □

The above information and relation (3.7) imply

$$|\langle \mathrm{d}\Psi_\lambda(u_n) - \mathrm{d}\Psi_\lambda(u), u_n - u\rangle| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.8}$$

Therefore

$$\lim_{n\to\infty} \int_\Omega \Big(|\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta w) + a(x)|u_n|^{p(x)-2}u_n(u_n - u)\Big)\mathrm{d}x = 0.$$

Now, Proposition 3.3 ensures that $\{u_n\}$ converges strongly to w in X. Since $\Psi_\lambda \in C^1(X, \mathbb{R})$, we conclude

$$\mathrm{d}\Psi_\lambda(u_n) \to \mathrm{d}\Psi_\lambda(u), \tag{3.9}$$

as $n \to \infty$.

Since $\Psi_\lambda \in C^1(X, \mathbb{R})$, we obtain

$$\mathrm{d}\Psi_\lambda(u_n) \to \mathrm{d}\Psi_\lambda(u) \quad \text{as} \quad n \to \infty. \tag{3.10}$$

Relations (3.7) and (3.10) show that $\mathrm{d}\Psi_\lambda(u) = 0$ and thus u is a weak solution for problem $(\mathbf{P}_\lambda)$. Moreover, by relation 3.7 it follows that $\Psi_\lambda(u) < 0$, hence u is nontrivial. Since $\Psi_\lambda(|u|) = \Psi_\lambda(u)$ then problem $(\mathbf{P}_\lambda)$ has a positive solution. This completes the proof of Theorem 3.4. □

3.3. Final comments

(i) Problem $(\mathbf{P}_\lambda)$ corresponds to a *subcritical* setting, as described by Remark 3.2. We consider that valuable research directions correspond either to the *critical* or to the *supercritical* framework (in the sense of Sobolev variable exponents). No results are known even for the *almost critical* case with *lack of compactness*. More precisely, with the same notations as in Remark 3.2, a very interesting open problem is to study the qualitative analysis of solutions of $(\mathbf{P}_\lambda)$ provided that there exists z_1, $z_2 \in \Omega$ such that

$$\max(r_1(z_1), s_1'(z_1)q(z_1)) = p^*(z_1) \quad \text{and} \quad \max(r_2(z_2)s_2'(z_2)\alpha(z_2)) = p^*(z_2)$$

but

$$\max(r_1(x), s_1'(x)q(x)) < p^*(x) \quad \text{for all } z \in \Omega \setminus \{z_1\}$$

and

$$\max(r_2(x)s_2'(x)\alpha(x)) < p^*(x) \quad \text{for all } z \in \Omega \setminus \{z_2\}.$$

(ii) Another very interesting research direction is to extend the approach developed in this paper to the abstract setting recently studied by Mingione et al. [4,7], namely *double-phase* problems with associated energies of the type

$$u \mapsto \int_\Omega [|\Delta u|^{p_1(x)} + V(x)|\Delta u|^{p_2(x)}]\mathrm{d}x$$

and

$$u \mapsto \int_\Omega [|\Delta u|^{p_1(x)} + V(x)|\Delta u|^{p_2(x)} \log(e + |x|)]\mathrm{d}x,$$

where $p_1(x) \le p_2(x)$, $p_1 \ne p_2$, and $V(x) \ge 0$. Considering two different materials with power hardening exponents $p_1(x)$ and $p_2(x)$, respectively, the coefficient $V(x)$ dictates the geometry of a composite of the two materials. When $V(x) > 0$ then $p_2(x)$-material is present, otherwise the $p_1(x)$-material is the only one making the composite.

These problems extend to a biharmonic setting with variable exponents the pioneering papers by Marcellini [20,21] on (p, q)-problems, which involve integral functionals of the type

$$u \mapsto \int_\Omega F(x, \nabla u)\mathrm{d}x.$$

The integrand $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfied unbalanced polynomial growth conditions of the type

$$|\xi|^p \lesssim F(x, \xi) \lesssim |\xi|^q \quad \text{with } 1 < p < q,$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

(iii) We suggest to extend the methods developed in this paper to the more general framework of Musielak–Orlicz–Sobolev spaces (see [26, Chaper 4] for a collection of stationary problems studied in these function spaces).

Acknowledgements

V.D. Rădulescu acknowledges the support through a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2016-0130.

References

[1] Antontsev, S.N., Shmarev, S.I.: A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions. Nonlinear Anal. **60**, 515545 (2005)
[2] Ayoujil, A., El Amrouss, A.: Continuous spectrum of a fourth-order nonhomogeneous elliptic equation with variable exponent. Electron. J. Differ. Equ. **24**, 112 (2011)
[3] Baraket, S., Rădulescu, V.: Combined effects of concave-convex nonlinearities in a fourth-order problem with variable exponent. Adv. Nonlinear Stud. **16**, 409–419 (2016)
[4] Baroni, P., Colombo, M., Mingione, G.: Non-autonomous functionals, borderline cases and related function classes. St. Petersb. Math. J. **27**, 347–379 (2016)
[5] Bouslimi, M., Kefi, K.: Existence of solution for an indefinite weight quasilinear problem with variable exponent. Complex Variables Elliptic Equ. **58**, 1655–1666 (2013)
[6] Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image processing. SIAM J. Appl. Math. **66**, 1383–1406 (2006)
[7] Colombo, M., Mingione, G.: Bounded minimisers of double phase variational integrals. Arch. Ration. Mech. Anal. **218**, 219–273 (2015)
[8] Edmunds, D., Rakosnik, J.: Sobolev embeddings with variable exponent. Stud. Math. **143**, 267–293 (2000)
[9] El Amrous, A., Moradi, F., Moussaoui, M.: Existence of solutions for fourth-order PDEs with variable exponents. Electron. J. Differ. Equ. **153**, 1–13 (2009)
[10] El Amrous, A., Ourraoui, A.: Existence of solutions for a boundary value problem involving $p(x)$-biharmonic operator. Bol. Soc. Paran. Mater. **31**, 179–192 (2013)
[11] Ekeland, I.: On the variational principle. J. Math. Anal. Appl. **47**, 324–353 (1974)
[12] Fan, X., Han, X.: Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$. Nonlinear Anal. **59**, 173–188 (2004)
[13] Ferrero, A., Warnault, G.: On solutions of second and fourth order elliptic equations with power-type nonlinearities. Nonlinear Anal. **70**, 2889–2902 (2009)
[14] Ge, B., Zhou, Q.-M., Wu, Y.-H.: Eigenvalue of the $p(x)$-biharmonic operator with indefinite weight. Z. Angew. Math. Phys. **66**, 1007–1021 (2015)
[15] Halsey, T.C.: Electrorheological fluids. Science **258**, 761–766 (1992)
[16] Harjulehto, P., Hästö, P., Le, U.V., Nuortio, M.: Overview of differential equations with non-standard growth. Nonlinear Anal. **72**, 4551–4574 (2010)
[17] Heidarkhani, S., Afrouzi, G.A., Moradi, S., Caristi, G., Ge, B.: Existence of one weak solution for $p(x)$-biharmonic equations with Navier boundary conditions. Z. Angew. Math. Phys.**67**(3), Art. 73 (2016)
[18] King, J.R.: Two generalisations of the thin film equation. Math. Comput. Model. **34**, 737–756 (2001)
[19] Kong, L.: On a fourth order elliptic problem with a $p(x)$-biharmonic operator. Appl. Math. Lett. **27**, 21–25 (2014)
[20] Marcellini, P.: On the definition and the lower semicontinuity of certain quasiconvex integrals. Ann. Inst. H. Poincaré Anal. Non Linéaire **3**, 391–409 (1986)
[21] Marcellini, P.: Regularity and existence of solutions of elliptic equations with (p, q)-growth conditions. J. Differ. Equ. **90**, 1–30 (1991)
[22] Mashiyev, R.A., Ogras, S., Yucedag, Z., Avci, M.: Existence and multiplicity of weak solutions for nonuniformly elliptic equations with non-standard growth condition. Complex Variables Elliptic Equ. **57**, 579–595 (2012)
[23] Mihailescu, M., Rădulescu, V.D.: On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. Proc. Am. Math. Soc. **135**, 2929–2937 (2007)
[24] Orlicz, W.: Über konjugierte Exponentenfolgen. Stud. Math. **3**, 200–212 (1931)
[25] Rădulescu, V.D.: Nonlinear elliptic equations with variable exponent: old and new. Nonlinear Anal. **121**, 336–369 (2015)

[26] Rădulescu, V.D., Repovš, D.D.: Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis, Monographs and Research Notes in Mathematics. Taylor & Francis, Chapman and Hall/CRC, Boca Raton (2015)
[27] Ružička, M.: Electrorheological Fluids: Modeling and Mathematical Theory. Springer, Berlin (2000)
[28] Zang, A.B., Fu, Y.: Interpolation inequalities for derivatives in variable exponent Lebesgue Sobolev spaces. Nonlinear Anal. **69**, 3629–3636 (2008)
[29] Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR. Izv. **9**, 33–66 (1987)
[30] Zhikov, V.V.: Lavrentiev phenomenon and homogenization for some variational problems. C. R. Acad. Sci. Paris Sér. I Math. **316**(5), 435–439 (1993)
[31] Zhikov, V.V.: On Lavrentiev's phenomenon. Russ. J. Math. Phys. **3**(2), 249–269 (1995)

Khaled Kefi
Community College of Rafha
Northern Border University
Arar
Kingdom of Saudi Arabia
e-mail: khaled_kefi@yahoo.fr

Khaled Kefi
Mathematics Department, Faculty of Sciences
Tunis El Manar
1060 Tunis
Tunisia

Vicenţiu D. Rădulescu
Department of Mathematics, Faculty of Sciences
King Abdulaziz University
P.O. Box 80203 Jeddah 21589
Kingdom of Saudi Arabia
e-mail: vicentiu.radulescu@imar.ro

Vicenţiu D. Rădulescu
Department of Mathematics
University of Craiova
Street A.I. Cuza 13
200585 Craiova
Romania

(Received: February 28, 2017; revised: March 28, 2017)